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# Space-Time Structure of Weak and Electromagnetic Interactions

David Hestenes

*The generator of electromagnetic gauge transformations in the Dirac equation has a unique geometric interpretation and a unique extension to the generators of the gauge group SU(2)×U(1) for the Weinberg-Salam theory of weak and electromagnetic interactions. It follows that internal symmetries of the weak interactions can be interpreted as space-time symmetries of spinor fields in the Dirac algebra. The possibilities for interpreting strong interaction symmetries in a similar way are highly restricted.*

## 1. INTRODUCTION

It has often been suggested that observed symmetries among elementary particles may well reflect symmetries of some fundamental dynamical equation or system of equations. The suggestion has gained credence with the recent successes of gauge theories, the Weinberg-Salam (W-S) model in particular. However, in current theories no definite connection has yet been established between the so-called *internal symmetries* of particle interactions and *space-time symmetries* of Lorentz invariant fields. This paper aims to show that the basis for a connection between internal symmetries and spacetime symmetries exists already in the Dirac equation for an electron.

The key step in the argument is the recognition that the generator of electromagnetic gauge transformations in the Dirac equation has a definite geometric interpretation. We show, in fact, that it is the generator of local spatial rotations which leave the Dirac current and spin invariant. All this is hidden in the conventional representation of the Dirac equation, but we employ a representation that makes it explicit.

Having established that the generator of electromagnetic gauge transformations has a geometric interpretation, we certainly expect the same for generators of weak gauge transformations if the unification of weak and electromagnetic interactions in a gauge theory is on the right track. Sure enough, we find that the group SU(2) × U(1) of the W-S model exists already as a symmetry group of spinor fields in space-time.

We find that the W-S model can be accommodated by generalization of the conventional spinors with a definite geometrical and physical meaning. The electron and its neutrino can then be interpreted as orthogonal eigenstates of a single lepton in the same way that orthogonal spin states are interpreted as distinct states of a single particle. This unification of electron and neutrino by representing them as components of a single spinor field is geometrically analogous to the unification of electric and magnetic fields by representing them as components of a single-rank two-tensor.

By adhering to the geometrical interpretation of the generators, we are led to a representation of the W-S model with new features. This provides some theoretical justification for the W-S model by establishing an intrinsic connection to the Dirac theory. This includes a justification for the gauge group in the W-S model by establishing a geometric interpretation of the group and definite reasons for preferring it to others. On the other hand, the experimental success of the W-S model provides some support for the argument presented here, for it will be seen that the SU(2) × U(1) group of the W-S model is a unique consequence of the argument, and competing groups which have been proposed cannot be accommodated within this theoretical framework.

The new features attributed here to the W-S model do not alter any experimental predictions of the model. However, they do suggest new ways to generalize the model while severely limiting the possibilities that need to be considered.

## 2. SPACE-TIME ALGEBRA

To help us identify and represent geometrical and physical properties of spinor fields, we shall employ a somewhat unconventional formalism. The basic definitions and results we need will merely be stated here, because the formalism and its rationale have been expounded at length in Refs. 1–4. Its relation to conventional Dirac theory is summarized in Ref. 4. An extensive mathematical development of the formalism is carried out in Ref. 5.

Let $\{\gamma_\mu,\ \mu = 0, 1, 2, 3\}$ be an orthonormal set of vectors in space-time. We shall represent all physical quantities as elements of the real Clifford algebra generated by the $\gamma_\mu$. Let us refer to this algebra as the *Space-Time Algebra* (STA), because all its elements have a geometric interpretation.

Representations of the $\gamma_\mu$ by $4\times 4$ matrices are called *Dirac matrices*. The *Dirac Algebra* is the matrix algebra over the field of complex numbers generated by the *Dirac* matrices. There are two good reasons for not formulating physics in terms of the Dirac algebra: First, matrix representations of the $\gamma_\mu$, are completely irrelevant to their interpretation as vectors. Second, imaginaries in the complex number field have no designated geometrical interpretation. Because it contains imaginary scalars the Dirac algebra has twice as many degrees of freedom as STA, twice as many, it turns out, as are needed for the purposes of physics. To see that this is so, we need a few definitions and relations.

The signature of space-time is expressed by the equations

$$\gamma_0^2 = 1, \qquad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1$$

We use the usual convention for raising and lowering indices, so

$$\gamma_\mu = \mathrm{g}_{\mu\nu}\gamma^\nu$$

where

$$\mathrm{g}_{\mu\nu} \equiv \gamma_\mu \cdot \gamma_\nu = \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)$$

An arbitrary element of STA will be referred to as a *multivector*. The special multivector

$$i = \gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_5$$

is called the *unit pseudoscalar*. We prefer the symbol $i$ to $\gamma_5$ for reasons which will become apparent.

