

Zitterbewegung in Quantum Mechanics

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The possibility that *zitterbewegung* opens a window to particle substructure in quantum mechanics is explored by constructing a particle model with structural features inherent in the Dirac equation. This paper develops a self-contained dynamical model of the electron as a lightlike particle with helical *zitterbewegung* and electromagnetic interactions. The model admits periodic solutions with quantized energy, and the correct magnetic moment is generated by charge circulation. It attributes to the electron an electric dipole moment rotating with ultrahigh frequency, and the possibility of observing this directly as a resonance in electron channeling is analyzed in detail. Correspondence with the Dirac equation is discussed. A modification of the Dirac equation is suggested to incorporate the rotating dipole moment.

PACS numbers: 10.03.65.-w

Keywords: *zitterbewegung*, geometric algebra, electron channeling, de Broglie frequency

I. INTRODUCTION

This paper continues a research program investigating implications of the *Real Dirac Equation* for the interpretation and extension of quantum mechanics. Details of the program have been reviewed elsewhere [1–3], so it suffices here to state the main ideas and conclusions to set the stage for the present study.

The program began with a reformulation of the Dirac equation in terms of *Spacetime Algebra* (Section II), which revealed geometric structure that is suppressed in the standard matrix version. In particular, it revealed that the generator of phase and electromagnetic gauge transformations is a spacelike bivector specified by electron spin. In other words, spin and phase are inseparably related — spin is not simply an add-on, but an essential feature of quantum mechanics. However, physical implications of this fact depend critically on relations of the Dirac wave function to physical observables, which are not specified by the Dirac equation itself. That started the present research program to investigate various possibilities.

A standard observable in Dirac theory is the Dirac current, which doubles as a probability current and a charge current. However, this does not account for the magnetic moment of the electron, which many investigators conjecture is due to a circulation of charge. What, then, is the nature of the charge circulation? High energy scattering experiments limit the size of the electron to less than 10^{-16} cm, [4] which rules out models of the electron as an extended body. After a lengthy analysis of the Dirac equation Bohm and Hiley conclude [5]: “the electron must still be regarded as a simple point particle whose only intrinsic property is its position.” Under this assumption, spin and phase must be expressed in the kinematics of electron motion. The charge circula-

tion that generates the magnetic moment can then be identified with the *zitterbewegung* of Schroedinger [6].

This raises the central question of the present research: Is the *zitterbewegung*, so construed, a real physical phenomenon, or is it merely a colorful metaphor? Although this question was motivated by structural features of the Dirac equation, it cannot be answered without attributing substructure to electron motion that is not specified by standard Dirac theory.

The main purpose of this paper is to formulate and study a well-defined particle model of the electron with *spin and zitterbewegung dynamics* motivated by the Dirac equation. Since the term *zitterbewegung* is quite a mouthful, I often abbreviate it to *zitter*, especially when it is used as an adjective.

We study the structure of the *zitter particle model* in considerable detail with the aim of identifying new experimental implications. The main conclusion is that the electron is the seat of a rapidly rotating electric dipole moment fluctuating with the *zitter* frequency of Schroedinger. As this frequency is so rapid, it is observable only under resonance conditions. It is argued that familiar quantum mechanical effects, such as quantized states, may be attributable to *zitter* resonance. In particular, the Landau levels for electron motion in a magnetic field correspond to exact periodic solutions. Moreover, the possibility of observing *zitter* directly as a resonance in electron channeling is analyzed and supported with experimental evidence that has not been explained by standard quantum mechanics.

The relation of the *zitter* particle model to the Dirac equation is also studied. The main conclusion is that, though *zitter* oscillations are inherent in the Dirac equation, they will not be manifested as an oscillating electric dipole without altering the definition of charge current. A simple modification of the Dirac equation to incorporate the altered definition is proposed.

In conclusion, the relation of the *zitter* particle model to the Dirac equation can be considered from two different perspectives. On the one hand, it can be regarded as a “quasiclassical” approximation that embodies struc-

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tural features of the Dirac equation in a convenient form for analysis. On the other hand, it can be regarded as formulating fundamental properties of the electron that are manifested in the Dirac equation in some kind of average form. The choice of perspective is left to the reader.

II. SPACETIME ALGEBRA

Spacetime algebra is thoroughly expounded elsewhere [1], so a brief description is sufficient here, mainly to define terms.

We represent Minkowski *spacetime* as a real 4-dimensional vector space \mathcal{M}^4 . The two properties of scalar multiplication and vector addition in \mathcal{M}^4 provide only a partial specification of spacetime geometry. To complete the specification we introduce an associative *geometric product* among vectors a, b, c, \dots with the property that the square of any vector is a (real) scalar. Thus for any vector a we can write

$$a^2 = aa = \epsilon|a|^2, \quad (1)$$

where ϵ is the *signature* of a and $|a|$ is a (real) positive scalar. As usual, we say that a is *timelike*, *lightlike* or *spacelike* if its signature is positive ($\epsilon = 1$), null ($\epsilon = 0$), or negative ($\epsilon = -1$). We can specify the signature of \mathcal{M}^4 as a whole, by adopting the axioms: (a) \mathcal{M}^4 contains at least one timelike vector; and (b) every 2-plane in \mathcal{M}^4 contains at least one spacelike vector.

To facilitate applications of STA to physics a few definitions and theorems are needed. From the *geometric product* uv of two vectors it is convenient to define two other products. The *inner product* $u \cdot v$ is defined by

$$u \cdot v = \frac{1}{2}(uv + vu) = v \cdot u, \quad (2)$$

while the *outer product* $u \wedge v$ is defined by

$$u \wedge v = \frac{1}{2}(uv - vu) = -v \wedge u. \quad (3)$$

The three products are therefore related by

$$uv = u \cdot v + u \wedge v. \quad (4)$$

This can be regarded as a decomposition of the product uv into symmetric and skewsymmetric parts, or alternatively, into scalar and bivector parts.

The inner and outer products can be generalized. One way is to define the outer product along with the notion of *k-vector* iteratively as follows: scalars are defined to be 0-vectors, vectors are 1-vectors, and bivectors, such as $u \wedge v$, are 2-vectors. For a given *k-vector* K , the integer k is called the *grade* of K . For $k \geq 1$, the outer product of a vector v with a *k-vector* K is a $(k+1)$ -vector defined in terms of the geometric product by

$$v \wedge K = \frac{1}{2}(vK + (-1)^k K v) = (-1)^k K \wedge v. \quad (5)$$

The corresponding inner product is defined by

$$v \cdot K = \frac{1}{2}(vK + (-1)^{k+1} K v) = (-1)^{k+1} K \cdot v, \quad (6)$$

and it can be proved that the result is a $(k-1)$ -vector. Adding (5) and (6) we obtain

$$vK = v \cdot K + v \wedge K, \quad (7)$$

which obviously generalizes (4). The important thing about (7), is that it decomposes vK into $(k-1)$ -vector and $(k+1)$ -vector parts.

By continuing as above, STA has been developed into a complete coordinate-free calculus for spacetime physics [1]. However, to hasten comparison with standard Dirac algebra, we interrupt that approach to introduce coordinates and a basis for the algebra. Let $\{\gamma_\mu; 0, 1, 2, 3\}$ be a *right-handed orthonormal frame* of vectors with γ_0 in the forward light cone. In accordance with (2), the components $g_{\mu\nu}$ of the metric tensor for this frame are given by

$$g_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu). \quad (8)$$

This will be recognized as isomorphic to a famous formula of Dirac's. The difference here is that the γ_μ are vectors rather than matrices. A *coframe* $\{\gamma^\mu\}$ is defined by the usual convention for raising and lowering indices: $\gamma_\mu = g_{\mu\nu} \gamma^\nu$.

The unit pseudoscalar i for spacetime is related to the frame $\{\gamma_\nu\}$ by the equation

$$i = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3. \quad (9)$$

It is readily verified from (9) that $i^2 = -1$, and the geometric product of i with any vector is anticommutative.

By multiplication the γ_μ generate a complete basis of *k-vectors* for STA, consisting of the $2^4 = 16$ linearly independent elements

$$1, \quad \gamma_\mu, \quad \gamma_\mu \wedge \gamma_\nu, \quad \gamma_\mu i, \quad i. \quad (10)$$

Multivectors with even grade compose a subalgebra of the STA generated by the bivectors $\{\sigma_k \equiv \gamma_k \gamma_0; k = 1, 2, 3\}$, so that

$$\sigma_1 \sigma_2 \sigma_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = i. \quad (11)$$

Any multivector can be expressed as a linear combination of these elements.

For example, a bivector F has the expansion

$$F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \quad (12)$$

with its "scalar components" $F^{\mu\nu}$ given by

$$F^{\mu\nu} = \gamma^\mu \cdot F \cdot \gamma^\nu = \gamma^\nu \cdot (\gamma^\mu \cdot F) = (\gamma^\nu \wedge \gamma^\mu) \cdot F. \quad (13)$$

Note that the two inner products in the middle term can be performed in either order, so a parenthesis is not needed; also, we use the usual convention for raising and lowering indices.

Alternatively, one can decompose the bivector into

$$F = \mathbf{E} + i\mathbf{B} = F^{k0} \sigma_k + \frac{1}{2} F^{kj} \sigma_j \sigma_k, \quad (14)$$

corresponding to the split of an electromagnetic field into "electric and magnetic parts."

The entire spacetime algebra is obtained from linear combinations of basis k -vectors in (10). A generic element M of the STA, called a *multivector*, can thus be written in the *expanded form*

$$M = \alpha + a + F + bi + \beta i = \sum_{k=0}^4 \langle M \rangle_k, \quad (15)$$

where α and β are scalars, a and b are vectors, and F is a bivector. This is a decomposition of M into its k -vector parts, with $k = 0, 1, 2, 3, 4$, where $\langle \dots \rangle_k$ means “ k -vector part.” Of course, $\langle M \rangle_0 = \alpha$, $\langle M \rangle_1 = a$, $\langle M \rangle_2 = F$, $\langle M \rangle_3 = bi$, $\langle M \rangle_4 = \beta i$.

Obviously, we can decompose M into even and odd parts given, respectively, by

$$\langle M \rangle_+ = \alpha + F + \beta i, \quad \langle M \rangle_- = a + bi. \quad (16)$$

Note that the even part commutes with the pseudoscalar i , while the odd part anticommutes with it.

Computations are facilitated by the operation of *reversion*. For M in the expanded form (15) the *reverse* \widetilde{M} can be defined by

$$\widetilde{M} = \alpha + a - F - bi + \beta i. \quad (17)$$

Note, in particular, the effect of reversion on the various k -vector parts: $\widetilde{\alpha} = \alpha$, $\widetilde{a} = a$, $\widetilde{F} = -F$, $\widetilde{i} = i$.

It is not difficult to prove that

$$(MN)^\sim = \widetilde{N}\widetilde{M}, \quad (18)$$

for arbitrary multivectors M and N . For scalar parts $\langle M \rangle \equiv \langle M \rangle_0$, we have

$$\langle \widetilde{M} \rangle = \langle M \rangle \quad \text{hence} \quad \langle MN \rangle = \langle NM \rangle. \quad (19)$$

The scalar part in STA corresponds to the trace in Dirac’s matrix representation of the algebra.

Besides the inner and outer products defined above, many other products can be defined in terms of the geometric product. We will need the *commutator product* $M \times N$, defined for any M and N by

$$M \times N \equiv \frac{1}{2}(MN - NM) = -N \times M. \quad (20)$$

If M is a bivector, this product is grade preserving. Also, the product is distributive on the geometric product, with the form of a “derivation”:

$$M \times (NP) = (M \times N)P + N(M \times P). \quad (21)$$

If the geometric product is replaced by the commutator product in this expression, it assumes the familiar form of a Jacobi identity.

For bivectors S and F we can now expand the geometric product as follows:

$$SF = S \cdot F + S \times F + S \wedge F \quad (22)$$

where $S \times F$ is a bivector and $S \wedge F$ is a pseudoscalar. Note how this differs from the expansion (7).

One great advantage of STA is that it provides coordinate-free representations of both tensors and spinors in the same system. Spinors can be represented as even multivectors serving as algebraic operators. Using reversion, it is easy to prove that every even multivector ψ satisfies

$$\psi \widetilde{\psi} = \rho e^{i\beta}. \quad (23)$$

Hence, for $\rho \neq 0$, ψ can be written in the *canonical form*

$$\psi = (\rho e^{i\beta})^{\frac{1}{2}} R, \quad (24)$$

where

$$R\widetilde{R} = \widetilde{R}R = 1. \quad (25)$$

Every such R determines a *Lorentz rotation* of a given multivector M :

$$R : M \rightarrow M' = RM\widetilde{R}, \quad (26)$$

and every Lorentz rotation can be expressed in this coordinate-free form. Construed as an operator in this sense, the quantity R is called a *rotor* while ψ is called a *spinor*. We shall describe particle kinematics and the Dirac wave function by spinors in this sense.

The set of all rotors form a multiplicative group called the *rotor group* or the *spin group* of spacetime. When $R = R(\tau)$ represents a one parameter family of rotors (hence of Lorentz rotations), “angular velocities” Ω and Ω' are defined by

$$\dot{R} = \frac{1}{2}\Omega R = \frac{1}{2}R\Omega', \quad (27)$$

or

$$\Omega = 2\dot{R}\widetilde{R} = R\Omega'\widetilde{R} \quad \text{and} \quad \Omega' = 2\widetilde{R}\dot{R}, \quad (28)$$

where the overdot indicates derivative. It follows from (25) and (15) that Ω and Ω' are bivectors.

We represent each spacetime point as a vector $x = x^\mu \gamma_\mu$ with rectangular coordinates $x^\mu = \gamma^\mu \cdot x$. The *vector derivative* with respect to the point x is defined by $\nabla = \partial_x = \gamma^\mu \partial_{x^\mu}$. As ∇ is a vectorial operator, we can use (7) to decompose the derivative of a k -vector field $K = K(x)$ into

$$\nabla K = \nabla \cdot K + \nabla \wedge K, \quad (29)$$

where the terms on the right can be identified, respectively, with the usual *divergence* and *curl* in tensor calculus.

Besides the STA definitions and relations given above, many others can be found in the references. We shall introduce some of them as needed.

III. CLASSICAL PARTICLES WITH SPIN

Classical models of the electron as a point particle with spin were first formulated by Frenkel [7] and Thomas [8], improved by Mathisson [9], and given a definitive form by Weyssenhoff [10]. They have been revisited from time to time by many investigators [11–13], including others to be cited below. They are of interest mainly for the insight they bring to the interpretation of quantum mechanics. But they are also of practical value, for example, in the study of spin precession [1, 14] and tunneling [15–17].

In Weyssenhoff’s analysis the models fall into distinct classes, differentiated by the assumption that the electron’s spacetime history is timelike in one and lightlike in the other. The timelike case has been studied by many investigators, such as those cited above. Ironically, the lightlike case, which Weyssenhoff regarded as far more interesting, has been generally ignored. Without being aware of his analysis, I arrived at similar conclusions from my study of real Dirac theory. This paper revisits Weyssenhoff theory with new mathematical tools to simplify and extend it.

Our first task is to reformulate Weyssenhoff theory in the language of spacetime algebra (STA) and show how it is simplified, clarified and extended. A decided advantage of STA is its uniform use of spinors in both classical and quantum theory. For one thing, this makes it easier to relate classical models to solutions of the Dirac equation [1]. Although we are most interested in the lightlike case, we treat the timelike case in enough detail to compare the two cases. Besides, the timelike models can be used for other particles with spin besides electrons, including atoms.

We consider a particle with spacetime *history* $z = z(\tau)$ and *proper velocity* [1]

$$u \equiv \dot{z} = \frac{dz}{d\tau}. \quad (30)$$

For the time being, we allow u to be either timelike, with constant $u^2 > 0$, or lightlike, with constant $u^2 = 0$, so in either case

$$\dot{u} \cdot u = 0. \quad (31)$$

For the time being, the time τ will be regarded as an arbitrary affine parameter.

Suppose the particle has *proper momentum* $p = p(\tau)$ and intrinsic angular momentum (*spin*) represented by a bivector $S = S(\tau)$. Equations of motion are obtained from general conservation laws for momentum and angular momentum. However, to formulate these laws correctly it is essential to note that just as p contributes an orbital momentum, S contributes an intrinsic part to the momentum, so we cannot make the usual assumption $p = mu$. Instead, the relation between p and u depends on the dynamics of motion. The same can be said about the relation between S and u , although we shall assume that it is restricted by the *kinematical constraint*

$$S \cdot u = 0. \quad (32)$$

This reduces the degrees of freedom in S to three at most.

The noncollinearity of p and u raises a question about how mass should be defined. Without prejudicing the issue, it is convenient to introduce a *dynamical mass* defined by

$$m \equiv p \cdot u. \quad (33)$$

This quantity is well defined for both timelike and lightlike particles, so lightlike particles can have mass in this sense. However, the value of this mass obviously depends on the choice of time parameter τ , which, so far, is not well defined in the lightlike case. This is our first hint that mass is intimately related to intrinsic time scaling of electron histories.

Without presupposing a relation between momentum and velocity, *momentum conservation* can be given the general form

$$\dot{p} = f, \quad (34)$$

where the *proper force* f describes transfer of momentum through external interactions.

Angular momentum conservation is governed by

$$\dot{S} = u \wedge p + \Gamma, \quad (35)$$

where the *proper torque* Γ describes angular momentum transfer through external interactions. To see where (35) comes from, we introduce the *orbital angular momentum* $p \wedge z$ so the *total angular momentum* is

$$J \equiv p \wedge z + S. \quad (36)$$

The angular momentum conservation then has the more familiar form

$$\dot{J} = f \wedge z + \Gamma, \quad (37)$$

where $f \wedge z$ is the orbital torque. The equivalence of (37) with (35) follows immediately by differentiating (36).