A multivector is said to be even (odd) if it commutes (anticommutes) with $i$. Any multivector $\Psi$ can be decomposed into even and odd parts; thus,

$$\Psi = \Psi_+ + \Psi_- \tag{1a}$$

$$\Psi_+ = \tfrac{1}{2}(\Psi - i\Psi i) \tag{1b}$$

$$\Psi_- = \tfrac{1}{2}(\Psi + i\Psi i) \tag{1c}$$

Any odd multivector can be expressed as the product of an even multivector with the vector $\gamma_0$.

The even multivectors in STA form an algebra called the *even subalgebra*. This algebra is itself a Clifford algebra generated by the three multivectors

$$\boldsymbol{\sigma}_k \equiv \gamma_k\gamma_0 \quad \text{for} \quad k = 1, 2, 3$$

Note that

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i = \gamma_0\gamma_1\gamma_2\gamma_3$$

It will be convenient to adopt the notation

$$\mathbf{a} = a_k\boldsymbol{\sigma}_k$$

for a linear combination of the $\boldsymbol{\sigma}_k$ with scalar coefficients $a_k$. Then any even multivector $M$ can be put in the form

$$M = \alpha + i\beta + \mathbf{a} + i\mathbf{b} \tag{2}$$

where $\alpha$ and $\beta$ are scalars. Since $i^2 = -1$ and $i\mathbf{b} = \mathbf{b}i$ this has the mathematical form of a complex scalar $\alpha + i\beta$ plus a complex vector $\mathbf{a} + i\mathbf{b}$.

Since the STA is generated by the $\gamma_\mu$, any multivector can be expressed as a polynomial in the vectors $\gamma_\mu$. The operation of reversing the order of vector in such a polynomial is called *reversion*, and we write $\widetilde{\Psi}$ to denote the result of this operation on a multivector $\Psi$. The operation

$$\Psi^\dagger = \gamma_0\widetilde{\Psi}\gamma_0 \tag{3}$$

corresponds exactly to *Hermitian conjugation* in the Dirac Algebra. Applied to Eq. (2) it gives

$$M^\dagger = \alpha - i\beta + \mathbf{a} - i\mathbf{b} \tag{4}$$

so it can be regarded as a kind of "complex conjugation."

## 3. THE DIRAC THEORY

The Dirac equation is one of the foundation stones of quantum electrodynamics. By studying its structure, we can identify a geometrical basis for the unification of weak and electromagnetic interactions.

In terms of STA the Dirac wave function can be represented as an even multivector $\psi$ for which the Dirac equation has the form (in natural units)

$$\gamma^\mu(\partial_\mu\psi - eA_\mu\psi i\boldsymbol{\sigma}_3) = -m\psi\gamma_0 i\boldsymbol{\sigma}_3 \tag{5}$$

where $e = |e|$ is the electron charge and the $A_\mu$ are components of the electromagnetic vector potential. We shall see how this relates to the conventional form of the Dirac equation below. It is most important to understand that on the right of (5) the vector $\gamma_0$ is an arbitrarily chosen timelike vector while $\gamma_3$ in $\boldsymbol{\sigma}_3 = \gamma_3\gamma_0$ can be any spacelike vector orthogonal to it. Furthermore, $\gamma_0$ and $\gamma_3$ need not be related to the coordinate vectors $\gamma^\mu$ in $\gamma^\mu\partial_\mu$ on the left side of (5), though we choose to relate them for mathematical simplicity.

Two fundamental "observables" in the Dirac theory are the *Dirac current*

$$j \equiv \psi\,\gamma_0\widetilde{\psi} \tag{6}$$

and the *spin current*

$$s \equiv \psi\,\gamma_3\widetilde{\psi} \tag{7}$$

Components of these currents are related to conventional expressions in the Dirac theory by

$$j_\mu = \gamma_\mu \cdot j = (\gamma_\mu\psi\,\gamma_0\widetilde{\psi})_S = (\psi^\dagger\,\gamma_0\gamma_\mu\psi)_S$$

and

$$s_\mu = \gamma_\mu \cdot (\psi\gamma_3\widetilde{\psi}) = (\boldsymbol{\sigma}_3\widetilde{\psi}\,\gamma_0\gamma_\mu\psi)_S = -(i\boldsymbol{\sigma}_3\psi^\dagger\,\gamma_0\gamma_\mu\gamma_5\psi)_S = -i'\Psi'^\dagger\,\gamma_0\gamma_\mu\gamma_5\Psi'$$

where $\Psi'$ is the column matrix representing a spinor, and $i'$ is the unit imaginary in the conventional Dirac algebra. The subscript $S$ in STA denotes scalar part, which corresponds to the trace in the matrix theory.

## 4. THE GAUGE GROUP OF THE DIRAC CURRENT

Dirac's equation (5) is invariant under the electromagnetic gauge transformation

$$\psi \to \psi\, e^{\frac{1}{2}i\boldsymbol{\sigma}_3\chi} \tag{8}$$

accompanied by $ea_\mu \to eA_\mu + \frac{1}{2}\partial_\mu\chi$. The corresponding gauge transformation on the conventional matrix representation of a Dirac spinor is

$$\Psi' \to e^{\frac{1}{2}i'\chi}\,\Psi' \tag{9}$$

The striking difference between (8) and (9) is that the generator $i'$ has no evident geometrical interpretation, while $i\boldsymbol{\sigma}_3 = i\gamma_3\gamma_0 = \gamma_2\gamma_1$ is the generator of rotations in a spacelike plane related to physical currents by (6) and (7). Furthermore, $i'$ has no evident relation to the generators of any large group while $i\boldsymbol{\sigma}_3$ belongs to the set of generators $i\boldsymbol{\sigma}_1$, $i\boldsymbol{\sigma}_2$, $i\boldsymbol{\sigma}_3$ of SU(2). However, we shall see that $i'$ really does correspond to $i\boldsymbol{\sigma}_3$, so its relation to the larger group is merely hidden rather than absent.

Now the basic idea of a unified gauge theory of weak and electromagnetic interactions is that the interactions are generated by a single gauge group. Since the generator $i\boldsymbol{\sigma}_3$ of electromagnetic gauge transformations belongs to STA, we should expect the same of the larger group. Thus we are led to consider

$$\psi \to \psi\, e^{\frac{1}{2}i\boldsymbol{\sigma}_1\chi_1} \qquad \text{and} \qquad \psi\, e^{\frac{1}{2}i\boldsymbol{\sigma}_2\chi_2} \tag{10}$$

as candidates for gauge transformations of weak interactions.

The transformations (8) and (10) leave the Dirac current invariant. The most general gauge transformation

$$\psi \to \psi U \tag{11a}$$

leaving the Dirac current invariant must satisfy $U^\dagger U = 1$, from which it follows that

$$U = e^{\frac{1}{2}i(\boldsymbol{\chi}+\beta)} \tag{11b}$$

Thus the invariance group of the Dirac current is the group SU(2) × U(1) of the Weinberg-Salam theory, but we have yet to relate it to weak interactions.

Note that the electromagnetic gauge transformation (8) leaves the spin current (7) invariant while the transformations (10) do not. Thus we can associate the spontaneous symmetry breaking of SU(2) by the electromagnetic interaction with the physical existence of a definite local spin direction.

## 5. IDEALS AND SPINORS

We can make (5) look more like the conventional form of the Dirac wave equation by multiplying it on the right by $(1+\gamma_0)(1-\boldsymbol{\sigma}_3)$ so it becomes

$$\gamma^\mu(\partial_\mu\Psi + e\Psi i A_\mu) = \Psi i m \tag{12}$$

where

$$\Psi = \psi\tfrac{1}{2}(1+\gamma_0)(1-\boldsymbol{\sigma}_3) \tag{13}$$

Note that

$$\Psi\widetilde{\Psi} = 0 = \Psi i\widetilde{\Psi}$$

But

$$\Psi\gamma_0\widetilde{\Psi} = \psi(1+\gamma_0)\widetilde{\psi} = \Psi\Psi^\dagger\gamma_0 \tag{14}$$

and

$$\Psi\gamma_0 i\widetilde{\Psi} = i\psi(\gamma_3+\gamma_3\gamma_0)\widetilde{\psi} = -\Psi i\Psi^\dagger\gamma_0 \tag{15}$$

Whence

$$j_\mu = (\gamma_\mu\psi\,\gamma_0\widetilde{\psi})_S = (\Psi^\dagger\,\gamma_0\gamma_\mu\Psi)_S \tag{16}$$

and

$$s_\mu = (\gamma_\mu\psi\,\gamma_3\widetilde{\psi})_S = (\Psi^\dagger\gamma_0\gamma_\mu\gamma_5\Psi i)_S \tag{17}$$

To enhance similarity with conventional expressions, we write $\gamma_5$ instead of $i$ when multiplying a spinor $\Psi$ the left as in (17).