Equation (37) tells us that J is a constant of motion for a free particle with spin. Spin is not separately conserved. According to (36), angular momentum can be exchanged back and forth between orbital and spin parts.

To get well defined equations of motion from the conservation laws we need to specify the interactions. For a particle with charge q and dipole moment $M = M(\tau)$, we consider

$$f = qF \cdot u + \nabla F \cdot M \quad (38)$$

where $F = F(z)$ is an applied electromagnetic (bivector) field [aptly called the *Faraday*] and the vector derivative $\nabla = \gamma^\mu \partial_\mu$ operates on F . The first term on the right of (38) is the standard “Lorentz force,” while the second is a force of “Stern-Gerlach” type. The latter is accompanied by the torque

$$\Gamma = F \times M. \quad (39)$$

The “interaction laws” (38) and (39) must be supplemented by a “constitutive equation” expressing M as a

function of S and u . In general M can include both electric and magnetic dipole moments. The general constitutive constraint

$$M \times S = 0 \quad (40)$$

is satisfied by the standard constitutive equation $M = gS$ for a magnetic moment, but also by the more general relation

$$M = gSe^{i\beta}. \quad (41)$$

which is worth noting, because it appears in Dirac theory, as we see later.

From the specific form of the interaction laws we can prove that the magnitude of the spin is a conserved quantity. First note that (32) implies

$$S \wedge S = 0 \quad \text{so} \quad S^2 = S \cdot S. \quad (42)$$

This follows from the identity

$$(S \wedge S)u = (S \wedge S) \cdot u = 2S \wedge (S \cdot u).$$

From (35) with (39) we get

$$\frac{1}{2} \frac{d}{d\tau} S^2 = S \cdot \dot{S} = S \cdot (u \wedge p) + S \cdot (M \times F).$$

The first term on the right vanishes because

$$S \cdot (u \wedge p) = (S \cdot u) \cdot p = 0,$$

and the second term vanishes by (40) and the identity

$$S \cdot (F \times M) = \langle SFM \rangle = F \cdot (M \times S). \quad (43)$$

Therefore

$$S^2 = -|S|^2 = \text{constant}. \quad (44)$$

The negative sign appears because (32) implies that S cannot be a timelike bivector, though it can be null.

Up to this point, our equations apply equally to timelike and lightlike particles. To go further we must consider each case separately. Three distinct possibilities have been studied in the past. To make contact with Dirac theory, most investigators have been attracted by the possibility of choosing $|S| = \hbar/2$ coupled with a timelike velocity. This case has been most thoroughly studied by Corben [11]. However, that model suffers from an ill-defined zitter radius, as we see below. The alternative possibility, coupling spin to a lightlike velocity overcomes that deficiency. I confess to struggling with that model for many years before I realized that for a lightlike particle the spin must be a lightlike bivector. Shortly thereafter I learned that Weyssenhoff had figured that out long before [10]. However, he was unable to generalize the free particle case to include interactions. Before solving that problem, let us see what can be learned from the free particle case.

IV. FREE PARTICLE MOTION

We gain insight into the ‘‘kinematics of spin’’ from the free particle solution. The solution has the same form for both timelike and lightlike cases, though we shall see that there is a crucial difference in physical interpretation.

For a free particle it follows from (34), (37) and (36) that p and $J = p \wedge z + S$ are constants of motion. Since $p^2 = m_e^2$, where m_e is the experimentally measured electron mass, we can write

$$p = m_e v, \quad (45)$$

with $v^2 = 1$,

We seek a general solution with constant dynamical mass $m_0 = p \cdot u = p \cdot \dot{z}$. Integrating this relation, we obtain

$$p \cdot (z - z_0) = m_0 \tau, \quad (46)$$

where z_0 is the particle position at $\tau = 0$. Conservation of angular momentum (36) gives us

$$(z - z_0) \wedge p = S(\tau) - S_0, \quad (47)$$

where S_0 is the initial value of the spin. Adding these equations and multiplying by $p^{-1} = p/p^2$ we get an equation for the *particle history*:

$$z = [S(\tau) - S_0] p^{-1} + m_0 p^{-1} \tau + z_0. \quad (48)$$

We can write this in the form

$$z(\tau) = r(\tau) + x(\tau), \quad (49)$$

where a *center of motion* is defined by

$$x = \frac{m_0}{p} \tau + z_0 - S_0 \cdot p^{-1}, \quad (50)$$

and a radius vector is defined by

$$r = S(\tau) \cdot p^{-1}. \quad (51)$$

Differentiating (50), we get the constant vector

$$\dot{x} = m_0 p^{-1} = \gamma v \quad \text{where} \quad \gamma = u \cdot v = m_0/m_e. \quad (52)$$

Thus, we identify τ as proper time in the ‘‘ p frame’’ scaled by a factor γ . In the timelike case we have $u^2 = \dot{z}^2 = 1$, so τ is the particle’s proper time and γ is a time dilation factor. In the lightlike case $u^2 = 0$, but $\gamma = u \cdot v = 1$ defines a unique projection of time τ onto the lightlike history.

Differentiating (51), we obtain

$$\dot{r} = \dot{S} \cdot p^{-1} = (u \wedge p) \cdot p^{-1} = u - \gamma v. \quad (53)$$

It follows that $\dot{r} \cdot r = 0$, so r^2 is a constant of motion, and $r = r(\tau)$ must be a rotating vector. Hence (49) describes a helix in spacetime centered on the timelike straight line (50), and r is its vector radius of curvature.

Consequently, we can write

$$\dot{r} = \Omega \cdot r = \Omega r, \quad (54)$$

where Ω is a constant bivector. And, using (53), for particle speed in terms of time $t = \gamma\tau$ we obtain

$$\gamma^{-2}|\dot{r}|^2 = \gamma^{-2}\Omega^2 r^2 = 1 - \gamma^{-2}u^2 \leq 1. \quad (55)$$

In the timelike case $u^2 = 1$, so the zitter speed $|\dot{r}|$ can have any value in the range $0 \leq |\dot{r}| < \gamma$. In the lightlike case $u^2 = 0$, so $|\dot{r}| = 1$ is fixed at the speed of light.

From this we conclude that, for all timelike and lightlike velocities \dot{z} , all aspects of the free particle motion are determined by the scalar, vector and bivector constants of motion, γ , p and Ω , with $\gamma = p \cdot \dot{x}/m_e$, $p^2 = m_e^2$ and $p \cdot \Omega = 0$. Determination of $\Omega^2 = -|\Omega|^2$ requires further assumptions about spin that will be introduced in the next section. The result is

$$|\Omega| = \omega_e \equiv \frac{2m_e c^2}{\hbar} \approx 1.5527 \times 10^{21} \text{ rad/sec}, \quad (56)$$

which is the *zitter frequency* of Schroedinger where m_e is the electron rest mass.

The free particle solutions show that the zitter radius has a unique value only in the lightlike model. If zitter motion generates the fixed magnitude of the observed electron magnetic moment, a fixed radius would seem to be required. For that reason, the lightlike model seems more promising physically. However, the timelike model is more closely related to the Dirac equation, as we shall see. So we proceed with analysis of both models.

V. LAGRANGIANS FOR A PARTICLE WITH SPIN

In Section III we postulated equations of motion and a number of relations among velocity, spin, momentum and mass variables. To ensure that all this constitutes a complete, coherent and self-consistent dynamical system we show that it can be derived from a single Lagrangian.

Conventional Lagrangians cannot be used for lightlike particles, owing to the absence of proper time as a natural parameter. Weyssenhoff [18] addressed this problem by deriving parameter invariant Lagrangians for both timelike and lightlike particles. However, he encountered a problem that limited his treatment to the free particle case. Krüger [19] resolved the problem for a lightlike particle by introducing a Lagrangian defined in terms of particle path curvature (i.e. acceleration) instead of velocity. However, his model does not have all the features that we are looking for, so we employ a different approach here.

Here we take full advantage of STA by using it to construct a spinor-based Lagrangian. We see that spinors are especially helpful for building lightlike constraints into the Lagrangian. The resulting spinor equations provide a superior dynamical model for a timelike or lightlike particle with spin, mass and zitter. Thus, we are able to

complete Weyssenhoff's program to construct dynamical equations for a lightlike particle with spin and electromagnetic interactions.

Continuous "motion" of a particle in spacetime is represented by a curve $z = z(\tau)$ and its derivative $\dot{z} = dz/d\tau$, where τ is an affine parameter for which the physical interpretation is initially unspecified. The kinematic structure of this curve is described by a spinor $\psi = \psi(\tau)$ and its derivative $\dot{\psi}$.

The dynamics of motion in an external electromagnetic field $F(z) = \nabla \wedge A$ with vector potential $A = A(z)$ is determined by a Lagrangian $L = L(z, \psi, P)$ of the form

$$L = \left\langle -\hbar\dot{\psi}\gamma_2\gamma_1\tilde{\psi} + P(\dot{z} - \psi\gamma_0\tilde{\psi}) + qA\psi\gamma_0\tilde{\psi} + \mu F\psi\gamma_2\gamma_1\tilde{\psi} \right\rangle, \quad (57)$$

with units: $c = 1$, $\hbar = m_c$ = fixed reference mass. The coupling constants are charge q and magnetic moment μ . A vectorial Lagrange multiplier P relates the two kinds of kinematic variable.

The Lagrangian (57) has been proposed by Doran and Lasenby [20], but focussed on the case $\psi\tilde{\psi} = 1$. We drop that constraint for two reasons: First, because it does not apply to the *lightlike case* where $\psi\tilde{\psi} = 0$, and second, because we want to understand the physical significance of constraints on $\psi\tilde{\psi}$ in the *timelike case*.

The main innovation in (57) is the *kinetic term* $\langle -\hbar\dot{\psi}\gamma_2\gamma_1\tilde{\psi} \rangle$. As we see later, this term is derivable by straightforward approximation from the Dirac Lagrangian, so it establishes a definite link to the Dirac equation.

The constraint term $\langle P(\dot{z} - \psi\gamma_0\tilde{\psi}) \rangle$ was introduced by Proca [21] who was the first to use Dirac spinors for describing classical particles. Barut and Zanghi [22] used the same approach to model electron zitter. Gull [23] translated their model into STA and noted that it lacked a magnetic moment. Therein lies a small surprise. The last two terms in (57) are standard interaction terms in Dirac theory, where the last is the so-called "Pauli term" for a particle with anomalous magnetic moment μ . However, we shall see that simple consistency between charge and momentum conservation implies a relation between q and μ that gives the correct g-factor, which originally emerged so mysteriously from the Dirac equation. Evidently, one must look elsewhere for explanation of the electron's anomalous magnetic moment.

The *method of multivector differentiation* is the simplest and most elegant way to derive equations of motion [20, 24]. For a Lagrangian that is homogeneous of degree one in derivatives, variation with respect to a multivector variable $X = X(\tau)$ yields the multivector Lagrange equation

$$\delta_X L \equiv \partial_X L - \partial_\tau(\partial_{\dot{X}} L) = 0, \quad (58)$$

where $\dot{X} = \partial_\tau X$.

Variation of the Lagrangian with respect to P obviously gives

$$\dot{z} = \psi\gamma_0\tilde{\psi} \equiv \rho u, \quad (59)$$

which defines a *particle velocity* u and time *scaling factor* ρ in terms of the spinor ψ .

The result of varying position vector z is the *force law*

$$\rho^{-1}\dot{p} = qF \cdot u + \nabla F \cdot M, \quad (60)$$

with *momentum vector* p defined by the canonical expression

$$p \equiv P - qA, \quad (61)$$

and electromagnetic moment bivector defined by

$$\rho M \equiv \mu\psi\gamma_2\gamma_1\tilde{\psi}. \quad (62)$$

Of course, the vector derivative ∇ in (60) operates only on F and not on M .

The result of varying spinor ψ in the Lagrangian is the dynamical spinor equation

$$\hbar\dot{\psi}\gamma_2\gamma_1 = -p\psi\gamma_0 + \mu F\psi\gamma_2\gamma_1. \quad (63)$$

The reverse of this equation is

$$\hbar\gamma_2\gamma_1\dot{\tilde{\psi}} = -\gamma_0\tilde{\psi}p + \mu\gamma_2\gamma_1\tilde{\psi}F. \quad (64)$$

These combine to give us

$$\rho^{-1}d_\tau(\rho S) = u \wedge p + gF \times S, \quad (65)$$

where $g = 2\mu/\hbar$ and the *spin bivector* S is defined by

$$\rho S \equiv \frac{\hbar}{2}\psi\gamma_2\gamma_1\tilde{\psi}. \quad (66)$$

Alternatively, one can cast (63) in the form

$$\hbar\dot{\psi}\gamma_0 = p\psi\gamma_2\gamma_1 + \mu F\psi\gamma_0, \quad (67)$$

which, when combined with its reverse, gives the equation of motion

$$\rho^{-1}d_\tau(\rho u) = \frac{4}{\hbar^2}p \cdot S + gF \cdot u, \quad (68)$$

coupling particle acceleration to spin and momentum.

The Lagrangian has delivered a system of coupled equations for particle velocity, spin and momentum. Some simplifications are still needed to facilitate physical interpretation and analysis. First, note that a unit vector e_1 is implicitly defined by

$$\omega e_1 = \frac{4}{\hbar^2}p \cdot S, \quad (69)$$

with

$$\omega = -\frac{4}{\hbar^2}p \cdot S \cdot e_1 = \frac{4}{\hbar^2}S \cdot (p \wedge e_1), \quad (70)$$

By this definition, $p \cdot e_1 = 0 = u \cdot e_1$. Accordingly, the particle equation of motion (68) takes the form

$$\frac{\hbar}{2}\rho^{-1}d_\tau(\rho u) = \omega e_1 + gF \cdot u. \quad (71)$$

Evidently, the term ωe_1 describes an intrinsic curvature of the particle history.

As in the free particle case, it is convenient to define a *dynamical mass*:

$$m = \dot{z} \cdot p = \rho u \cdot p. \quad (72)$$

From the three *equations of motion* (60), (68) and (66), we easily derive

$$\dot{m} = gF \cdot d_\tau(\rho S) + g\rho(\dot{z} \cdot \nabla F) \cdot S = g d_\tau(\rho S \cdot F). \quad (73)$$

Thus, we obtain a first integral of motion

$$m = m_0 + g\rho S \cdot F, \quad (74)$$

where $m_0 = m_e\gamma$, with the restrictions on γ noted in connection with (55).

Finally, to complete this section on common features of timelike and lightlike models, we consider rationale for adopting the g-factor ratio $g = q/m_e$ in agreement with Dirac theory. First, note that both the particle equation (68) and the spin equation (66) have the same coupling constant g , so any anomalous g-factor (or mass) in one necessarily applies to the other. Second, assuming that the field F acts directly on the particle only, the term $qF \cdot u$ in the momentum equation (60) must refer to the same interaction as $gF \cdot u$ in the particle equation. Therefore, the mass $m_e = q/g$ should refer to the same constant mass $m_e = (p^2)^{1/2}$ obtained in the free particle limit. Evidently, as in the Dirac equation, the anomalous μ cannot be incorporated as a free parameter. Accordingly, we henceforth regard charge q and mass $m_e = q/g$ as the basic coupling constants in the Lagrangian (57).

VI. OBSERVABLES AND GAUGE TRANSFORMATIONS

Consider first the *timelike case*, for which, according to (24), the spinor has the general form

$$\psi = (\rho e^{i\beta})^{\frac{1}{2}} R, \quad (75)$$

where ρ and β are scalars, and rotor $R = R(\tau)$ determines a one parameter family of Lorentz rotations

$$e_\mu = e_\mu(\tau) = R\gamma_\mu\tilde{R} \quad (76)$$

that transforms a fixed orthonormal frame of vectors into an intrinsic *comoving frame* following the particle. The comoving frame is coupled to the particle velocity by

$$\dot{z} = \psi\gamma_0\tilde{\psi} = \rho u, \quad (77)$$

Note that the duality factor $e^{i\beta/2}$ has canceled in (77), because $\tilde{i} = i$ and $i\gamma_\mu = -\gamma_\mu i$. However, in the spin bivector (66) it survives to give us

$$S = \frac{\hbar}{2}\rho R\gamma_2\tilde{R}R\gamma_1\tilde{R}e^{i\beta} = \frac{\hbar}{2}\rho e^{i\beta}e_2e_1 \quad (78)$$

The strange duality factor appearing here occurs also in the Dirac theory, where its physical significance has remained obscure for a long time [2].

The time evolution of the frame $e_\mu(\tau)$ is determined by a rotor equation of the form

$$\dot{R} = \frac{1}{2}\Omega R. \quad (79)$$

By differentiating the normalization condition $R\tilde{R} = 1$, one can prove that $\Omega = \Omega(\tau)$ is a bivector-valued function. Specification of this function completely determines the frame kinematics. For, using (79) to differentiate (76), we get

$$\dot{e}_\mu = \Omega \cdot e_\mu. \quad (80)$$

The single rotor equation (79) is much easier to solve than this set of four differential equations. Accordingly, determining an explicit functional form for $\Omega = \Omega(\tau)$ will be central to our analysis of particle motion.

Now consider what (75) tells us when inserted in the Lagrangian (57):

$$L = -\hbar\rho \left\langle e^{i\beta}\dot{R}\gamma_2\gamma_1\tilde{R} \right\rangle + P \cdot (\dot{z} - \rho R\gamma_0\tilde{R}) + q\rho A \cdot (R\gamma_0\tilde{R}) + \mu\rho F \cdot (e^{i\beta}R\gamma_2\gamma_1\tilde{R}). \quad (81)$$

Note that the derivatives $\dot{\rho}$ and $\dot{\beta}$ have dropped out of the first term, because they contribute a vanishing scalar part. Thus, $\rho = \rho(\tau)$ is a positive scale factor that can be removed from the Lagrangian with a change of the term $P \cdot \dot{z}$ to $P \cdot z'$, where z' is the derivative with respect to the new time variable.