The operation of antiparticle conjugation can be defined by

$$\Psi \to \Psi_C \equiv i\Psi\gamma_0 i = -i\Psi i\gamma_0 \tag{18}$$

For applying it to the Dirac equation (12), we find that the equation for $\Psi_C$ differs from that of $\Psi$ only by the sign of the charge. The same result would be obtained by the more general definition $\Psi_C = i\Psi\gamma_0 ie^{i\alpha}$, but the choice $\alpha = \pi/2$ made in (18) has an especially simple algebraic interpretation. Considering (1a), (1b), (1c), we see that antiparticle conjugation defined by (18) *interchanges even and odd parts of the wave function and changes their relative sign*. This feature becomes important when we see that even and odd parts have a distinct physical significance in the Weinberg-Salam model. We know already that the distinction between even and odd multivectors has definite geometrical significance.

Now the easiest way to make contact with the conventional formulation of the Dirac theory is to suppose that $\Psi'$ is the matrix representation of $\Psi$ in the Dirac algebra and that $\Psi'$ is an eigenfunction of the matrix $i = \gamma_5$ on the right so that

$$\Psi'\gamma_5 = \Psi' i' = i'\Psi' \tag{19}$$

where the unit imaginary $i'$ is the eigenvalue of $\gamma_5$. Then taking the matrix representation of (12) and using (19) we obtain

$$\gamma^\mu(\partial_\mu + ei' A_\mu)\Psi' = mi'\Psi' \tag{20}$$

Consistent with (19), we may assume that all nonzero elements of the $4 \times 4$ matrix $\Psi'$ are concentrated in a single row, because the operators in (20) do not mix rows. Therefore (20) is indeed equivalent to the conventional Dirac equation, the only difference being that the wave function is represented as an element of a minimal left ideal of the Dirac algebra instead of a column matrix. This difference, however, suggests a generalization.

## 6. LEPTON ISOSPACE

A *left ideal* is a subset of an algebra which is invariant under left multiplication by all elements of the algebra. It is *minimal* if it contains no smaller left ideals. The four columns of a matrix $\Psi'$ in the Dirac algebra are four linearly independent minimal ideals spanning the algebra. Now, the electron and its neutrino belong to a family of leptons obeying Dirac-like equations. Formally, we could represent the wave functions of four distinct leptons as columns of a Dirac matrix $\Psi'$ so that the columns constitute states in a "lepton isospace." Then matrices multiplying $\Psi'$ on the left, as in (20), couple spin and energy components of individual lepton wave functions while matrices multiplying $\Psi'$ on the right must be regarded as "isospace operators" coupling distinct leptons. So far all this is completely formal, and representing the lepton family as a $4 \times 4$ square matrix has no more physical significance than representing it as a 16-component column matrix. However, the situation is different if we use STA instead of the Dirac algebra.

To begin with, STA has only two instead of four linearly independent minimal left ideals, because it has only half as many elements as the Dirac algebra. Therefore, our proposed lepton family can contain no other particles besides the electron and its neutrino. The electron wave function $\Psi$ in (13) belongs already to a minimal left ideal; let us now denote it by $\Psi_e$ to distinguish it from the neutrino wave function $\Psi_\nu$. Denoting the composite lepton wave function by $\Psi$, we have

$$\Psi = \Psi_\nu + \Psi_e \tag{21}$$

According to (13) we must have

$$\Psi_e \boldsymbol{\sigma}_3 = -\Psi_e \tag{22a}$$

The "orthogonality" of $\Psi_\nu$ to $\Psi_e$ is assured by

$$\Psi_\nu \boldsymbol{\sigma}_3 = \Psi_\nu \tag{22b}$$

Then

$$\Psi_\nu = \Psi \tfrac{1}{2}(1 + \boldsymbol{\sigma}_3) \tag{23a}$$

and

$$\Psi_e = \Psi \tfrac{1}{2}(1 - \boldsymbol{\sigma}_3) \tag{23b}$$

Thus, the electron and its neutrino are eigenstates of $\boldsymbol{\sigma}_3$ in lepton isospace.

Now

$$\boldsymbol{\sigma}_\pm \equiv \tfrac{1}{2}\boldsymbol{\sigma}_1(1 \pm \boldsymbol{\sigma}_3) = \tfrac{1}{2}(\boldsymbol{\sigma}_1 \mp i\boldsymbol{\sigma}_2) \tag{24}$$

acting on the right side of the lepton wave function (21) are raising and lowering operators in lepton isospace. The most general operator acting in this isospace is an even multivector $M$, and it decomposes into the irreducible parts shown in (2), namely, an *isoscalar* $\alpha$, an *isovector* $\mathbf{a}$, an *isobivector* (= isopseudovector) $i\mathbf{b}$, and an isopseudoscalar $i\beta$.