The parameter β in (81) is not constrained by dynamics, but it contributes an arbitrary duality rotation to the definition of spin (78). We are free to eliminate that arbitrariness by assuming

$$e^{i\beta} = \pm 1. \quad (82)$$

Switch from the positive to negative sign with this factor expresses an important discrete symmetry of the Lagrangian when coupled with charge conjugation $q \rightarrow -q$ and time reversal $\tau \rightarrow -\tau$. The net effect on the Lagrangian is to reverse the signs of q and μ along with the orientation of the spacetime history $z(\tau)$. In other words, it is *antiparticle conjugation*. It thus conforms to Feynman's interpretation of antiparticles as "particles going backward in time." The relation of particle current $\psi\gamma_0\psi$ to particle velocity \dot{z} is not so obvious in Dirac theory, because the Dirac Lagrangian lacks a term involving \dot{z} .

A. Gauge Invariance

The spin and velocity observables $\psi\gamma_2\gamma_1\tilde{\psi}$ and $\psi\gamma_0\tilde{\psi}$ are invariant under the rotor gauge transformation

$$\psi \rightarrow \psi' = \psi U, \quad (83)$$

provided the rotor U commutes with $\gamma_2\gamma_1$ and γ_0 . Hence, U must have the form

$$U = e^{\gamma_2\gamma_1\chi/\hbar}, \quad (84)$$

where $\chi = \chi(\tau)$ is an arbitrary scalar function. This is identical in algebraic form to a gauge transform in the Dirac equation. Applying it to the Lagrangian (57), we see that the kinetic term becomes

$$-\hbar \left\langle \psi'\gamma_2\gamma_1\tilde{\psi}' \right\rangle = -\hbar \left\langle \psi\gamma_2\gamma_1\tilde{\psi} \right\rangle + \dot{\chi} \left\langle \psi\tilde{\psi} \right\rangle. \quad (85)$$

In the timelike case, the additional term must be canceled by a transformation of the vector potential:

$$qA \rightarrow qA' = qA - \nabla\chi. \quad (86)$$

Thus, for $\left\langle \psi\tilde{\psi} \right\rangle = \rho$ and $\left\langle \psi\gamma_0\tilde{\psi} \right\rangle = \dot{z}$, we have

$$\rho\dot{\chi} = \dot{z} \cdot \nabla\chi. \quad (87)$$

Evidently, gauge invariance requires $\rho = 1$, thereby fixing the time scale on the particle history to the time variable in spinor space.

This is an intriguing if not ironic result. Recall that Weyl originally introduced gauge transformations as an extension of General Relativity to incorporate length scale invariance. Subsequently, he renounced that idea and introduced the notion of gauge invariance that has become standard in quantum mechanics. Here we have come full circle to find electromagnetic gauge invariance associated with time scale invariance. Perhaps Weyl had the right idea in the first place.

Geometrically, a gauge transformation rotates the vectors e_1 and e_2 while leaving the spin $S = (\hbar/2)e_2e_1$ and velocity $u = e_0$ unchanged. This corresponds to a phase shift in standard quantum mechanics. A unique gauge is set by requiring the vector e_1 defined by (69) to satisfy

$$e_1 = R\gamma_1\tilde{R} \quad \text{with} \quad p \cdot e_1 = 0. \quad (88)$$

So, from (70)

$$\omega = (2/\hbar)(e_2e_1) \cdot (p \wedge e_1) = (2/\hbar)p \cdot e_2. \quad (89)$$

We stick with this choice henceforth. The same conclusion obtains in the lightlike case, so we need not belabor the point.

B. Observables for the Lightlike Case

In the lightlike case, the null condition $\psi\tilde{\psi} = 0$ can be satisfied by multiplying an even multivector with the form (75) on an idempotent such as

$$P_{\pm} = \frac{1}{2}\gamma_{\pm}\gamma_0 = \frac{1}{2}(1 + \gamma_2\gamma_0), \quad (90)$$

where $\gamma_{\pm} \equiv \gamma_0 \pm \gamma_2$. Note the properties of idempotence and orthogonality

$$P_{\pm}^2 = P_{\pm}, \quad P_+\tilde{P}_+ = P_+P_- = 0. \quad (91)$$

Also note

$$\gamma_2\gamma_0\gamma_{\pm} = \pm\gamma_{\pm} \quad (92)$$

and $i\gamma_2\gamma_0 = \gamma_3\gamma_1$; whence

$$\rho e^{i\beta}P_{\pm} = e^{\pm\gamma_3\gamma_1\beta}P_{\pm} \quad (93)$$

Thus, the null spinor has the general form:

$$\psi = \rho^{1/2}RP_+, \quad (94)$$

where R is a rotor and no duality factor is needed, because (93) shows that it can be absorbed into R .

The particular form for P_+ was chosen in relation to γ_0 so that

$$P_+\gamma_0\tilde{P}_+ = \frac{1}{4}\gamma_+\gamma_0\gamma_+ = \frac{1}{2}\gamma_+. \quad (95)$$

Hence

$$\psi\gamma_0\tilde{\psi} = \frac{\rho}{2}R\gamma_+\tilde{R} = e_0 + e_2 \equiv u = \dot{z}, \quad (96)$$

and the normalization $\rho = 2$ has been chosen for simplicity.

In a similar way, we can reduce the form for spin. Thus,

$$P_+\gamma_2\gamma_1\tilde{P}_+ = \frac{1}{2}\gamma_+\gamma_0\gamma_2\gamma_1 = \frac{1}{2}\gamma_+\gamma_1. \quad (97)$$

Now we can define the spin bivector by

$$S = \psi\gamma_2\gamma_1\tilde{\psi} = \frac{\hbar}{2}R\gamma_+\gamma_1\tilde{R} = \frac{\hbar}{2}ue_1. \quad (98)$$

The simplifications (96) and (98) can be inserted into (57) to put the Lagrangian for a lightlike particle in the form

$$L = \left\langle -\hbar\dot{R}\gamma_+\gamma_1\tilde{R} + P(\dot{z} - R\gamma_+\tilde{R}) + qAR\gamma_+\tilde{R} + \mu FR\gamma_+\gamma_1\tilde{R} \right\rangle, \quad (99)$$

In this version of the Lagrangian, the relation of the lightlike model to the timelike model and the Dirac equation is not so evident. Its form invariance under the gauge transformations defined by (83), (84) and (86) with $\psi = R$ is readily verified. Unlike the timelike case, the velocity is not gauge invariant in the lightlike case. Rather, it is transformed to $u' = R'\gamma_+\tilde{R}' = R(\gamma_0 + U\gamma_2\tilde{U})\tilde{R} = e_0 + e_2'$. The gauge transformation singles out the e_2e_1 plane as special. In the timelike case, that plane is associated with spin $S = e_2e_1\hbar/2$. But in the lightlike case $S = ue_1\hbar/2$, and, as we shall see, e_2e_1 is special as a component of the rotational velocity Ω .

VII. ROTOR EQUATION: TIMELIKE CASE

The basic features of our two models for a charged particle with spin are now fully specified, but further simplification is possible by consolidating the equations of motion into a single rotor equation. Although the lightlike case is more interesting, we focus first on the timelike case as a basis for comparison.

Inserting the rotor derivative (79), the spinor equation (63) now reads

$$\frac{\hbar}{2}\Omega R\gamma_2\gamma_1 = -pR\gamma_0 + \mu FR\gamma_2\gamma_1. \quad (100)$$

It follows immediately that

$$\Omega S = -pu + gFS, \quad (101)$$

with *velocity*

$$u = R\gamma_0\tilde{R} = e_0 = \dot{z}, \quad (102)$$

and *spin*

$$S = \frac{\hbar}{2}R\gamma_2\gamma_1\tilde{R} = \frac{\hbar}{2}e_2e_1 = isu, \quad (103)$$

where, using $i = e_0e_1e_2e_3$, a *spin vector*

$$s = \frac{\hbar}{2}e_3 \quad (104)$$

has been defined in exact analogy to Dirac theory.

Inserting the expansion $\Omega S = \Omega \cdot S + \Omega \times S + \Omega \wedge S$ into (101) and separating parts of homogenous grade we find:

$$\Omega \cdot S = -u \cdot p + gF \cdot S = -m_0, \quad (105)$$

$$\Omega \times S = u \wedge p + gF \times S = \dot{S}, \quad (106)$$

$$\Omega \wedge S = gF \wedge S. \quad (107)$$

Here we see that the mass integral of motion $m - gF \cdot S = m_0 = -\Omega \cdot S$ can be interpreted as constraining projection of the spin onto the rotational velocity to be a constant of motion.

Equation (101) is easily solved algebraically for any one of the observables S, u, p . But the most significant fact is that it can be solved for

$$\Omega = ips^{-1} + gF = 2\dot{R}\tilde{R}, \quad (108)$$

This gives us immediately

$$\dot{e}_{\mu} = (ips^{-1}) \cdot e_{\mu} + gF \cdot e_{\mu}, \quad (109)$$

which shows explicitly how the four equations of motion for the e_{μ} are coupled. Of course, it is simpler to solve the single equation of motion for R , so we examine what it takes to do that.

Note that the vanishing pseudoscalar part of (108) implies

$$p \cdot s^{-1} = 0. \quad (110)$$

Also, for a free particle,

$$\Omega^2 = p^2 s^2 = -\omega_0^2, \quad (111)$$

as advertised in Section IV.

The zitter term ips^{-1} in (108) requires further analysis. Since $p \cdot e_1 = p \cdot e_2 = 0$, the vector p lies in the $e_0 e_2$ plane, so

$$p = p \cdot e_0 e_0 - p \cdot e_2 e_2 = mu - m_2 e_2, \quad (112)$$

with

$$m_2 = p \cdot e_2 = [m^2 - p^2]^{1/2} = \frac{\hbar}{2}\omega, \quad (113)$$

where the last equality recalls the definition of ω in (89).

Now the zitter term can be written

$$\begin{aligned} ips^{-1} &= p \cdot (is^{-1}) = \frac{2}{\hbar} p \cdot (e_0 e_2) e_1 \\ &= \frac{2}{\hbar} [m e_2 e_1 - m_2 e_0 e_1]. \end{aligned} \quad (114)$$

Inserting this into (108), we get the rotor equation of motion in the form

$$\dot{R} = \frac{g}{2} F R + R \left[\frac{m}{\hbar} \gamma_2 \gamma_1 - \frac{m_2}{\hbar} \gamma_0 \gamma_1 \right]. \quad (115)$$

Thus, we have reduced our problem to ascertaining the temporal dependence of m and m_2 . That requires use of the momentum equation.

From (108) we find

$$\Omega \cdot p = gF \cdot p. \quad (116)$$

Then the momentum equation (60) gives us

$$\dot{p} - gF \cdot p = \dot{m} e_0 - \dot{m}_2 e_2 = F \cdot (qu - gp) + g\nabla F \cdot S.$$

Dotting this with e_0 and e_2 gives us equations for \dot{m} and \dot{m}_2 . We found and solved the equation for \dot{m} before. To solve for \dot{m}_2 , we use $u \wedge p = m_2 e_0 e_2$ in the spin equation (106) to get

$$e_2 \cdot F \cdot (qu - gp) = g \left(\frac{m_0 - m}{m_2} \right) F \cdot \dot{S}.$$

Hence,

$$\dot{m}_2 = g \left(\frac{m_0 - m}{m_2} \right) F \cdot \dot{S} + g(e_2 \cdot \nabla F) \cdot S. \quad (117)$$

We need not bother here with solving for m_2 as a function of R , as we will not be using the result.

The rotor equation (115) with the equations for \dot{m} and \dot{m}_2 compose a closed system of differential equations that can be solved for the rotor $R = R(\tau)$. We defer discussing methods of solution to the lightlike case.

VIII. ROTOR EQUATION: LIGHTLIKE CASE

For the lightlike case, we proceed in close analogy to the timelike case and expand on its unique features.

From the Lagrangian (99), we derive the rotor equation

$$\hbar \dot{R} \gamma_+ \gamma_1 = \frac{\hbar}{2} \Omega R \gamma_+ \gamma_1 = -p R \gamma_+ + \mu F R \gamma_+ \gamma_1. \quad (118)$$

It follows immediately that

$$\Omega S = -pu + gFS, \quad (119)$$

with *lightlike velocity*

$$u = R \gamma_+ \tilde{R} = e_0 + e_2 = \dot{z}, \quad (120)$$

and *lightlike spin*

$$S = \frac{\hbar}{2} R \gamma_+ \gamma_1 \tilde{R} = \frac{\hbar}{2} u e_1 = m_e r_e u = isu, \quad (121)$$

where the *spin vector* $s = \frac{\hbar}{2} e_3$ is defined as before. Also, a *zitter radius vector* r_e has been defined by

$$r_e = -\lambda_e e_1, \quad (122)$$

with *zitter radius*

$$\lambda_e = \omega_e^{-1} = \frac{\hbar}{2m_e} = |r_e|. \quad (123)$$

This makes the spin $S = m_e r_e \wedge \dot{z}$ look like an orbital angular momentum for a particle with momentum $m_e \dot{z}$, but moving at the speed of light. More about that below.

One other form for the spin has special significance:

$$S = \frac{\hbar}{2} e_0 (e_1 + ie_3) = (m_e r_e + is) e_0, \quad (124)$$

Let's call this the *spin-zitter split* of S for physical reasons to be explained.

Equation (119) looks just like (101), and it can be expanded in the same way to get expressions for $\Omega \cdot S$, $\Omega \wedge S$ and, in particular,

$$\dot{S} = \Omega \times S = u \wedge p + gF \times S. \quad (125)$$

However, (119) cannot be solved for Ω , because S and u do not have multiplicative inverses. The best we can do is multiply (119) on the right by γ_1 to get the particle equation

$$\dot{u} = \Omega \cdot u = \omega e_1 + gF \cdot u, \quad (126)$$

with

$$m = p \cdot u = \hbar\omega/2 \quad (127)$$

determined by the gauge choice $p \cdot e_1 = 0$, as noted before.

One way to derive an expression for Ω is to use the theorem:

$$\Omega = \frac{1}{2} \dot{e}_\mu e^\mu = \frac{1}{2} (\dot{e}_0 e_0 - \dot{e}_1 e_1 - \dot{e}_2 e_2 - \dot{e}_3 e_3). \quad (128)$$

But this requires separate equations for the \dot{e}_μ . To obtain such equations, we first use $\dot{u} = \dot{e}_0 + \dot{e}_2$ to split (126) into

$$\dot{e}_0 = gF \cdot e_0 + a, \quad (129)$$

$$\dot{e}_2 = gF \cdot e_2 - a + \omega e_1, \quad (130)$$

where the vector a remains to be determined.

Now we invoke the momentum equation

$$\dot{p} = qF \cdot u + \nabla\Phi, \quad (131)$$

where a *spin-zitter potential* for the gradient force is defined by

$$\Phi = gF \cdot S = qF \cdot (r_e \wedge u). \quad (132)$$

Several alternative forms can be obtained from the various expressions for spin in (121) or (124). As we have seen before, the gauge choice that gives us (127) implies that p lies in the $u \wedge e_2 = e_0 e_2$ plane. Hence, we can parametrize p in the form

$$p = m_1 e_0 - m_2 e_2 = m_1 u - m e_2, \quad (133)$$

so $m = p \cdot u = m_1 + m_2$ and

$$u \wedge p = m e_2 e_0. \quad (134)$$

In the free particle limit $m_1 = m_e$.

Now we insert (133) into (131) to get

$$m_1 \dot{e}_0 - m_2 \dot{e}_2 + \dot{m}_1 e_0 - \dot{m}_2 e_2 = qF \cdot u + \nabla\Phi. \quad (135)$$

Then we eliminate \dot{e}_0 and \dot{e}_2 with (129) and (130) to solve for

$$a = \left(\frac{m_e - m_1}{m_e m} \right) qF \cdot u + \frac{q}{m_e} F \cdot e_2 + \frac{2m_2}{\hbar} e_1 + m^{-1} (\nabla\Phi + \dot{m}_2 e_2 - \dot{m}_1 e_0), \quad (136)$$

Equation (135) is also easily solved for

$$\dot{m}_1 = \left(\frac{m_2 + m_e}{m_e m} \right) qF \cdot \dot{S} + e_0 \cdot \nabla\Phi, \quad (137)$$

$$\dot{m}_2 = \left(\frac{m_1 - m_e}{m_e m} \right) qF \cdot \dot{S} + e_2 \cdot \nabla\Phi. \quad (138)$$

This gives us

$$\dot{m} = \dot{m}_1 + \dot{m}_2 = \frac{q}{m_e} (F \cdot \dot{S} + S \cdot \dot{F}) = \dot{\Phi}. \quad (139)$$

in agreement with our previous result (73).

Inserting the equations for \dot{m}_1 and \dot{m}_2 into (136), we obtain an explicit expression for the vector a in terms of the field F and its derivatives, so inserting this into (129) and (130) gives the specific equations for \dot{e}_0 and \dot{e}_2 that we sought.

To clarify the structure of the vector a , we write $a_\mu = a \cdot e_\mu$ and verify from (136) that $a_0 = a_2 = 0$, so

$$a = -a_1 e_1 - a_3 e_3. \quad (140)$$

Furthermore,

$$m a_1 = gF_{01}(m_e - m_1) + gF_{21}(m_e + m_2) + m_2 \omega + \partial_1 \Phi \quad (141)$$

and

$$m a_3 = gF_{03}(m_e - m_1) + gF_{23}(m_e + m_2) + \partial_3 \Phi, \quad (142)$$

where $F_{\mu\nu} = F \cdot (e_\mu \wedge e_\nu)$ and $\partial_\mu = e_\mu \cdot \nabla\Phi$.

These results are completely general. Note, however, the simplification in the expressions for a_1 and a_2 if we assume $m_1 = m_e$ with $m_2 = m - m_e$. Of course that propagates back to simplify (136) with $\dot{m}_1 = 0$ in (137) and $\dot{m}_2 = \dot{m}$ in (138). This guarantees the free particle limit $m = m_1 = m_e$. However, from (137) we see that, for a uniform field F it implies $F \cdot \dot{S} = F \cdot (u \wedge p) = 0$, which is too restrictive for a general solution, though we see below that it is an interesting special case.

Next, to derive separate equations of motion for the spin vector e_3 and the zitter vector e_1 , we write

$$\frac{2}{\hbar} S = e_0(e_1 + i e_3) = e_0 Q \quad (143)$$

to put (125) in the form

$$\dot{e}_0 Q + e_0 \dot{Q} = \omega e_2 e_0 + g[(F \times e_0)Q + e_0(F \times Q)].$$

Eliminating \dot{e}_0 with (129) and using (140), we reduce it to the form

$$\dot{Q} - gF \times Q = -\omega e_2 - e_0 a Q. \quad (144)$$

Then, separating vector and pseudovector parts, we get the desired equations

$$\dot{e}_1 = gF \cdot e_1 - \omega e_2 - a_1 u, \quad (145)$$

$$\dot{e}_3 = gF \cdot e_3 - a_3 u. \quad (146)$$

Finally, we insert (129), (130), (145) and (146) into (128) and use $F = (F \cdot e_\mu) e^\mu / 2$ to get

$$\Omega = gF + \omega e_2 e_1 + a u. \quad (147)$$

This is the expression for Ω that we were looking for. Let's consider its physical interpretation.

Clearly the term $\omega e_2 e_1$ generates circular motion or *zitter* in the *zitter plane*. This is confirmed by inserting (147) into the equations for velocity(126) and spin (125), where it contributes the terms $(\omega e_2 e_1) \cdot u = \omega e_1$ and

$$(\omega e_2 e_1) \times S = \frac{\hbar}{2} \omega (e_2 e_1) \times (u e_1) = m e_2 e_0 = u \wedge p.$$

The bivector au does not contribute to either equation.

The integral of the *zitter frequency* ω determines a *zitter phase angle* φ , as defined by

$$\omega = \omega_e + \frac{2}{\hbar} \Phi = \dot{\varphi} = \frac{d\varphi}{d\tau}, \quad (148)$$

We shall see that this angle is completely analogous to the phase angle in the Dirac equation and, *á fortiori*, to the phase in Schroedinger's wave function. Note that the phase angle may depend on the electromagnetic vector potential, but the frequency shift is gauge invariant, as it has the form of a dynamical flux integral on the circulating particle history.

The *first curvature* κ_1 of a particle history measures the bending rate orthogonal to its velocity. In the particle equation of motion (126) the direction of helical bending is given by the *zitter vector* e_1 . Hence, the first curvature is given by

$$\kappa_1 \equiv -\dot{u} \cdot e_1 = \omega - \frac{q}{m_e} F \cdot (ue_1) = \omega_e = \lambda_e^{-1}. \quad (149)$$

We have seen that this quantity is rigorously constant! However, it is composed of two parts: the first can be regarded as an *intrinsic curvature* while the second is an *extrinsic curvature* due to external forces. This tells us that, in response to external forces, the intrinsic curvature is adjusted to maintain an overall constant value. Its inverse is the curve's radius of curvature, which we recognize as the free particle zitter radius.

The dynamical mass

$$m = \frac{\hbar}{2} \omega = m_e + \Phi \quad (150)$$

can be interpreted as energy stored in the zitter motion, consisting of a *self-energy* m_e and an interaction energy Φ . Inserting the spin-zitter split (124) into (132), we can express the *spin-zitter interaction energy* in the form

$$\Phi = \langle e_0 F (qr_e + igs) \rangle. \quad (151)$$

This separates Φ into a magnetic part and an electric part. The timelike vector e_0 can be regarded as specifying an instantaneous rest system for the electron.

The magnetic part $g \langle Fise_0 \rangle$ will be recognized as the usual Zeeman interaction with the same magnetic moment as specified by Dirac theory. When inserted into the momentum equation (132), we see that it doubles as a Stern-Gerlach force.

The electric part $qr_e \cdot (e_0 \cdot F)$ is a dipole interaction with a rapidly rotating dipole moment vector $qr_e = -q\lambda_e e_1$. Let's call it the *zitter interaction energy*. This is a distinctively new feature of the zitter model, with testable predictions that distinguish it from Dirac theory, as we shall see.

Our analysis of the interaction term in (150) suggests that the self energy m_e may be due to electromagnetic interaction, but the interaction must be of magnetic type $m_e e_2 e_1$ to generate the observed free electron spin and magnetic moment. Longstanding problems with infinite electric self energy are thereby avoided. This suggestion is more than unfounded speculation. Rather it serves as a constraint to be satisfied by a solution to the self interaction compatible with the zitter model. That is a problem for the future.

We are now prepared to compare the zitter term $\omega_e e_2 e_1 + au$ in (147) with the corresponding term ips^{-1} in (108). Although the latter term may look more compact and physically perspicuous, it needs further reduction to determine its temporal dependence, as explained in the analysis leading to (115) and (117). In contrast, the term in (147) is ready for direct integration. However, note from (114) that the zitter rotation velocity $\omega_e e_2 e_1$ is common to both timelike and lightlike velocities, so they must have some predictions in common. In particular, they have the same relation to phase angle in the Dirac equation, though only the lightlike model agrees with the Dirac free particle limit.

If the zitter terms in (147) and (108) are dropped altogether, then in both models the rotational velocity Ω reduces to

$$\Omega = gF = 2\dot{R}\tilde{R} \quad (152)$$

and (126) gives us

$$\dot{u} = gF \cdot u, \quad (153)$$

which is precisely the classical Lorentz force for $u = \dot{z}$. So let us refer to (152) as the *classical approximation*. However, note that (152) also implies

$$\dot{s} = gF \cdot s, \quad (154)$$

which gives spin precession with the same g-factor as the Dirac equation. Thus, the classical approximation is not simply the limit $\hbar \rightarrow 0$ as often stated. Indeed, the spin precession in (154) has been measured in high precision experiments on single electrons [25], with a result close to

$$g = \frac{e}{m_e} \left(1 + \frac{\alpha}{4\pi} \right), \quad (155)$$

where α is the fine structure constant. Calculation of the correction term is outside the purview of the present model. It is mentioned here to set a definit limit on the "classical approximation." Conceivably, the anomalous moment is a consequence of electron zitter, as omission of zitter defines the classical approximation. Whether that would be consistent with the accepted calculation using QED remains to be seen.

IX. ZITTER CENTER, REST FRAME AND MEAN VALUES

Zitter fluctuation in electron position is small (of order $\lambda_e \approx 10^{-13}\text{m}$) and rapid (of order $\omega_e = \lambda_e^{-1}c \approx 10^{21}/\text{s}$), so some sort of mean position $x(\tau) = \bar{z}(\tau)$ must be defined to separate fluctuations from the main trend. This is akin to defining the "guiding center" of a helical orbit in plasma physics, or the "center of curvature" for a geometric curve, or even a "center of mass," though the mass is not a distributed quantity in this case. Without committing to one interpretation, let's refer to $x(\tau)$ as

the *zitter center*. Among several possibilities, we consider two alternative definitions.

First, the fact that λ_e is a constant of motion suggests the *rigid radius definition*:

$$x = x(\tau) \equiv z - r_e, \quad (156)$$

where $r_e = -\lambda_e e_1$ is the zitter radius vector defined before.

Clearly $x(\tau)$ is a timelike curve, so we need to ascertain how its proper time relates to the time variable τ . From

$$u = \dot{z} = \dot{x} + \dot{r}_e, \quad (157)$$

we get $u \cdot (\dot{x} + \dot{r}_e) = u^2 = 0$, and (126) gives us

$$u \cdot \dot{x} = -u \cdot \dot{r}_e = -\lambda_e \dot{u} \cdot e_1 = 1. \quad (158)$$

Therefore

$$\dot{x}^2 = 1 - \dot{x} \cdot \dot{r}_e. \quad (159)$$

This tells that τ oscillates about the proper time of the zitter center with half the zitter period, except in the free particle case where $\dot{x} \cdot \dot{r}_e = 0$.

To relate \dot{x} to the particle dynamics, we use $\hbar/2 = m_e \lambda_e$ to write (145) in the form

$$m_e \dot{x} = m_e \dot{r}_e - qF \cdot r_e + m_e a \cdot r_e u. \quad (160)$$

Inserting this into (133) with $m_1 = m_e$, we get the momentum vector in the interesting form

$$p = m_e \dot{x} + qF \cdot r_e - m_e a \cdot r_e u. \quad (161)$$

For a free particle this reduces to the classical expression $p = m_e \dot{x}_e$, as required.

We could get an equation of motion for \dot{x} by inserting (161) into the momentum equation (131), but a simpler and exact approach is to write the particle equation (126) in the form

$$\ddot{z} = \ddot{x} + \ddot{r}_e = \frac{2m}{\hbar} e_1 + \frac{q}{m_e} F \cdot (\dot{x} + \dot{r}_e).$$

Hence

$$m_e \ddot{x} - qF \cdot \dot{x} = - (m_e \ddot{r}_e + m \omega_e^2 r_e) + qF \cdot \dot{r}_e, \quad (162)$$

The right side shows explicitly how the zitter drives deviations from the Lorentz force law. To calculate those deviations, we need a separate equation of motion for the zitter. Unfortunately, the separation of zitter from the Lorentz force is not as clean as it looks in (162), because the field $F = F(z) = F(x + r_e)$ acts at the location of the particle.

The alternative approach is to define the zitter center $x = x(\tau)$ indirectly by specifying its velocity

$$v = \dot{x} \equiv e_0 = R \gamma_0 \tilde{R}. \quad (163)$$

This has both advantages and drawbacks. First

$$v^2 = v \cdot u = v \cdot \dot{z} = 1 \quad (164)$$

implies that we can regard τ as proper time on the history $x(\tau)$, and that projects to a unique time defined on the particle history $z(\tau)$. Second, the definition (163) is analogous to the spinor form of the Dirac current, so it facilitates comparison with the Dirac equation. As drawbacks, we note the the equation of motion (129) for v lacks physical transparency, and the zitter radius $r \equiv z - x$ is only indirectly determined by integrating

$$\dot{r} = u - v = e_2. \quad (165)$$

However, the similarity of (159) and (164) suggests that $r = r_e$ may often be a good approximation. The issue deserves to be examined systematically.

A. Zitter averages and approximations

As zitter fluctuations are so rapid, it is most convenient to separate them from zitter means, which are more directly observable. The velocity $v = \dot{x}$ defines an instantaneous rest frame for the electron at each time τ , so we define the zitter mean as an average over the free particle zitter period $T_e = 2\pi/\omega_e$ that keeps the *zitter center velocity* v and the spin vector $s = (\hbar/2)e_3$ fixed. With an overline to denote average value, basic zitter means are specified by the expressions

$$v = \dot{x} = \bar{u} = e_0, \quad \bar{e}_1 = 0 = \bar{e}_2, \quad \bar{e}_3 = e_3. \quad (166)$$

Consequently, the mean of the *spin bivector* (121) is

$$\bar{S} = \frac{\hbar}{2} \bar{u} \bar{e}_1 = \frac{\hbar}{2} e_2 \bar{e}_1 = i s v. \quad (167)$$

This approximation ignores variations in zitter radius and mass over a zitter period.

Since the electromagnetic field $F(z) = F(x + r)$ acts at the location of the particle, to get its effect on the zitter center, we expand with respect to the zitter radius vector. Accordingly, the average field at the zitter center is given by

$$\bar{F}(z) = F(x) + \bar{r} \cdot \nabla F(x) + \frac{1}{2} \overline{(r \cdot \nabla)^2} F(x) + \dots \quad (168)$$

With $\bar{r} = 0$, we have $\bar{F} = F(x)$ to first order.

Neglecting second order effects gives us the mean mass

$$\bar{m} = m_e + \bar{\Phi} \quad \text{with} \quad \bar{\Phi} = g \bar{S} \cdot F(x) = g \bar{F} \cdot (i s v). \quad (169)$$

The mass shift $\bar{\Phi}$ will be recognized as a generalization of the Zeeman interaction in atomic physics.

To lowest order, momentum reduces to

$$\bar{p} = m_e v + \bar{\Phi} \bar{e}_2, \quad (170)$$

Consequently, the momentum equation (131) can serve as an equation of motion for the zitter center.

Projecting out the effect of the mass derivative and averaging, we obtain from (131) a mean equation

$$m_e \dot{v} = q \bar{F} \cdot v + v \cdot (v \wedge \nabla) \bar{\Phi}. \quad (171)$$

This is recognized as the classical equation for a charged particle with intrinsic spin, so it can be regarded as the *classical limit of the zitter model*. With $\overline{F} = F(x)$, the first term on the right is the classical Lorentz force, while the second is the Stern-Gerlach force.

For slowly varying electromagnetic fields there is not much difference between equations (131) and (170), save that replacement of the Lorentz force $q\overline{F} \cdot v$ by $q\overline{F} \cdot u$ produces a *wobble* of v about its mean value. The big difference in the equations comes with electromagnetic fields oscillating with a frequency close to the zitter frequency, for then resonance can occur, as we shall see.

Our model is not complete without an equation of motion for the spin. From (170) we get $\overline{u} \wedge \overline{p} = \overline{u} \wedge \overline{p}$, so the spin equation (125) reduces to

$$\dot{\overline{S}} = g\overline{F} \times \overline{S}. \quad (172)$$

This is just the classical approximation mentioned before, but augmented by the Stern-Gerlach force in (170). We leave consideration of higher order approximations to another time.

B. Spin-Zitter Split

Aside from the identification $v = \dot{x}$, a special role for $v = e_0$ is indicated by the way it splits the complete spin in (124) into a slowly precessing component

$$\overline{S} = (S \wedge v)v = isv. \quad (173)$$

and a rapidly varying zitter component

$$m_e r_e = -\frac{\hbar}{2} e_1 = S \cdot v. \quad (174)$$

Altogether,

$$S = Sv^2 = (S \cdot v + S \wedge v)v = (m_e r_e + is)v. \quad (175)$$

This is called a spacetime split of S with respect to v [1].

The specification of spin vector here is algebraically identical to the one in Dirac theory [1]. The rapidly rotating unit vector $\hat{r} = -e_1$ is also inherent in Dirac theory, but its physical significance has been overlooked. In Dirac theory, identification of the radius vector and the spin vector as parts of a null bivector S has not been made heretofore. The implication is that the spin observable in Dirac theory is only the zitter average \overline{S} of the full spin S . Nevertheless, zitter is still present in the Dirac equation as a rotating phase factor in the wave function, and we shall see that there are indeed possibilities to detect it experimentally.

The “ v -split” of the spin in (175) entails a v -split of the electromagnetic dipole moment into electric and magnetic parts:

$$\begin{aligned} \frac{q}{m_e} S &= \mathbf{d}_v + i\boldsymbol{\mu}_v \quad \text{with} \\ \mathbf{d}_v &= qr_e v, \quad \boldsymbol{\mu}_v = \frac{q}{m_e} sv = -i\frac{q}{m_e} \overline{S}. \end{aligned} \quad (176)$$

Note that the electric dipole has constant magnitude $|\mathbf{d}_v| = q\lambda_e = q\hbar/2m_e$, so it only rotates. A corresponding v -split of the external electromagnetic field has the form $F = \mathbf{E}_v + i\mathbf{B}_v$, where electric and magnetic parts are given by

$$\mathbf{E}_v = \frac{1}{2}(F - vFv), \quad i\mathbf{B}_v = \frac{1}{2}(F + vFv). \quad (177)$$

Of course, these are electric and magnetic fields as “seen” in the instantaneous rest frame of the zitter center, not to be confused with fields in a “lab frame.”

The induced mass shift can now be expressed in the physically perspicuous form

$$\Phi = \frac{q}{m_e} S \cdot F = \mathbf{d}_v \cdot \mathbf{E}_v - \boldsymbol{\mu}_v \cdot \mathbf{B}_v. \quad (178)$$

This is a general result, holding exactly in every application of the zitter model. As \mathbf{d}_v is a rotating vector, this result is consistent with the well established experimental fact that the electron does not have a static electric dipole moment. Of course, if the \mathbf{E}_v field is slowly varying, the effective shift is reduced to the Zeeman interaction $\overline{\Phi} = -\boldsymbol{\mu}_v \cdot \mathbf{B}_v$.

However, the spin split described here has a drawback, namely, that the reference direction v is constantly changing with time, so it is difficult to compare spin directions at different times. We show next how that drawback can be eliminated by examining the spin split more thoroughly from a different point of view that is closely related to the way that spin precession is analyzed in relativistic quantum mechanics.