In contrast to a representation by Dirac matrices, our STA representation of isospace has a definite physical significance. We have already seen that $\boldsymbol{\sigma}_3$ has a special relation to the electron spin, so this explains the significance of identifying the electron and the neutrino as eigenstates of $\boldsymbol{\sigma}_3$ rather than, say, $\boldsymbol{\sigma}_1$ or $\boldsymbol{\sigma}_2$. Note that the $\gamma_\mu$ in the isovectors $\boldsymbol{\sigma}_k = \gamma_k\gamma_0$ *must* be interpreted as space-time vectors rather than operators in some abstract "particle space," although the indices of $\gamma_0$ and the $\boldsymbol{\sigma}_k$ are not to be associated with space-time coordinates here; rather, they refer to structural features of the lepton wave function and its differential equation.

We can see now that the appearance of $i$ on the right of $\Psi$ in the Dirac equation (12) means that it is an operator in isospace. The $i$ is formally an imaginary number, since it commutes with the $\boldsymbol{\sigma}_k$ and $i^2 = -1$. However, geometrically, it is the unit pseudoscalar for isospace as well as for spacetime. This imbues complex conjugation with a geometrical significance and relates inversion in isospace to inversion in space-time.

## 7. THE WEINBERG-SALAM MODEL

We are now in a position to show that STA provides an ideal basis for the Weinberg-Salam model of weak and electromagnetic interactions. We can do this by translating into STA some key features of the W-S model from the review article by Abers and Lee.[(6)]

The matrix representation $\Psi'$ of the lepton wave function can be separated into left- and right-handed parts in the way; thus

$$\Psi' = L' + R' \tag{25a}$$

where

$$L' = \tfrac{1}{2}(1 - i'\gamma_5)\Psi' \tag{25b}$$

$$R' = \tfrac{1}{2}(1 + i'\gamma_5)\Psi' \tag{25c}$$

According to (19),

$$i'\gamma_5\Psi' = \gamma_5\Psi' i' = i\Psi' i$$

Comparison with (1a), (1b), (1c) shows, therefore, that left- and right-handed wave functions in the Dirac algebra correspond respectively to even and odd multivectors in STA. Now we have all the correlates with matrix algebra, and we can work exclusively with STA.

Separation of the lepton wave function (21) into left/right (even/odd) parts gives

$$\Psi = L + R = L_\nu + L_e + R_\nu + R_e \tag{26}$$

In the W-S model

$$R_\nu = R\tfrac{1}{2}(1 + \boldsymbol{\sigma}_3) = \tfrac{1}{4}(\Psi + i\Psi i)(1 + \boldsymbol{\sigma}_3) = 0 \tag{27}$$

and $L$ and $R$ are assumed to behave differently under gauge transformations. The gauge group of the left-handed part is

$$L \to L e^{\frac{1}{2}i(\boldsymbol{\chi} + \beta)} \tag{28}$$

This is the *invariance group* SU(2) × U(1) of the *lepton current* $L\gamma_0\widetilde{L}$. Since $L$ is even, the gauge transformation of $L$ and its invariant $L\gamma_0\widetilde{L}$ are mathematically equivalent to the gauge transformation (11a), (11b) of the electron wave function $\psi$ and its invariant, the Dirac current $\psi\gamma_0\widetilde{\psi}$. Physically, of course, $L$ and $\psi$ describe different things, although they are related.

The lepton wave function $L$ determines four currents in lepton isospace,

$$\gamma_0\widetilde{L}\gamma_\mu L = L^\dagger\gamma_0\gamma_\mu L = l^0_\mu + \mathbf{l}_\mu \tag{29a}$$

where

$$l^0_\mu = \tfrac{1}{2}(\gamma_0\widetilde{L}\gamma_\mu L + \widetilde{L}\gamma_\mu L\gamma_0) = (\gamma_0\widetilde{L}\gamma_\mu L)_S = \gamma_\mu \cdot (L\gamma_0\widetilde{L}) \tag{29b}$$

is an *isoscalar current* and

$$\mathbf{l}_\mu = \tfrac{1}{2}(\gamma_0\widetilde{L}\gamma_\mu L - \widetilde{L}\gamma_\mu L\gamma_0) \tag{29c}$$

is an *isovector current*.

The transformation of the right-handed wave function $R$ corresponding to (28) is

$$R \to R e^{i\beta} \tag{30}$$

It is the invariance group of $R\gamma_0\widetilde{R} = R_e\gamma_0\widetilde{R}_e$ corresponding to the isoscalar current $r^0_\mu = \gamma_\mu \cdot (R\gamma_0\widetilde{R}) = (\gamma_0\widetilde{R}\gamma_\mu R)_S$.