C. Zitter and Spin in the electron rest frame

Transformation of the time dependent instantaneous rest frame specified by velocity $v = v(\tau)$ to an inertial rest frame specified by a constant vector γ_0 is a *boost*, specified by

$$\begin{aligned} v &= L\gamma_0\tilde{L} = L^2\gamma_0 \quad \text{or} \\ L^2 &= v\gamma_0 = v_0(1 + \mathbf{v}), \end{aligned} \quad (179)$$

where $v_0 = v \cdot \gamma_0$ is the relativistic time dilation factor relating particle proper time τ to inertial time t , that is,

$$v_0 \equiv v \cdot \gamma_0 = \frac{1}{(1 + \mathbf{v}^2)^{\frac{1}{2}}} = \frac{dt}{d\tau}, \quad \mathbf{v} = \frac{d\mathbf{x}}{dt} \quad (180)$$

is the *relative velocity* of the particle in the inertial reference frame. Let us refer to this frame as *the electron rest frame*.

We can solve (179) to get an explicit expression for the rotor L that generates the boost [1]:

$$L = (v\gamma_0)^{\frac{1}{2}} = \frac{1 + v\gamma_0}{[2(1 + v_0)]^{\frac{1}{2}}} = \frac{1 + v_0 + v_0\mathbf{v}}{[2(1 + v_0)]^{\frac{1}{2}}}. \quad (181)$$

However, it is usually easier to work with the simpler form for L^2 in (179).

The deboost of the v -split for spin S in (175) to a split in the electron rest frame is now given by

$$S = LS_0\tilde{L} \quad \text{where} \quad S_0 = -m_e\mathbf{r} + i\mathbf{s}, \quad (182)$$

with electric and magnetic moments defined by

$$\frac{q}{m_e}S_0 = -q\mathbf{r} + i\frac{q}{m_e}\mathbf{s} \equiv \mathbf{d} + i\boldsymbol{\mu}. \quad (183)$$

The deboost of the particle velocity has the form

$$uv = L(1 + \mathbf{u})\tilde{L}. \quad (184)$$

Since spin and velocity are related by $Su = 0$, from (182) and (184) we obtain

$$S_0(1 + \mathbf{u}) = (m_e\mathbf{r} + i\mathbf{s})(1 + \mathbf{u}) = 0. \quad (185)$$

Separating parts of homogeneous grade, we ascertain

$$\mathbf{r} \cdot \mathbf{u} = 0, \quad \mathbf{s} \cdot \mathbf{u} = 0 \quad (186)$$

and

$$\mathbf{s} = im_e\mathbf{r}\mathbf{u} = -m_e\mathbf{r} \times \mathbf{u}. \quad (187)$$

This makes spin look like a classical orbital angular momentum, but with the wrong sign. However, the velocity $\mathbf{u} = \dot{\mathbf{z}}$ is not generally the derivative of the radius vector \mathbf{r} , as we shall see.

It is often useful to express the relation of spin to zitter in terms of a rotating orthonormal frame:

$$\mathbf{e}_k \equiv \tilde{L}e_kvL = U\boldsymbol{\sigma}_k\tilde{U}, \quad (188)$$

where $\mathbf{e}_1 = -\hat{\mathbf{r}}$, $\mathbf{e}_2 = \mathbf{u}$ and $\mathbf{e}_3 = \hat{\mathbf{s}}$. However, it is physically more enlightening to first study the dynamics of spin and zitter more directly.

To treat interaction with external fields in the electron rest frame, we need the deboost of an electromagnetic field:

$$F_0 \equiv \tilde{L}FL = \mathbf{E}_0 + i\mathbf{B}_0, \quad (189)$$

where

$$F = \mathbf{E} + i\mathbf{B} = LF_0\tilde{L} \quad (190)$$

expresses the field in terms of electric and magnetic fields defined in the lab frame.

Note that

$$\begin{aligned} \langle SF \rangle &= \langle LS_0\tilde{L}F \rangle = \langle S_0\tilde{L}FL \rangle \\ &= \langle S_0F_0 \rangle = \langle (\mathbf{d} + i\boldsymbol{\mu})(\mathbf{E}_0 + i\mathbf{B}_0) \rangle. \end{aligned} \quad (191)$$

Whence the spin potential (= mass shift) assumes the form

$$\Phi = \frac{q}{m_e}S \cdot F = \frac{q}{m_e}S_0 \cdot F_0 = \mathbf{d} \cdot \mathbf{E}_0 - \boldsymbol{\mu} \cdot \mathbf{B}_0. \quad (192)$$

However, this is not the final form required for physical interpretation.

We still need explicit expressions for the deboosted fields \mathbf{E}_0 and \mathbf{B}_0 in terms of lab fields \mathbf{E} and \mathbf{B} . Using (179) we have, with obvious notation,

$$F_0 = \tilde{L}FL = F_{\parallel} + F_{\perp}L^2 \quad (193)$$

where

$$F_{\parallel}\mathbf{v} = \mathbf{v}F_{\parallel}, \quad F_{\perp}\mathbf{v} = -\mathbf{v}F_{\perp}. \quad (194)$$

Whence

$$\mathbf{E}_0 = \mathbf{E}_{\parallel} + v_0\mathbf{E}_{\perp} + v_0\mathbf{v} \times \mathbf{B}, \quad (195)$$

where

$$\mathbf{E}_{\parallel} = (\mathbf{E} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}, \quad \mathbf{E}_{\perp} = \mathbf{E} - \mathbf{E}_{\parallel}. \quad (196)$$

Similarly,

$$\mathbf{B}_0 = \mathbf{B}_{\parallel} + v_0\mathbf{B}_{\perp} - v_0\mathbf{v} \times \mathbf{E}. \quad (197)$$

Insertion into (192) gives explicit expressions for the interaction of the electron's electric and magnetic dipoles with external fields.

Physical interpretation of spin dynamics is facilitated by transforming the spin equation of motion (125) to the instantaneous rest frame. Thus,

$$\dot{S} = L(\dot{S}_0 + \Omega_0 \times S_0)\tilde{L} = L(m_e\mathbf{e}_2 + \frac{q}{m_e}F_0 \times S_0)\tilde{L}, \quad (198)$$

where the rotational velocity Ω_0 is determined by differentiating the boost (181), with the result [1]:

$$\begin{aligned} L\Omega_0\tilde{L} &= \Omega_v \equiv 2\dot{L}\tilde{L} = \frac{\dot{v} \wedge (v + \gamma_0)}{1 + v_0} \\ &= v_0\dot{\mathbf{v}} + \frac{\dot{v}_0\mathbf{v}}{1 + v_0} + i\frac{v_0^2\dot{\mathbf{v}} \times \mathbf{v}}{1 + v_0}. \end{aligned} \quad (199)$$

As is evident in equation (179), this quantity acts like an effective electromagnetic field induced by acceleration of the rest frame. The magneticlike term at the right side of the equation is responsible for the classical *Thomas precession* of the spin.

Inserting the split (182) with (183) into (198), we have

$$\begin{aligned} \dot{S}_0 &= m_e\dot{\mathbf{r}} + i\dot{\mathbf{s}} \\ &= m_e\mathbf{e}_2 + (F_0 - \frac{m_e}{q}\Omega_0) \times (\mathbf{d} + i\boldsymbol{\mu}). \end{aligned} \quad (200)$$

Now we introduce the split

$$\mathbf{a} + i\mathbf{b} \equiv F_0 - \frac{m_e}{q}\Omega_0 = \tilde{L}(F - \frac{m_e}{q}\Omega_v)L, \quad (201)$$

noting that expressions for \mathbf{a} and \mathbf{b} can be obtained from (199) by the deboost (193). Using

$$\begin{aligned} (\mathbf{a} + i\mathbf{b}) \times (\mathbf{d} + i\boldsymbol{\mu}) \\ = \mathbf{a} \wedge \mathbf{d} - \mathbf{b} \wedge \boldsymbol{\mu} + i(\mathbf{a} \wedge \boldsymbol{\mu} + \mathbf{b} \wedge \mathbf{d}) \end{aligned} \quad (202)$$

with $\mathbf{a} \wedge \mathbf{d} = i(\mathbf{a} \times \mathbf{d})$, we split (200) into coupled equations of motion for zitter and spin:

$$m_e \dot{\mathbf{r}} = m\mathbf{e}_2 + \boldsymbol{\mu} \times \mathbf{a} + \mathbf{d} \times \mathbf{b}, \quad (203)$$

$$\dot{\mathbf{s}} = \mathbf{a} \times \mathbf{d} + \boldsymbol{\mu} \times \mathbf{b}. \quad (204)$$

These equations are helpful for analyzing the dynamical behavior of spin and zitter vectors.

The spin equation (204) is most familiar. Its last term $\boldsymbol{\mu} \times \mathbf{b}$ is recognized as the usual spin precession torque, including the Thomas precession, as already noted. The term $\mathbf{a} \times \mathbf{d}$ is something new. First note that

$$\dot{\mathbf{s}} \cdot \mathbf{s} = (\mathbf{a} \times \mathbf{d}) \cdot \mathbf{s} = (\mathbf{d} \times \mathbf{s}) \cdot \mathbf{a} = 0 = \mathbf{e}_2 \cdot \mathbf{a}. \quad (205)$$

To prove that this quantity does indeed vanish, we use (193):

$$\begin{aligned} m\mathbf{a} \cdot \mathbf{e}_2 &= \langle L\mathbf{a}\tilde{L}u \wedge p \rangle = \langle (F - \frac{m_e}{q}\Omega_v)u \wedge p \rangle \\ &= \frac{m_e m}{q} [\dot{u} \cdot v + \dot{v} \cdot u] = 0. \end{aligned} \quad (206)$$

It follows from this constraint that we can write

$$\mathbf{a} \times \mathbf{d} = \hat{\mathbf{s}} \times \mathbf{d} (\hat{\mathbf{s}} \cdot \mathbf{a}). \quad (207)$$

This torque has several interesting properties. First, it rotates \mathbf{s} toward the vector \mathbf{e}_2 , which is the direction of zitter circulation in the rest frame. Second, the torque decreases in magnitude until it vanishes at $\mathbf{s} \cdot \mathbf{a} = 0$. Third, its zitter average is zero if \mathbf{a} is slowly varying. These properties suggest that the term is a good candidate for a spin polarization torque.

The zitter equation (203) has similar properties, of course, with the additional term $m\mathbf{e}_2$ expressing the high frequency zitter rotation. We only note that the $\boldsymbol{\mu} \times \mathbf{a}$ torque ensures that \mathbf{r} rotates toward \mathbf{e}_2 along with \mathbf{s} .

X. ZITTER SOLUTIONS AND QUANTIZATION

We now consider a general strategy for solving the rotor equation of motion (147), which, using $e_\mu = R\gamma_\mu\tilde{R}$, can be written in the form

$$\dot{R} = \frac{g}{2}FR + \frac{1}{2}R(\omega\gamma_2\gamma_1 + a'\gamma_+), \quad (208)$$

where

$$\begin{aligned} \omega &= \omega_e + gF \cdot (ue_1) \\ &= \omega_e + \left\langle gFR\gamma_+\gamma_1\tilde{R} \right\rangle = \frac{2m}{\hbar} = \dot{\varphi}, \end{aligned} \quad (209)$$

and, from (136),

$$a' = \tilde{R}aR = -a_1\gamma_1 - a_3\gamma_3, \quad (210)$$

with rotation angles $\dot{\varphi}_1 = -a_1$ and $\dot{\varphi}_3 = -a_3$ determined by (141) and (142). Note that equation (209) gives $\dot{\varphi}$ as

a function of R . The same is true for the functional dependence of $\dot{\varphi}_1$ and $\dot{\varphi}_3$. Hence, these three scalar equations are coupled to the rotor equation (208), and a closed solution is not to be expected in most cases.

The terms on the right side of the equation (208) have an obvious physical interpretation. The first term describes the “external” interaction with an electromagnetic field F , while the second term has two parts describing the “internal” dynamics of the zitter. The first part describes rotation in the “zitter plane” with zitter frequency ω , while the second maintains orientation with respect to the zitter center and feeds energy in and out through the gradient force.

An obvious strategy for solving the rotor equation of motion is to separate external and internal dynamics as follows: For a given electromagnetic field F , the rotor $R = R(\tau)$ can be factored into

$$R = LU, \quad (211)$$

where rotor L satisfies the equation

$$\dot{L} = \frac{g}{2}FL, \quad (212)$$

and rotor U satisfies

$$\dot{U} = \frac{1}{2}U(\omega\gamma_2\gamma_1 + a'\gamma_+). \quad (213)$$

One advantage of this separation is that, by itself, (212) is the rotor equation for the classical approximation, and exact solutions have already been found for several kinds of electromagnetic field [1, 26]. Unfortunately, they all presume that motion is along a timelike curve $x(\tau)$, whereas, in the present case $F = F(z) = F(x+r)$ is given along the lightlike curve $z(\tau)$, so a problem arises in relating $z(\tau)$ to the zitter center $x(\tau)$. A good way to solve that problem is with a zitter average of the field as given in (168). Then the equation for U can be solved separately.

Every method of solving the rotor equations must exploit properties of the given field F , which, as specified in (190) can always be expressed in terms of an electric and magnetic field in the “lab frame” $F = \mathbf{E} + i\mathbf{B}$. Alternatively, every non-null field has an intrinsic decomposition of the form

$$F = \alpha f_3 f_0 + \beta f_2 f_1 = (\alpha + i\beta) f_3 f_0, \quad (214)$$

where the f_μ are eigenvectors of the stress energy tensor

$$T(f_\mu) \equiv -\frac{1}{2}Ff_\mu F, \quad (215)$$

with eigenvalues determined by the invariant

$$F^2 = \mathbf{E}^2 - \mathbf{B}^2 + i\mathbf{E} \cdot \mathbf{B} = \alpha^2 - \beta^2 + 2i\alpha\beta. \quad (216)$$

An interesting class of rotor solutions is specified by the conditions

$$e_\mu(\tau) = f_\mu(z(\tau)). \quad (217)$$

An example is given below.

A. Periodic solutions in a uniform field

The analysis of zitter motion is greatly simplified when F is uniformly constant, for then (212) has the simple solution

$$L = \exp\left(\frac{g}{2}F\tau\right). \quad (218)$$

Since $FL = LF$ in this case, (209) reduces to

$$\omega = \omega_e + g \left\langle FU\gamma_+\gamma_1\tilde{U} \right\rangle. \quad (219)$$

The analysis is thus reduced to solving (213) for the spinor U . However, it is more instructive to solve the particle equation of motion directly.

Equations (137) and (138) show that even for a uniform field the dynamical mass will oscillate unless $F \cdot \dot{S} = mF \cdot (e_2e_0) = 0$. We restrict our analysis to the case of constant mass. That is most simply achieved by imposing the condition $F \cdot e_0 = 0$. Then we can adopt (217) to conclude from (214) and (216) that F is a ‘‘magneticlike’’ field that can be put in the form

$$F = \beta e_1e_2, \text{ with } \beta = (\mathbf{E}^2 - \mathbf{B}^2)^{1/2}, \mathbf{E} \cdot \mathbf{B} = 0. \quad (220)$$

Now, we look for solutions of (162) that decouple the orbital motion on the left from the zitter motion on the right to give us

$$m_e\ddot{x} - qF \cdot \dot{x} = 0, \quad (221)$$

and

$$m_e\ddot{r}_e + m\omega_e^2 r_e - qF \cdot \dot{r}_e = 0. \quad (222)$$

The zitter equation can be solved by assuming that $\dot{r}_e = \Omega_e \cdot r_e = \Omega_e r_e$ where $\Omega_e = \omega_e e_2e_1$, so the last two terms are combined by using the constant of motion $m - q\beta/\omega_e = m_e$. Thus, the zitter equation is reduced to a harmonic oscillator with natural frequency ω_e .

The orbital equation (221) is more interesting. It has a familiar form, so we can integrate it immediately to get

$$\dot{x} = v_0 + \dot{r}, \quad (223)$$

where v_0 is a constant with $v_0 \cdot F = 0$, and

$$\dot{r} = gF \cdot r = g\beta(e_1e_2)r \quad (224)$$

is the equation for a radius vector r rotating in the e_1e_2 plane. It is identical in form to equation (54) for the zitter of a timelike particle. The crucial point is that the time scale is determined by the condition $\dot{x}^2 = \gamma^2$. Hence, for $t = \tau/\gamma$ as the proper time for the particle center, the center circulates with orbital frequency

$$\omega_t = \frac{\gamma|\dot{r}|}{|r|} = \gamma g\beta = \gamma(\omega - \omega_e), \quad (225)$$

where (219) has been applied. The factor γ is thus a free parameter determining the frequency and hence the radius of the orbit.

Now consider the possibility of fixing the orbital frequency so it is commensurate with the frequency shift $\omega - \omega_e$. In other words, we impose the condition

$$\gamma = \frac{\omega_t}{g\beta} = n, \quad (226)$$

where n is a positive integer. According to (220), β is the magnitude of the effective magnetic field, so $\omega_L = g\beta = q\beta/m_e$ is the familiar Lamor frequency for precession in a magnetic field.

This possibility is very interesting! It shows that zitter provides a criterion for quantizing periodic orbits; in this case, so the quantum condition is identical to the one for quantized Landau levels for the Schroedinger equation, except for absence of a zero point energy. It is noteworthy that this energy spectrum has been observed in the *quantum Hall effect* [27] (also without a zero point energy). This example suffices to show that the zitter model has the potential to explain quantization of electron states and its experimental manifestations. Of course, much more work remains to see how far this possibility can be pushed.

Three more points: First, Quantum Hall devices include a constant electric field \mathbf{E} orthogonal to the magnetic field \mathbf{B} to drive a flow of electrons. That is already incorporated in (220), which also says that the presence of the electric field shifts the magnitude of the quantizing magnetic field. That shift may be a new and testable experimental prediction, as I have not seen it mentioned in the literature, though I am no expert in the field.

Second, The above analysis carries through if the condition $F \cdot e_0 = 0$ is replaced by the weaker condition $F \cdot (e_0e_1) = 0$. The result again allows a a quantized solution, but it is uniformly accelerated by a constant ‘‘electriclike’’ field αe_3e_0 .

Third, and potentially most important, the quantum condition (226) may generalize to a resonance between the frequency shift $\omega - \omega_e$ and the orbital frequency ω_t for any bound state. Perhaps this can fulfill Schroedinger’s original expectation that stationary atomic states can be explained as resonances, specifically, as resonances of the electron’s orbital motion with its internal clock.

XI. AN EXPERIMENTAL TEST OF THE ZITTER MODEL

The formulation and structural analysis of the zitter model is now complete and ready for confrontation with experiment. The *Zitter Model* describes the dynamics of a charged electron moving along a helical spacetime history at the speed of light. We have seen that such a particle must have a lightlike spin bivector that generates a slowly varying spacelike spin and magnetic moment along with a rapidly rotating electric dipole. Properties of the spacelike spin in the model are in agreement with standard quantum mechanics. However, the ‘‘zitter frequency’’ of the rotating dipole is so high that the

only prospects for experimental test are in resonant interactions. In this section we see how the zitter model can explain a resonance in an electron channeling experiment that has not predicted by standard quantum mechanics. Whether quantum mechanics can account for this resonance without modification remains to be seen, but I doubt it.

A. Zitter model for a static potential

The interaction of atoms and crystals with an electron is usually modeled with a static potential. For that purpose we introduce a static vector potential defined in the inertial frame of γ_0 at each time t and position \mathbf{x} by

$$qA = V\gamma_0 \quad \text{where} \quad V = V(\mathbf{x}). \quad (227)$$

This determines an electromagnetic field

$$qF = \nabla V \wedge \gamma_0 = -\gamma_0 \nabla V = -\gamma_0 \wedge \nabla V \equiv -\nabla V. \quad (228)$$

This, in turn, determines an electromagnetic force on the particle

$$qF \cdot u = \nabla V u \cdot \gamma_0 - \gamma_0 \dot{V}, \quad \text{where} \quad \dot{V} = u \cdot \nabla V. \quad (229)$$

Inserting this into the momentum equation (131) we obtain

$$\frac{d}{d\tau}(p + V\gamma_0) = -\nabla(Vu \cdot \gamma_0 + \Phi). \quad (230)$$

Next, we introduce the space-time split [1] $p\gamma_0 = p \cdot \gamma_0 + p \wedge \gamma_0 \equiv p_0 + \mathbf{p}$ and separate spatial and temporal parts. This yields a *conserved energy*:

$$E = p_0 + V \quad (231)$$

and, after factoring out $v_0 \approx u \cdot \gamma_0$, a momentum force law

$$\frac{d\mathbf{p}}{dt} = -\nabla(V + v_0^{-1}\Phi), \quad (232)$$

where $v\gamma_0 = v_0(1 + \mathbf{v})$ with $\mathbf{v} = d\mathbf{x}/dt$ and $\mathbf{p} = mv_0\mathbf{v}$.

We still need an explicit form for the spin potential. From (229) we get

$$\Phi = \lambda_e qF \cdot (ue_1) = u \cdot \gamma_0 e_1 \cdot \nabla V - \gamma_0 \cdot e_1 \dot{V}. \quad (233)$$

However, it is better to use (195) and (196) in (192) to get the form

$$\begin{aligned} \Phi &= q\lambda_e[\mathbf{e} \cdot (\mathbf{E}_{\parallel} + v_0\mathbf{E}_{\perp}) + \hat{\mathbf{s}} \cdot (v_0\mathbf{v} \times \mathbf{E})] \\ \text{with} \quad q\mathbf{E} &= -\nabla V, \end{aligned} \quad (234)$$

where \mathbf{e} denotes the unit zitter vector. This has the advantage of clearly separating the high frequency zitter from the low frequency spin contributions, and it exhibits the zitter radius λ_e as governing the strength of the interaction.

These results can be applied to experimental search for observable effects of electron zitter. As the energy conservation law (231) is identical to the usual one, zitter will

be manifested only in momentum fluctuations, which are so small and rapid that they are observable only in resonances. Consider *Mott scattering* by a Coulomb field, for example. In low and high energy ranges the cross section will not be significantly affected by the very high frequency zitter rotation, so the standard result should be obtained [28]. However, in an intermediate range where the impact parameter is on the order of a zitter wavelength, the zitter structure of the particle trajectory should have a significant effect on the scattering. How big an effect awaits calculation!

In the mean time, a new possibility for amplifying zitter resonances in crystals has recently appeared in electron channeling experiments, to which we now turn.

B. Zitter resonance in electron channeling

When a beam of electrons is channeled along a crystal axis, each electron is subject to periodic impulses from atoms along the axis. When the energy of the beam is adjusted so the crystal period matches the period of the electron's zitter dipole, a resonant interaction may be expected to alter the distribution of transmitted electrons. Indeed, a pioneering experiment in search of such a resonance has already been performed, but without knowledge of the dipole interaction mechanism described here. The anticipated resonance was observed at close to the de Broglie frequency, which is precisely half the zitter frequency [29]. Our purpose here is to show how this result can be explained quantitatively by the zitter model. This confluence of theory and experiment provides ample grounds for repeating the experiment with greater accuracy to confirm the results and look for details suggested by the theory.

C. Experimental specifications

The anticipated resonant energy is easily calculated from the de Broglie's (circular) frequency $\omega_B = m_e c^2 / \hbar$. One of de Broglie's original insights was that the frequency of a moving electron observed in a laboratory will be $\omega_L = \omega_B / \gamma$, where $\gamma = v_0$ is the relativistic time dilation factor. The distance traversed during a clock period is $d = 2\pi c\beta / \omega_L = \hbar p / (m_e c)^2$. For the silicon crystal used in the experiment, the interatomic distance along the $\langle 110 \rangle$ direction is $d = 3.84 \text{ \AA}$, which implies a resonant momentum $p = 80.874 \text{ MeV}/c$.

In channeling the maximum crystal potential is a few hundred electron volts at most, so in the 80 MeV region of interest to us, the effective electron mass $M \equiv \gamma m_e = E/c^2$ is constant to an accuracy of 10^{-5} , and $\gamma = 158$.

In axial channeling electrons are trapped in orbits spiraling around a crystal axis. To a first approximation, the crystal potential can be modeled as the potential for a chain of atoms, so it has the form

$$V(\mathbf{x}) = V(r, z) = U(r)P(2\pi z/d), \quad (235)$$

where $\mathbf{x}(t) = \mathbf{r} + z\sigma_z$ is the particle position from the first atom in the chain, with $r = |\mathbf{r}|$. The longitudinal potential $P(2\pi z/d) = P(\omega_0 t)$ is periodic with a *tunable frequency* $\omega_0 = 2\pi\dot{z}/d$ that varies with the energy E . Note that at the expected resonance $\dot{z} = d/T_L$, so $\omega_0 = 2\pi/T_L = \omega_B/\gamma$ corresponds to the de Broglie frequency.

Our problem is to calculate perturbations on the transverse component of the momentum vector, as that can remove electrons from stable orbits in the beam. The transverse component of equation (232) has the familiar form of a nonrelativistic equation:

$$M\ddot{\mathbf{r}} = -\hat{\mathbf{r}}(P\partial_r U + \gamma^{-1}\partial_r\Phi), \quad (236)$$

while the longitudinal component has the form

$$M\ddot{z} = -(U\partial_z P + \gamma^{-1}\partial_z\Phi), \quad (237)$$

where now the overdot indicates differentiation with respect to “lab time” t . In the energy range of interest, it is easy to show that oscillations in the transverse velocity can be ignored, so we regard \dot{z} as a constant tunable velocity close to the speed of light as already assumed above.

D. Crystal potential

To proceed further, we need an explicit model of the crystal potential. For analytic simplicity, we approximate the potential by the first two terms in a Fourier expansion with respect to the reciprocal lattice vector. So write (235) in the more specific form

$$V = U(r)(1 + \cos\omega_0 t), \quad (238)$$

where the coefficient of the second term is set to make the potential vanish between atoms. The first term is the potential for a uniformly charged string, which (in its simplest version) has Lindhard’s form [30, 31]:

$$U(r) = -k \ln[1 + (Ca/r)^2], \quad (239)$$

where $Ca = 0.329\text{\AA}$, and (with $Z = 14$ for silicon) the coupling constant $k = Ze^2/d = 52.5$ eV is the product of electron charge with charge per unit length of the string. The constant $a = 0.885Z^{-1/3}a_0 = (0.885)(14)^{-1/3}(0.529\text{\AA}) = .190\text{\AA}$ is the Fermi-Thomas screening radius, and the constant $C^2 = 3$ is a fairly accurate fit over the range of interest.

The string potential is defined as an average over atomic potentials; thus,

$$U(r) = \int_{-\infty}^{\infty} V_{\text{atom}}[(r^2 + z^2)^{1/2}] \frac{dz}{d}, \quad (240)$$

where the screened atomic potential is given by

$$V_{\text{atom}}(R) = -\frac{Ze^2}{R} \varphi(R/a). \quad (241)$$

To get (239), Lindhard used the screening function

$$\varphi(R/a) = 1 - [1 + (Ca/R)^2]^{-1/2}. \quad (242)$$

Actually, the experiment is not very sensitive to the shape of the potential, so long as it is sufficient to bind the electron to an orbit around the crystal axis. In the first approximation $V = U(r)$ the projection of the electron orbit onto a transverse plane looks like a precessing ellipse or rosette. The second term $U \cos\omega_0 t$ is ignored in most channeling calculations, as it merely produces small harmonic oscillations on the radius of the precessing ellipse. However, the periodicity of the second term is essential for resonance in the zitter perturbation of interest here.

E. Classical channeling orbits

Ignoring the zitter perturbation in (236) for the time being, we seek to ascertain the effect of the periodic factor $P(\omega_0 t) = 1 + \cos\omega_0 t$ on the orbital motion. With an obvious change of notation, we can represent the radius vector in the complex form $\mathbf{r} = re^{i\theta}$, where the imaginary i is the bivector generator of rotations in the transverse plane. Then equation (236) assumes the complex form

$$M\ddot{\mathbf{r}} = -U'Pe^{i\theta} = -\frac{U'}{r}(1 + \cos\omega_0 t)\mathbf{r}, \quad (243)$$

with $U' \equiv \partial_r U$.

We are interested only in radial oscillations, so we use conservation of angular momentum $L = Mr^2\dot{\theta}$ to separate out the rotational motion. With the periodic driving factor omitted, equation (243) admits the energy integral

$$E_{\perp} = \frac{1}{2}Mr\dot{r}^2 + W(r) \quad (244)$$

where $W(r) = \frac{L^2}{2Mr^2} + U(r)$.

Let us expand this around a circular orbit of radius r_0 , and for quantitative estimates take $r_0 = 0.50\text{\AA}$ as a representative intermediate radius. For $\mathbf{r}_0 = re^{i\theta_0}$, equation (243) gives us

$$\dot{\theta}_0^2 = \frac{U'_0}{Mr_0} = \frac{L^2}{M^2r_0^4}, \quad (245)$$

where $U'_0 = U'(r_0)$. In terms of $x = r - r_0$, expansion of (244) gives us

$$E_{\perp} = \frac{1}{2}M\dot{x}^2 + W_0 + \frac{1}{2}W''_0x^2, \quad (246)$$

where

$$W_0 = U_0 + r_0U'_0/2, \quad W'_0 = U'_0 - U'_0 = 0, \quad (247)$$

$$W''_0 = U''_0 + 3U'_0/r_0,$$

with $U_0 = -18.9$ eV,

$$U' = \partial_r U = \frac{2k}{r} \left[\frac{(Ca/r)^2}{1 + (Ca/r)^2} \right], \quad (248)$$

with $r_0 U'_0 = 31.7$ eV, and

$$U'' = -\frac{U'}{r} \left[\frac{3 + (Ca/r)^2}{1 + (Ca/r)^2} \right], \quad (249)$$

with $r_0^2 U''_0 = -76.0$ eV.

Differentiating (246) and reinserting the periodic driving factor, we obtain the desired equation for radial oscillations:

$$\ddot{x} + \Omega_0^2(1 + \cos \omega_0 t)x = 0, \quad (250)$$

where, for mass M at the expected resonance,

$$\begin{aligned} \Omega_0^2 &\equiv \frac{W''_0}{M} = \frac{3U'_0 + r_0 U''_0}{Mr_0} \\ &= \frac{3 \times 31.7 - 76.0}{80.9 \times 10^6} \left(\frac{c^2}{r_0^2} \right), \end{aligned} \quad (251)$$

so $\Omega_0 = 4.21 \times 10^{15} \text{ s}^{-1}$. This should be compared with the $\theta_0 = 4.75 \times 10^{15} \text{ s}^{-1}$ from (245) and the expected resonant frequency $\omega_0 = \omega_B/\gamma = 4.91 \times 10^{18} \text{ s}^{-1}$. We note that the distance traveled in one orbital revolution is $d_r = 2\pi c/\theta_0 = 3.97 \times 10^3 \text{ \AA} = 0.397 \mu\text{m}$. Thus, the orbit makes 2.52 revolutions in passing through the one micron crystal, so the orbital revolutions are only weakly coupled to the high frequency radial oscillations.

Equation (250) is a special case of *Mathieu's equation*:

$$\ddot{x} + q(1 + h \cos \omega t)x = 0. \quad (252)$$

According to *Floquet's Theorem*, this equation has solutions of the general form [32, 33]

$$x(t) = e^{st} \sum_{n=-\infty}^{\infty} a_n e^{i n \omega t}. \quad (253)$$

Substitution into (252) yields a three term recursion relation that can be solved for s and ratios of the Fourier coefficients a_n . In general, the *Floquet exponent* $s = s(q, h)$ is a complex constant, so it determines whether solutions are unstable or periodic.

Since $\Omega_0 \ll \omega_0$ in equation (250), its Floquet exponent $s = i\Omega$ is pure imaginary, and its recursion relations give

$$\begin{aligned} \Omega^2 &= \frac{3}{2} \Omega_0^2, \quad \text{and} \\ \frac{a_n}{a_{n-1}} &= \frac{a_{-n}}{a_{-n+1}} = \frac{\Omega_0^2}{2n^2 \omega_0^2} \quad \text{for } n \geq 1. \end{aligned} \quad (254)$$

Therefore, the solution is dominated by the first order term, with the particular form

$$\begin{aligned} x(t) &= a(\cos \Omega t) \cos \omega_0 t = \frac{a}{2}(\cos \omega_+ t + \cos \omega_- t), \\ \omega_{\pm} &= \omega_0 \pm \Omega. \end{aligned} \quad (255)$$

As might have been anticipated, this describes a harmonic oscillator with high frequency ω_0 and a slowly varying amplitude with frequency Ω , which is equivalent

to a sum of two oscillators with frequencies ω_{\pm} separated by $\omega_+ - \omega_- = 2\Omega$.

We shall see that, at the resonant frequency $\omega_0 = \omega_B/\gamma$, the frequency shift $\Omega = (0.857 \times 10^{-3})\omega_0$ is the right order of magnitude to contribute to experimental effects. Moreover, this quantity has been estimated at the particular radius $r = 0.50 \text{ \AA}$, and it may be larger by an order of magnitude for smaller radii of experimental relevance. Accordingly, a distribution of Ω values will contribute to the experiment.

A limit on the maximum radius of a channeled electron is set by the requirement that $W(r)$ in (244) must be negative for a bound orbit. A sharper limit is set by neighboring atoms. The total crystal potential can be modeled as a sum of chain potentials isomorphic to the one for the channeling axis. For larger radii perturbation from other chains can induce transition to a neighboring chain, with the result that channeled electrons jump randomly from chain to chain. We are interested in this effect only to the extent that it affects the distribution of electrons transmitted by a single channel. However, a quantitative estimate of such transitions will not be attempted here.

In silicon, the closest chain to a $\langle 110 \rangle$ channel is at a distance of 1.36 \AA with atoms located at positions $z_{n+1/2} = (n + 1/2)d$ alternating with the positions $z_n = nd$ along the channel. Considering the slow precession of a channeled orbit at the resonant frequency, this chain will resonate with it for hundreds of atomic steps. In fact, the interaction might lock onto the orbit to prevent precession during resonance. This may indeed have a substantial effect on channeled electrons, but we will not investigate it further here.

F. Zitter perturbations

Now we are prepared to consider the effect of zitter perturbations on the orbit. To evaluate the zitter potential Φ , we use equation (234) with $\mathbf{E}_{\parallel} = (\mathbf{E} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}$ and $\hat{\mathbf{v}} \approx \boldsymbol{\sigma}_z = \nabla z$, and we ignore the spin term (because it vanishes on averaging over spin directions). By the way, we don't need to use the spin equation of motion in our calculations; we only need the fact that it implies that the unit zitter vector \mathbf{e} rotates rapidly in a plane that precesses slowly with the spin \mathbf{s} .

Inserting the electric potential into (234), we get

$$\Phi = -\lambda_e [\gamma \mathbf{e} \cdot \hat{\mathbf{r}} U' P + \mathbf{e} \cdot \boldsymbol{\sigma}_z U P']. \quad (256)$$

Since $\hat{\mathbf{r}}$ is effectively constant over a zitter period, we can make the zitter oscillations explicit by writing $\mathbf{e} \cdot \hat{\mathbf{r}} = \cos(\omega_Z t/\gamma + \delta)$ and $\mathbf{e} \cdot \boldsymbol{\sigma}_z = \sin(\omega_Z t/\gamma + \delta')$, where ω_Z is the zitter frequency. Thus,

$$\begin{aligned} \Phi &= -\lambda_e [\gamma U' P \cos(\omega_Z t/\gamma + \delta) \\ &\quad + U P' \sin(\omega_Z t/\gamma + \delta')], \end{aligned} \quad (257)$$

where the smaller second term can be neglected.