The *gauge invariant derivatives* of $L$ and $R$ are

$$D_\mu L = \partial_\mu L + L\,\frac{i}{2}\,(B_\mu\,\mathrm{g}' - \mathbf{A}_\mu\,\mathrm{g}) \tag{31a}$$

$$D_\mu R = \partial_\mu R + R i B_\mu\,\mathrm{g}' \tag{31b}$$

where $g'$ and $g$ are coupling constants, $B_\mu$ is an isoscalar (space-time) vector field, and $\mathbf{A}_\mu$ is an isovector. This corresponds to the gauge invariant lepton lagrangian

$$\begin{aligned}\mathcal{L}_{\text{leptons}} &= (i\gamma_0 \widetilde{R}\gamma^\mu(\partial_\mu R + RB_\mu i\mathrm{g}'))_S \\ &\quad + \left(i\gamma_0 \widetilde{L}\gamma^\mu\left(\partial_\mu L + LB_\mu \frac{i}{2}\mathrm{g}' - L\mathbf{A}_\mu \frac{i}{2}\mathrm{g}\right)\right)_S \\ &= (i\Psi^\dagger\gamma_0\gamma^\mu\partial_\mu\Psi)_S + \tfrac{1}{2}\mathrm{g}\,\mathbf{l}_\mu \cdot \mathbf{A}_\mu - \mathrm{g}'(\tfrac{1}{2}l^0_\mu + r^0_\mu)B^\mu \end{aligned} \tag{32}$$

The invariant lagrangian for the gauge fields has the usual form, namely

$$\mathcal{L}_{\text{gauge}} = -\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} - \tfrac{1}{4}\mathbf{F}_{\mu\nu} \cdot \mathbf{F}^{\mu\nu} \tag{33}$$

where

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{34}$$

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + \mathrm{g}\,\mathbf{A}_\mu \times \mathbf{A}_\nu \tag{35}$$

and

$$\mathbf{A}_\mu \times \mathbf{A}_\nu = -\frac{i}{2}\left(\mathbf{A}_\mu\mathbf{A}_\nu - \mathbf{A}_\nu\mathbf{A}_\mu\right)$$

It is now a trivial matter to complete the transcription of the W-S model into STA, so the reader may be referred to Abers and Lee.[6]

## 8. DISCUSSION

We have seen that the algebraic ingredients of the W-S model are already present in the Dirac theory. In particular, the generators of the gauge group SU(2) × U(l) can be identified as elements of the Dirac algebra, which relates them to the space-time structure of a spinor field. Thus, we have formulated the W-S model as a generalization of the Dirac equation entirely within the context of the Dirac algebra, or rather, within its geometric representation STA. To accomplish this, we have been led to identify the electron-neutrino doublet as a pair of orthogonal left ideal components of a single lepton wave function. This has a number of implications for model building and interpretation in elementary particle theory.

To begin with, in the geometric representation of the W-S model, the doublet wave function (21) is necessarily an irreducible unit, because there are no more than two linearly independent ideals in the algebra. Therefore, the doublet cannot be generalized to, say, a triplet, and the gauge group SU(2) × U(1) cannot be generalized to SU(3) without giving up the geometric representation along with its unique relation to space-time. Thus, the geometric representation imposes severe constraints on possible generalizations of the W-S model.

The composition of electron and neutrino wave functions into a single lepton wave function $\Psi = \Psi_\nu + \Psi_e$ is geometrically akin to the composition of electric and magnetic fields into a single skew tensor of rank two. This becomes rather obvious on noting that the multivector $\Psi$ has its values in the space of skew tensors on space-time. The even (left-handed) part $L$ of $\Psi$ is composed of tensors with even rank; thus, we can write the Lorentz invariant expansion

$$L = \alpha + \tfrac{1}{2} L^{\mu\nu}\gamma_\nu\gamma_\mu + \frac{1}{4!} L^{\mu\nu\alpha\beta}\gamma_\beta\gamma_\alpha\gamma_\nu\gamma_\mu \tag{36}$$

where the $\gamma_\mu$ can be regarded as the vector basis for a coordinate system in space-time, so that the scalar $\alpha$ is a tensor of rank zero, the $L^{\mu\nu}$ are scalar components of a rank-two skew tensor (bivector), and the $L^{\mu\nu\alpha\beta}$ are components of a rank-four skew tensor (pseudoscalar). For orthogonal coordinates, the last term in (36) reduces to

$$\frac{1}{4!} L^{\mu\nu\alpha\beta}\gamma_\beta\gamma_\alpha\gamma_\nu\gamma_\mu = L^{3210}\gamma_0\gamma_1\gamma_2\gamma_3 \equiv \beta i$$

In a similar way, the odd (right-handed) part $R$ of $\Psi$ is the composite of tensors with the odd ranks one and three.