The spin potential has a twofold effect on electron motion: first, as a shift in zitter frequency according to equation (148); second, as a perturbation of the momentum in equation (236). For the frequency shift we get the explicit expression

$$\omega_Z = \omega_e - \frac{\gamma U'}{m_e c} \cos(\omega_e t / \gamma + \delta). \quad (258)$$

The modulus of the oscillating term has the estimated mean value $\gamma U' c / m_e c^2 = (158)(31.7 \text{ eV} / 0.5 \text{ \AA})(3 \times 10^{18} \text{ \AA}^{-1} / (0.511 \times 10^6 \text{ eV})) = 1.96 \times 10^{16} \text{ s}^{-1}$. Compared to $\omega_e = 1.55 \times 10^{21} \text{ s}^{-1}$, this quantity is too small by 10^{-3} to play a role in the present experiment. However, the value of U' , estimated here at a radius of $r_0 = 0.5 \text{ \AA}$, may be regarded as lower bound, as the logarithmic potential is a very soft approximation to a realistic potential, which is much sharper at the screening radius $a = 0.190 \text{ \AA}$ where the logarithmic potential is invalid. We shall see that close resonant encounters with atomic nuclei play a dominant role in the experiment, so the oscillating term might indeed contribute a mean frequency shift on the order of $10^{-2} \omega_e$, which could show up in experimental data. Replacement of the soft periodic modulus $P(\omega_0 t) = (1 + \cos \omega_0 t)$ by one that is more sharply peaked at the atomic sites may strengthen this conclusion. That being said, for the rest of our analysis, it suffices to assume $\omega_Z = \omega_e$.

Inserting (257) into (236), we get an explicit expression for the zitter perturbation term:

$$\begin{aligned} -\gamma^{-1}(\nabla \Phi)_\perp &= -\gamma^{-1} \hat{\mathbf{r}} \partial_r \Phi \\ &= \lambda_e \hat{\mathbf{r}} U'' P \cos(\omega_Z t / \gamma + \delta). \end{aligned} \quad (259)$$

G. Zitter resonance

The task remains to show that this interaction can produce a resonant amplification of the electron's orbit. Inserting it into equation (236) with a convenient choice of phase and writing $\omega = \omega_Z / \gamma$, we get the equation of motion

$$M \frac{d^2 \mathbf{r}}{dt^2} = -\mathbf{r} \frac{U'}{r} (1 + \cos \omega_0 t) \left(1 + \frac{\lambda_e}{R} \cos \omega t\right), \quad (260)$$

where $R \equiv -U'/U'' = r[1 + (Ca/r)^2]/[3 + (Ca/r)^2]$ is an *effective (screened) radius*. Ignoring the amplitude modulation as determined in (255), we can reduce this to a radial equation

$$\ddot{x} + \omega_0^2 \left(1 + \frac{\lambda_e}{R} \cos \omega t\right) x = 0. \quad (261)$$

Of course, we can replace ω_0 in this equation by ω_\pm to get two separate resonant peaks.

For $h = \lambda_e / R$ constant, (261) is Mathieu's equation (252), so let us solve it using $h = \lambda_e / R_0 = 1.931 \times 10^{-3} / 0.208 = 9.283 \times 10^{-3}$ as an approximation. For small values of h such as this, equation (261) has a first order resonance at $\omega = 2\omega_0$. An easy way to see that is to

regard (261) as an equation for a driven harmonic oscillator with driving force proportional to its amplitude. In the experiment, the "natural frequency" ω_0 was adjusted by continuously varying the initial energy of the electron until a resonance was found. As for a periodically driven harmonic oscillator, resonance occurs when the driving frequency equals the natural frequency. In this case, that means $\omega - \omega_0 = \omega_0$ or $\omega_0 = \omega / 2 = \omega_Z / 2\gamma$, which explains why the resonance occurs at half the zitter frequency — a surprising result until contemplation shows that it is an obvious feature of parametric resonance!

More generally, it can be shown that (261) has resonances at $2\omega_0 = n\omega$, for $n = 1, 2, \dots$. We demonstrate that explicitly for the first order resonance by truncating the series in (253) to the form [34]

$$x = e^{st} \left[a \cos \frac{\omega t}{2} + b \sin \frac{\omega t}{2} \right]. \quad (262)$$

For a resonant solution, the Floquet exponent s must be real and positive. [Actually, to get this form from (253) we should use the Floquet coefficient $s - i\omega/2$ and incorporate its imaginary part in the series. Thus, there is some ambiguity in the definition of Floquet exponent.]

To validate the trial solution and evaluate its parameters, we insert it into the differential equation. Using trigonometric identities such as $2 \cos A \cos 2A = \cos A + \cos 3A$ to reduce products to sums and dropping higher order terms, we obtain

$$\begin{aligned} s\omega b + [s^s + \omega_0^2 - (\omega/2)^2 + (h/2)\omega_0^2]a &= 0, \\ s\omega a - [s^s + \omega_0^2 - (\omega/2)^2 - (h/2)\omega_0^2]b &= 0. \end{aligned}$$

For a near resonance solution we write $\omega = 2\omega_0 + \epsilon$ and neglect second order terms to get

$$\begin{aligned} sb - \frac{1}{2}[\epsilon - \frac{1}{2}h\omega_0]a &= 0, \\ sa + \frac{1}{2}[\epsilon + \frac{1}{2}h\omega_0]b &= 0. \end{aligned}$$

These equations can be solved for the coefficients provided

$$s^2 = \frac{1}{4}[(\frac{1}{2}h\omega_0)^2 - \epsilon^2] > 0. \quad (263)$$

Thus, we have a resonance with width

$$\Delta\omega = 2\epsilon = h\omega_0 = 9.283 \times 10^{-3} \omega_0, \quad (264)$$

or

$$\Delta p = h(80.9 \text{ MeV}/c) = 0.751 \text{ MeV}/c. \quad (265)$$

And for the amplification factor at resonance we have

$$st = \frac{1}{4}h\omega_0 t = \frac{\pi}{2}hn = 1.46 \times 10^{-2}n, \quad (266)$$

where n is the number of atoms traversed in a resonant state. Since $\ln 2 = 0.693$, this implies that the amplitude is doubled in traversing about 50 atoms.

Now note that the value of $h = \lambda_e / R_0$ used in (264) and (266) applies to only the subclass of orbits for which

$r_0 = 0.50\text{\AA}$. For smaller radii the values can be much larger. In principle, to get the width of the ensemble of orbits we should replace h in (264) by its average \bar{h} over the ensemble. However, the result will probably not differ much from the typical value we have chosen.

Similarly, the amplitude factor in (266) will have a distribution of values, and the doubling factor will be reached much faster for orbits with smaller radii. Presumably, random perturbations (such as thermal fluctuations of the nuclei) will limit the resonant state coherence length to some mean value \bar{n} . Consequently, states with smaller r_0 will be preferentially ejected from the beam.

Even more to the point, the perturbation parameter $h = \lambda_e/R$ is not constant as we tentatively supposed but increases rapidly as the electron approaches a nucleus. In resonance the value of h close to each nucleus dominates the perturbation, so its effective mean value is much smaller than the estimate for constant r_0 . Evidently, resonant interaction may eject electrons with small r_0 in just a few atomic steps.

H. Experimental implications

The predicted resonance width in (265) is in fair agreement with the width in the channeling experiment data [29], considering uncertainties in the value of h and such factors as thermal vibrations that may contribute to damping. Damping can only widen the width and destroy the resonance if it is too severe.

We need to explain how the orbital resonance is manifested in the experimental measurements. Two scintillators, SC2 and SC3, were employed to detect the transmitted electrons. The larger detector SC3, with a radius about 3 times that of SC2, served as a monitor while the smaller detector served as a counter for a central portion of the beam. The measured quantity was the ratio of SC2 to SC3 counts as momentum was varied in small steps (0.083%) over a 2% range centered at the expected resonance momentum 80.9 MeV. An 8% dip was observed at 81.1 MeV. Orbital resonances may contribute to this effect in at least two ways: first, and perhaps most important, by increasing the probability of close encounter with a nucleus that will scatter the electron out of the beam; second, by increasing the duration of eccentric orbits outside the central region. Overall, resonant interactions will be strongest on electrons confined to the central region. These observations suffice for a qualitative explanation of the observed dip. A quantitative calculation will not be attempted here.

The most problematic feature of the experiment is the 0.226 MeV/c difference between observed and predicted resonance energies. If estimation of the experimental error was overly pessimistic, that indicates a physical frequency shift. The most likely origin for such a shift is the frequency split in (255). The experiment was not sufficiently accurate to resolve separate peaks for the two frequencies, so the peaks would merge to broaden the

measured resonance width. However, the peak for ω_+ is likely to be higher than the peak for ω_- owing to greater probability for ejection from the beam. Hence, the center of the merged distribution will be displaced to a higher frequency. If this explanation is correct, then an increase in experimental resolution will separate the two peaks, and their relative heights will measure the relative probability of ejection at the two frequencies.

Though the string approximation to the crystal potential (239) is useful for semiquantitative analysis of zitter resonance, as we have seen, it breaks down completely at radii near the screening radius $a = 0.190\text{\AA}$; for then, as Lindhard [31] has shown, the electron's collision time with a nucleus is comparable to the travel time between atoms. In picturesque terms, the electron begins to "feel" individual atoms rather than a continuous string. Within this domain, our analysis of electron motion remains qualitatively the same, but a more realistic crystal potential is needed for accurate quantitative estimates. A hard lower limit on the radius is determined by the mean radius of nuclear zeropoint vibrations, which Debye theory estimates as 0.05\AA [30].

If the idea of zitter resonance is taken seriously, there are many opportunities for new theoretical and experimental investigations. Increasing the resolution by three orders of magnitude will open the door to refined studies of frequency shifts, line splitting, spin effects and Zeeman splitting, all of which are inherent in zitter theory [35]. As has been noted, the most straightforward prediction of the zitter model is a second order resonance near 161.7 MeV/c. In a first approximation, it can be analyzed in much the same way as here, though removal of electrons from the center of the beam may be enhanced by such processes as pair creation.

A more detailed analysis of zitter resonance in channeling requires close attention to experimental conditions, so that will be addressed elsewhere. Classical particle models have long been used for channeling calculations with considerable success. Besides being simpler and more transparent than quantum mechanical models, they often give better results at high energies. For present purposes, the zitter model differs from the usual classical model only by the zitter dipole interaction. As we see below, though zitter is inherent in the Dirac electron theory, it is unlikely that the channeling resonance effect can be derived from the Dirac equation without some modification such as projection into a Majorana state.

XII. ZITTERBEWEGUNG IN DIRAC THEORY

This section describes the intimate relation of the zitter model to the Dirac equation. The relation should not be surprising, as it was study of Dirac theory that led to the zitter model in the first place. Conversely, we shall see that the zitter model suggests modification of Dirac theory to incorporate deeper zitter substructure with new physical implications.

Our first task is to match up variables and dynamical equations in the zitter model with observables and dynamics in Dirac theory. An exact match is not to be expected, as the particle based zitter model can only be related to the Dirac field theory by some sort of averaging or projection process. Nevertheless, the comparison reveals specific similarities and differences that must be addressed in establishing a firm connection between zitter model and Dirac theory. This leads to suggestions for modifying the Dirac theory and further research to resolve outstanding issues.

In the language of STA, the *real Dirac equation* has the form

$$\begin{aligned} \gamma^\mu \partial_\mu (\psi \gamma_2 \gamma_1 \hbar - q A_\mu \psi) &= m_e \psi \gamma_0, \\ \text{or } \nabla \psi i \sigma_3 \hbar - q A \psi &= m_e \psi \gamma_0, \end{aligned} \quad (267)$$

where the Dirac wave function is a *real spinor field*

$$\psi = \psi(x) = (\rho e^{i\beta})^{1/2} R. \quad (268)$$

This version of the Dirac equation is fully equivalent to the standard matrix version [1], but it has great advantages for analyzing the structure of the Dirac theory as shown in the following.

The Dirac wave function determines a frame field of *local observables*

$$\psi \gamma_\mu \tilde{\psi} = \rho e_\mu, \quad \text{where } e_\mu = R \gamma_\mu \tilde{R} = e_\mu(x) \quad (269)$$

and $\rho = \rho(x)$ is interpreted as a scalar probability density, in accordance with the interpretation of the *Dirac current* $\psi \gamma_0 \tilde{\psi} = \rho e_0$ as a probability current. The vector fields

$$v \equiv e_0 = R \gamma_0 \tilde{R} = v(x), \quad (270)$$

and

$$s \equiv \frac{1}{2} e_3 = \frac{1}{2} \hbar R \gamma_3 \tilde{R} = s(x) \quad (271)$$

are interpreted as local velocity and spin observables for the electron. Note that these quantities are algebraically identical to the expressions for velocity and spin in the zitter model. Likewise for the other observables

$$u \equiv e_0 + e_2 = R \gamma_+ \tilde{R} = u(x), \quad (272)$$

and

$$\bar{S} \equiv i s v = \frac{1}{2} R i \sigma_3 \hbar \tilde{R} = \bar{S}(x). \quad (273)$$

The difference is that the observables here are vector and bivector fields, whereas in the zitter model they are defined on a particle history. Our next task is to compare them dynamically.

A. Zitterbewegung along Dirac histories

The mass term in the Dirac equation (267) can be written in the form

$$m_e \psi \gamma_0 = m_e (\rho e^{i\beta})^{1/2} R \gamma_0 \tilde{R} R = m_e v e^{-i\beta} \psi. \quad (274)$$

Whence, the Dirac equation can be reformulated in the compact form

$$D \psi \equiv \gamma^\mu D_\mu \psi = 0, \quad (275)$$

where a gauge invariant *coderivative* is defined by

$$D_\mu = \partial_\mu + \frac{1}{2} \Omega_\mu, \quad (276)$$

with the bivector-valued connexion

$$\Omega_\mu \equiv \frac{2}{\hbar} (m_e v_\mu e^{-i\beta} + q A_\mu) e_2 e_1. \quad (277)$$

The purpose of introducing this coderivative is not to reduce the Dirac equation to the maximally compact form (275), but to reveal that all the essential physics is contained in the connexion Ω_μ .

Our aim is to compare zitter dynamics along the history of a zitter center with Dirac dynamics along “streamlines” of the Dirac current. To that end, we evaluate the *directional coderivative* along a Dirac history as follows. From

$$D \psi \gamma_0 = D(v e^{-i\beta} \psi) = [D(v \psi) - i(\nabla \beta) v \psi] e^{-i\beta} = 0$$

we obtain

$$D(v \psi) = 2v \cdot D \psi + (Dv) \psi = i(\nabla \beta) v \psi. \quad (278)$$

This separates into the familiar conservation law for the Dirac current and a dynamical equation for rotations along a Dirac streamline:

$$\dot{\rho} + \rho D \cdot v = D \cdot (\rho v) = \nabla \cdot (\rho v) = 0, \quad (279)$$

$$v \cdot DR = \dot{R} + \frac{1}{2} \Omega(v) R = -\frac{1}{2} [D \wedge v + i(v \wedge \nabla \beta)] R. \quad (280)$$

The overdot indicates the directional derivative $v \cdot \nabla$ and (277) gives us

$$\Omega(v) = (\omega_e \cos \beta + \frac{2q}{\hbar} A \cdot v) e_2 e_1 + (\omega_e \sin \beta) e_0 e_3 \quad (281)$$

$$D \wedge v = \nabla \wedge v + (\omega_e \sin \beta) e_3 e_0. \quad (282)$$

When these are inserted into (280), the boosts in the $e_0 e_3$ plane cancel to give us

$$\dot{R} = \frac{1}{2} [(\omega_e \cos \beta + \frac{2q}{\hbar} A \cdot v) e_1 e_2 - \nabla \wedge v - i(v \wedge \nabla \beta)] R. \quad (283)$$

This is an exact result. It does indeed exhibit the familiar zitter rotation in the $e_2 e_1$ -plane, though the frequency seems to be different.

We still need to evaluate the curl of the velocity field to appreciate its effect on the dynamics. A general expression has been derived elsewhere [2, 3], but our purpose here is served by the eikonal approximation, expressed by

$$\nabla \psi \gamma_2 \gamma_1 \hbar = \nabla (\psi_0 e^{-\gamma_2 \gamma_1 \phi / \hbar} \gamma_2 \gamma_1 \hbar) = (\nabla \phi) \psi. \quad (284)$$

Inserting this into the Dirac equation (267), we obtain

$$\nabla \phi = m_e v e^{-i\beta} - q A. \quad (285)$$

This implies $e^{i\beta} = \pm 1$, where the choice of sign depends on the chosen sign of charge. We adopt that approximation only in this equation, as the parameter β is too important to ignore completely [2]. Then the curl of (285) gives us

$$-m_e \nabla \wedge v = q \nabla \wedge A = qF. \quad (286)$$

This is equivalent to the Lorentz force equation for a fluid of charge with uniform density, as seen by “dotting” with v , to get

$$v \cdot (\nabla \wedge v) = v \cdot \nabla v = -\frac{q}{m_e} v \cdot F = \frac{q}{m_e} F \cdot v. \quad (287)$$

More generally, we insert (286) into (280) and use (282) to get the spinor equation of motion

$$\dot{R} = \frac{1}{2} [\omega_v e_1 e_2 + \frac{q}{m_e} F - i(v \wedge \nabla \beta)] R, \quad (288)$$

with $\omega_v \equiv \omega_e \cos \beta + (2q/\hbar)A \cdot v$. This equation must be compared with the analogous expression (208) for rotational velocity in the zitter model. The general form is very similar, but we are not equipped to account for the differences for reasons to be discussed.

The zitter frequencies in the two equations are not equivalent, but they do have the same free particle limit ω_e . The bivector $i(v \wedge \nabla \beta)$ also contributes to the rotation rate, as implied by

$$v \cdot [i(v \wedge \nabla \beta)] = -i(v \wedge v \wedge \nabla \beta) = 0.$$

However, the physical significance of this term remains obscure. The apparent absence of a Stern-Gerlach force in the Dirac version (288) is noteworthy, but we cannot be sure that it is not buried in terms that we do not understand.

Another way to compare the zitter model with the Dirac theory is through their Lagrangians. The Dirac equation (267) is derivable from the Lagrangian

$$L = \left\langle \hbar \nabla \psi i \gamma_3 \tilde{\psi} + q A \psi \gamma_0 \tilde{\psi} + m_e \psi \tilde{\psi} \right\rangle. \quad (289)$$

Inserting (268) and using $i\gamma_3 = \gamma_2 \gamma_1 \gamma_0$ with (274), we see that the “kinetic term” can be written

$$\left\langle \hbar v \nabla \psi \gamma_2 \gamma_1 \tilde{\psi} e^{-i\beta} \right\rangle. \quad (290)$$

Restricting the derivative to streamlines we have $v \nabla \psi = v \cdot \nabla \psi = \dot{\psi}$, and comparison with the Lagrangian (57) is straightforward. Note that the additional factor in (290) cancels the corresponding factor in (81). However, this does not resolve the puzzling role of β in the Dirac equation [2].

It might be thought that the Dirac equation is more fundamental than the zitter model because interaction comes from the vector potential A alone and interaction with the field F arises only indirectly, for example in the manner described above. A famous consequence of this is the derivation of the gyromagnetic ratio $g = 2$. In contrast, the Lagrangian (57) for the zitter model appears

to presume the electron magnetic moment with separate coupling constants for A and F interactions. Note, however, that precisely two independent constants are presumed in both models. The rest mass is presumed in the Dirac equation, but that is replaced by the charge to mass ratio in the zitter model. It remains to be seen which is more fundamental.

B. Zitterbewegung substructure

In preceding sections we saw that a rotating electric dipole is the hallmark of zitterbewegung, so one wonders why it has attracted so little attention in accounts of Dirac theory. In the original paper introducing his equation [36], Dirac concluded that the electron has both a magnetic and an electric moment, the magnetic moment being the same as in the Pauli model. However, he said, “The electric moment, being a pure imaginary, we should not expect to appear in the model. It is doubtful whether the electric moment has any physical meaning.”

It is worth translating Dirac’s argument into STA, especially since its mathematical content has been retained in the current literature. One simply “squares” the operator on the wave function in (267) to get

$$(-\hbar^2 \nabla^2 + q^2 A^2) \psi - 2q(A \cdot \nabla + \frac{1}{2}F) \psi i \sigma_3 \hbar = m_e^2 \psi, \quad (291)$$

where $F = \nabla A = \nabla \wedge A$ with the *Lorenz condition* $\nabla \cdot A = 0$. This is the Klein-Gordon equation with an extra term that explicitly shows the action of bivector F “rotating” the wave function. The interaction energy density associated with this term is proportional to

$$\begin{aligned} \frac{1}{2} \langle F \psi i \sigma_3 \hbar \tilde{\psi} \rangle &= \langle F \bar{S} \rho e^{i\beta} \rangle \\ &= -\rho (\mathbf{B}_v \cdot \mathbf{s} \cos \beta + \mathbf{E}_v \cdot \mathbf{s} \sin \beta), \end{aligned} \quad (292)$$

where the v -split introduced in (177) has been used on the right hand side. The $\mathbf{B}_v \cdot \mathbf{s} \cos \beta$ term is recognized as the Pauli term except for the strange $\cos \beta$ factor. The $\mathbf{E}_v \cdot \mathbf{s} \sin \beta$ term is what Dirac identified as an imaginary dipole moment. He never mentioned the electric dipole again. In his influential textbook [37] he simply suppressed the offending term by a subterfuge advertised as a change in representation. Then he killed the term with an approximation that amounts to $\sin \beta = 0$ and never looked back. We find an alternative resolution of this “dipole problem” below.

Dirac was soon convinced by Schroedinger [6] that zitterbewegung is foundational to electron theory and he argued the case vigorously in his textbook [37]. As his argument is still widely accepted [38], it deserves comment here.

Dirac introduces a position operator by identifying a velocity operator as its time derivative, and he followed Schroedinger in integrating the equation for the free particle case. He identifies his α_k matrices as velocity operators and claims that their eigenvalues ± 1 correspond

to measured values of electron velocity, asserting: “we can conclude that *a measurement of a component of the velocity of a free electron is certain to lead to the result $\pm c$.*” (Dirac’s italics) From the STA point of view this argument and its implications are bogus, for reasons explained elsewhere [1]. However, we agree with the assumption that the electron moves with the speed of light. We differ in identifying the local electron velocity with the null vector $u = R\gamma_+\tilde{R}$ introduced above.

We also agree with Dirac in associating spin with zitter circulation. Dirac concludes: “Our argument is valid only provided the position of the particle is an observable. If this assumption holds, the particle must have a spin angular momentum of half a quantum.” However, Dirac’s analysis of zitter and spin never went beyond the free particle case. He overlooked (or dismissed) the inference that his account of spin arising from charged particle circulation implies a rapidly rotating electric dipole. Though zitter obviously arose from wave function phase oscillations in Schroedinger’s free particle analysis, Dirac never considered a general connection of zitter circulation to wave function phase. The present study can be regarded as an extension of Dirac’s analysis to incorporate these features in a general theory of zitter in quantum mechanics.

As the Dirac equation has an unsurpassed record of success in QM and QED applications, it is imperative to reconcile it with any proposals about zitter. If the zitter model describes substructure in electron motion that is not captured by the Dirac equation, it must at least be related to the Dirac wave function by some sort of averaging process. Without attempting a definitive reconciliation, let us note some issues that must be addressed.

We know that the conserved Dirac current ρv determines a congruence of curves (or streamlines) for every solution of the Dirac equation. As Bohm and Hiley have argued at length [5], each of these curves $x = x(\tau)$ can be regarded as a possible path for the electron weighted by a relative probability $\rho = \rho(x)$ that the electron actually followed that path. This is a viable particle interpretation of quantum mechanics. However, a refinement is necessary to account for zitterbewegung, which suggests that the actual particle paths are lightlike helices with tangent vector $u = u(x)$ at each spacetime point. The simplest refinement would have each of these lightlike paths winding around a Dirac streamline, but this possibility is questionable without deriving it from the zitter model, at least approximately, by a well defined averaging process (yet to be determined)!

A crucial problem is to justify the weighting of paths by the probability density $\rho = \rho(x)$. A new possibility is suggested by the fact that the analogous quantity in our zitter particle model is a timescale factor $\rho = \rho(\tau)$. Thus, the putative probability density in the Dirac equation might be derivable as a time scale weighting on a congruence of particle paths! In any case, time scaling in the zitter model must be reconciled with the probability interpretation in the Dirac theory.

It may be that a suitable averaging process relating particle histories to the Dirac equation will involve time averaging as well as ensemble averaging with constraints. Analogy with the zitter model already suggests that Dirac observables for velocity v and spin \bar{S} correspond to zitter time averages, but the phase of the Dirac wave function is directly comparable to the zitter phase.

The Schroedinger equation is a nonrelativistic approximation to the Dirac equation that freezes spin but preserves zitter oscillations in the phase of the wave function and in coupling of phase to amplitude, as specified, surprisingly, by the mysterious parameter β [26, 39]. As shown in these references, assuming $\beta = 0$ completely decouples phase from amplitude and so eliminates all QM effects. Clearly, therefore, the role of β must be accounted for in any averaging process.

The fact that the kinematic state of a particle with zitter is described by a rotor while the Dirac wavefunction is also a spinor suggests that the QM *superposition principle* can be construed as an average over particle rotor states. Indeed, it is easy to prove that the sum of two rotors R_1 and R_2 necessarily has the form (268) for a spinor:

$$R_1 + R_2 = (\rho^{i\beta})^{1/2} R. \quad (293)$$

The undeniable success of the Schroedinger equation suggests that, approximately at least, the superposition is an average over phase factors, such as Feynman’s sum over paths. This certainly produces the coupling of phase to amplitude so characteristic of QM. Note that an average over rotors blurs any average over paths, because path velocity is a bilinear function of the path rotor. Finally, it should be mentioned that gauge invariance provides a strong constraint on assembly of phases for distinct particle paths into a coherent ensemble.

Clearly, deriving the Dirac equation from zitter substructure is a nontrivial problem. Happily, we don’t have to wait for a solution to make progress in studying the zitterbewegung. Here is a promising alternative approach:

C. Putting zitterbewegung into the Dirac equation

We have seen that physical interpretation of the Dirac equation is crucially dependent on identification of a particle velocity observable, which requires theoretical assumptions beyond the Dirac equation itself. Historically, the mass term in the Dirac equation led to the conservation law for the Dirac current and its interpretation as a probability current, with an implicit identification of particle velocity. In the STA version of the Dirac equation (267), explicit appearance of the vector γ_0 shows that the velocity vector $v = R\gamma_0\tilde{R}$ is inherent in the structure of the equation. However, we have identified the vector $u = R\gamma_+\tilde{R}$ as a better candidate for electron velocity. This suggests a slight modification of the Dirac equation to replace v by u . Accordingly, we change the mass term

$m_e\psi\gamma_0$ to

$$m_e\psi\frac{1}{2}(\gamma_0 + \gamma_2) = m_e\psi_+\gamma_0, \quad (294)$$

where

$$\psi_+ \equiv \psi\frac{1}{2}(1 + \sigma_2). \quad (295)$$

In common parlance, this is a projection of a 4-component spinor ψ into a 2-component spinor ψ_+ . As only these components are now relevant to the electron velocity, we should perform the same projection on the other components of the Dirac equation. That requires modification of the vector potential term to achieve a 2-component equation. One of several possibilities gives us the modified Dirac equation

$$\nabla\psi_+i\sigma_3\hbar - qA\psi_+\sigma_3 = m_e\psi_+\gamma_0. \quad (296)$$

Let's refer to this as the *zitter Dirac equation*. Note that it has the essential property of invariance under projection from the right by $\frac{1}{2}(1 - \sigma_2)$.

To be assured that this modification of the Dirac equation has not damaged its essential physical meaning, we note that the zitter free particle solution for a congruence of histories with $\omega_e\tau/2 = p \cdot x/\hbar$ is also a solution of the Dirac equation, which, when substituted into (296) with arbitrary initial conditions, yields the algebraic relation

$$pu = m_e(1 - e_2e_0). \quad (297)$$

This is identical to the relation between momentum and velocity found in (127) and (134) for the zitter model. In contrast to the prosaic relation $p = m_e e_0$ from the free particle solution to the ordinary Dirac equation, this relation includes zitter in the vector e_2 , as it rotates with the zitter phase. Thus, the form of the zitter Dirac equation brings the null velocity observable to the fore.

It is also readily shown that the zitter Dirac equation has the same electromagnetic gauge invariance as the ordinary Dirac equation. Moreover, the gauge group can be generalized to incorporate electroweak interactions, as explained in [35]

Now check the observables. With respect to the projected wave function, the observables (272), (273) become

$$\psi_+\gamma_+\tilde{\psi}_+ = 2\psi_+\gamma_0\tilde{\psi}_+ = \psi\gamma_+\tilde{\psi} = \rho u, \quad (298)$$

and

$$\begin{aligned} \psi_+i\sigma_3\hbar\tilde{\psi}_+ &= \psi\frac{1}{2}(1 + \sigma_2)i\sigma_3\hbar\tilde{\psi} \\ &= \frac{\hbar}{2}\psi\gamma_+\gamma_1\tilde{\psi} = \rho\frac{\hbar}{2}ue^{-i\beta}e_1 = \rho S. \end{aligned} \quad (299)$$

Thus we get the same zitter velocity u , but the spin bivector \bar{S} is replaced by a null spin bivector S , which is identical in form to the spin (78) in the point particle model, and, as in (121), the duality factor can be absorbed into a rotation of the vector e_1 . We see immediately that the interaction energy density (292) is replaced by

$$\begin{aligned} \frac{g}{2} \langle F\psi_+i\sigma_3\hbar\tilde{\psi}_+ \rangle &= g \langle F\rho S \rangle \\ &= \rho(\mathbf{E}_v \cdot \mathbf{d} - \mathbf{B}_v \cdot \boldsymbol{\mu}), \end{aligned} \quad (300)$$

in perfect accord with equation (178) of the zitter model. Thus, the change in observables by projection on the wave function appears to eliminate the parameter β and its problems of physical interpretation.

Further insight comes from the following Lagrangian for the zitter Dirac equation:

$$\begin{aligned} L_{zD} = 2 \left\langle [-\hbar\nabla\psi_+i\gamma_0 \right. \\ \left. + qA\psi_+\gamma_0 + m_e\psi_+\sigma_3]\tilde{\psi}_+ \right\rangle. \end{aligned} \quad (301)$$

Note that the interaction term has the usual form $A \cdot J = \langle AJ \rangle$, but now the charge current is a null vector field

$$J = 2q\psi_+\gamma_0\tilde{\psi}_+ = q\psi\gamma_+\tilde{\psi} = q\rho u, \quad (302)$$

as expected. It follows from the zitter Dirac equation that this current is conserved, though it is not conserved in standard Dirac theory.

The usual Dirac current is not obtainable as a bilinear observable of the wave function ψ_+ . However, as noted before, it can be obtained as a zitter average $v = \bar{u}$ of the zitter velocity. This suggests that one should try to derive the zitter Dirac equation, rather than the Dirac equation itself, from the zitter model. But that possibility will not be explored here.

The zitter Dirac equation offers a new perspective on the significance of negative energy in Dirac theory. Recall that negative energy solutions were first regarded as a serious defect of the Dirac equation. Schroedinger showed that they are essential for a general solution even in the free particle case when he “discovered zitterbewegung” as interference between positive and negative components of a wave packet. In an audacious effort to save the theory, Dirac identified the negative energy states with an “anti-electron” and invoked the Pauli principle to suppress them (hole theory). Miraculously, the positron was discovered shortly thereafter, so the defect was transmuted to a spectacular triumph!

A standard conclusion from all this is expressed by the following quotations [38]: “The zitterbewegung demonstrates in a real sense a single particle theory is not possible.” “The difficulties with the negative energy states of the Dirac equation almost of necessity demand a many-body theory.” “*Hole theory is a many-body theory* describing particles with positive and negative charge. The simple probability interpretations of the wave functions acclaimed in a single-particle theory cannot be true any longer, because the creation and annihilation of electron-positron pairs must be taken in account in the wave function.” In other words, quantum field theory is needed to explain zitterbewegung!

For a different perspective on negative energy we note that $\psi^C \equiv \psi\sigma_2$ is the charge conjugate solution of the real Dirac equation [1]. Hence we can cast the zitter wave function (295) in the form

$$\psi_+ \equiv \frac{1}{2}(\psi + \psi^C). \quad (303)$$

This expresses Dirac's negative energy solution as an essential component of the zitter rather than an antiparticle. It is an alternative splitting of Dirac's 4-component wave function into a pair of 2-component wave functions for different particle states. The physical issue is this: Which components of the Dirac wave function should be identified with the electron? The zitter component ψ_+ describes an electron with zitter motion. Standard quantum field theory splits the zitter into positive and negative energy components and then reassembles it later from pair creation and annihilation. In most calculations the end result will be the same, because both approaches start from the same Dirac equation. Contrary to the standard Dirac equation (267), the zitter Dirac equation (296) is consistent with a single particle model of zitter without the strenuous expedient of field quantization. This is not to deny that some version of quantum field theory is necessary to account for creation and annihilation of particles. The problem is to devise experiments that identify the basic particle states. The channeling resonance already discussed may be one such experiment.

The zitter wave function ψ_+ has only four degrees of freedom, whereas it is commonly believed that all eight degrees of freedom in the Dirac wave function are needed to describe an electron. However, note that three degrees of freedom suffice to specify the null velocity $u = \dot{z}$, and additional degrees are not needed to describe spin and phase, because they are intrinsic properties of the null history. This leaves one scalar variable to serve as a probability density or time scale factor.

Standard theory suggests that the remaining four components should represent positron states through second quantization. However, zitter theory opens the possibility of identifying them with the electron neutrino and thereby relating zitter to electroweak theory. That option has been discussed elsewhere [35].

XIII. CONCLUSIONS

The zitterbewegung, if it turns out to be physically real, is belated confirmation of de Broglie's original hypothesis [40] that the electron has an internal clock with period precisely equal to twice the zitter period, precisely the relation between the period of a rotor and that of a vector it rotates.

As we have seen, the physical signature of zitter is a rotating electric dipole with ultra high frequency. If this exists, its implications for quantum mechanics will be far-reaching. Evidently it can be incorporated in Dirac theory by subtle changes in the specification of observables and the structure of the Dirac equation.

Experimental confirmation of the zitter should stimulate research on its proposed incorporation into electroweak theory. Then study of zitter self-interaction should look for excitations explaining the three lepton families. Finally, the strong analogy between electroweak interactions of leptons and quarks suggests that one should investigate modifications of zitter structure to model quarks and strong interactions. All this should go hand-in-hand with development of zitter field theory and reconciliation of it with quantum field theory.

Acknowledgments

I am indebted to Richard Clawson for insightful observations about analysis of the zitter model and to Michel Gouanère for explaining details of his channeling experiment and data analysis.

Note. Many of the papers listed in the references are available online at <<http://modelingnts.la.asu.edu>> or <<http://www.mrao.cam.ac.uk/~clifford/>>.

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