It may be thought that the decomposition (36) of a spinor doublet into tensors blurs the distinction between tensors and spinors and is without physical meaning. The tensor components such as $L^{\mu\nu}$ are not directly observable, because observables are bilinear functions of $L$ and $R$. Also, the gauge transformation (28) mixes tensor components of different rank. However, the distinguished roles of $L$ and $R$ in the W-S model show that there is physical significance to the decomposition of a spinor into tensors of even and odd ranks, so a further decomposition into tensor components might also have some significance. The fact that antiparticle conjugation (18) interchanges even and odd parts provides additional physical meaning to the even-odd decomposition.

Note that the decomposition of (36) into electron and neutrino spinors by applying the projection operators $\frac{1}{2}(1 \pm \boldsymbol{\sigma}_3)$ as in (23a), (23b) is a decomposition of the 8-dimensional space of even-rank skew tensors into two 4-dimensional subspaces by singling out a preferred timelike plane specified by $\boldsymbol{\sigma}_3 = \gamma_3\gamma_0$. Alternatively, the 6-dimensional space of bivectors can be decomposed into 3-dimensional subspaces by singling out a preferred timelike vector $\gamma_0$ as in (2). This, of course, is how the electromagnetic field tensor is decomposed into electric and magnetic parts.

We have shown that any Dirac spinor can be represented as a multivector in the space of skew tensors on space-time, so it is admissible to refer to the multivector-valued lepton wave function $\Psi$ as a spinor field on space-time. Lepton dynamics is expressed by the *structure* of the spinor field $\Psi$ which, in turn, is determined by the *structure* of a differential equation describing its interaction with other fields. In a pure gauge theory the *structure* of the interactions is determined by a gauge group. Since we have shown how to formulate the W-S model entirely in terms of the Space-Time Algebra, we can interpret the W-S model as a geometric theory in which leptonic interactions are described by the gauge structure of a spinor field in space-time. This suggests the hypothesis that *all interactions of elementary particles can be described by gauge structures of spinor fields in space-time*. The last words "in space-time" deserve emphasis, because most gauge theories describe gauge structures in some abstract "internal space" rather than in the space of skew tensors on space-time as here. The restriction of gauge theories to space-time structures severely limits the theoretical possibilities.

A few words should be said about possibilities for generalizing the W-S model to achieve a more comprehensive theory of interactions described by space-time structures. We have seen that the group SU(2) × U(1) of the W-S model is the invariance group of $L\gamma_0\widetilde{L}$, so this is the largest gauge group for an even spinor field. It is natural, therefore, to consider the invariance group of $\Psi\gamma_0\widetilde{\Psi}$|width0pt where $\Psi$ has both odd and even parts. This group has the structure of the orthogonal group O(4), though its generators are elements of STA; its gauge invariant derivative is[2]

$$D_\mu\Psi = \partial_\mu\Psi + \Psi[i(\mathbf{A}_\mu + B_\mu) + \gamma_0(\mathbf{C}_\mu + i\mathbf{D}_\mu)] \tag{37}$$

Here we have six new isovector vector bosons $\mathbf{C}_\mu - i\mathbf{D}_\mu$, in addition to the W-S bosons $\mathbf{A}_\mu + B_\mu$. Of course, equation (37) is inconsistent with the W-S model, because it couples $\mathbf{A}_\mu$ and $B_\mu$ in the same way to both even and odd parts of $\Psi$. But it is of interest to consider larger gauge groups from which the W-S model might arise by symmetry breaking. The explicit appearance of $\gamma_0$ in (37) implies that the bosons $\mathbf{C}_\mu + i\mathbf{D}_\mu$ couple even and odd parts of $\Psi$, so they would contribute directly to the lepton mass and might provide an alternative to the Higgs mechanism. On the other hand, the $\mathbf{C}_\mu + i\mathbf{D}_\mu$ might be considered for a role in strong interactions. The role of $\gamma_0$ in antiparticle conjugation (18) should also be taken into account in developing a theory along this line.

If we are right to interpret the W-S model as specifying a particular space-time structure for spinor fields, then we have established a geometrical basis for weak and electromagnetic interactions. Indeed, these interactions can then be represented in various ways as an affine connection for the space-time manifold.[3] One can hardly avoid asking whether it is also possible to represent strong interactions by a space-time structure of spinor fields, for then the classification of physical interactions would correspond to a classification of geometric structures on space-time.

The local space-time representation of isospace employed in this article was first introduced in Ref. 1 and further developed in Ref. 8. However, in those articles it was identified with the isospace of strong interactions. In this article we have adduced better reasons for identifying it with lepton isospace, but the possibility remains that the two kinds of isospace are geometrically related in some way. The concept of "lepton isospace" can be generalized immediately to "weak isospace" which applies to baryons as well as leptons, for, as explained by Abers and Lee,[(6)] the W-S model can be successfully applied to quarks if a "charmed quark" is introduced to form "weak quark doublets." From our geometric version of the W-S model it follows that the arrangement of quarks into weak doublets associates the quarks with a space-time structure like that of the electron-neutrino doublet. It also suggests relations between the gauge groups for weak and strong interactions which are still unexplored.

If strong interactions are to be given a geometric interpretation related to that for weak interactions, then there should be a natural representation of the group SU(3) in terms of STA. So there is! Any even multivector $M$ can be expressed as the sum

$$M = F + \phi \tag{38a}$$

of a bivector $F$ and a "complex" scalar-pseudoscalar

$$\phi = \alpha + i\beta \tag{38b}$$

[2] Proved in Section 24 and Appendix C of Ref. 1, where gravitational interactions are also derived from gauge invariance.

[3] An unconventional way to relate the structure of spinor fields to geometry of a space-time manifold is discussed in Ref. 7.

**TABLE 1.** SU(3) Generators $\lambda_k$ on the Space of Bivectors $\{F = \mathbf{a} + i\mathbf{b}\}$

| Linear opeators | Matrix representation |
|---|---|
| $\lambda_1(F) = \frac{1}{2}(\boldsymbol{\sigma}_1 F \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_2 F \boldsymbol{\sigma}_1)$ | $\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$ |
| $\lambda_2(F) = \frac{1}{2}(\boldsymbol{\sigma}_3 F - F \boldsymbol{\sigma}_3)$ | $\begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}$ |
| $\lambda_3(F) = \frac{1}{2}(\boldsymbol{\sigma}_1 F \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 F \boldsymbol{\sigma}_2)$ | $\begin{pmatrix} 0 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix}$ |
| $\lambda_4(F) = \frac{1}{2}(\boldsymbol{\sigma}_1 F \boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_3 F \boldsymbol{\sigma}_1)$ | $\begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}$ |
| $\lambda_5(F) = \frac{1}{2}(\boldsymbol{\sigma}_2 F - F \boldsymbol{\sigma}_2)$ | $\begin{pmatrix} & & -i \\ & 0 & \\ i & & \end{pmatrix}$ |
| $\lambda_6(F) = \frac{1}{2}(\boldsymbol{\sigma}_2 F \boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_3 F \boldsymbol{\sigma}_2)$ | $\begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$ |
| $\lambda_7(F) = \frac{1}{2}(\boldsymbol{\sigma}_1 F - F \boldsymbol{\sigma}_1)$ | $\begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}$ |
| $\lambda_8(F) = -\frac{1}{2\sqrt{3}}(F + 3\boldsymbol{\sigma}_3 F \boldsymbol{\sigma}_3)$ | $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$ |

If a timelike vector $\gamma_0$ is singled out, then, as expressed already in (2), the bivector $F$ has the induced decomposition

$$F = \mathbf{a} + i\mathbf{b} \tag{38c}$$

furthermore, $M$ has the natural norm

$$(M^\dagger M)_S = (F^\dagger F)_S + \phi^\dagger \phi = \mathbf{a}^2 + \mathbf{b}^2 + \alpha^2 + \beta^2 \tag{39}$$

The invariance group of this norm is SU(4), and the subgroup SU(3) is the invariance group of $(F^\dagger F)_S$. Thus, a timelike vector determines a unique decomposition of an even multivector into an SU(3) triplet $F$ and a singlet $\phi$. It is interesting and perhaps not totally irrelevant to note that the energy density of an electromagnetic field can be expressed in the form[4] $(F^\dagger F)_S$ so it has SU(3) as an invariance group. The eight generators of SU(3)

[4] Section 9 of Ref. 1.

$\lambda_1, \lambda_2, \ldots, \lambda_8$ are linear operators $\lambda_k = \lambda_k(F)$ on the space of bivectors, and they can be expressed solely in terms of STA as shown in Table 1. The table also gives the "standard matrix representation" of the $\lambda_k$ with matrix elements

$$\lambda_{kij} \equiv (\boldsymbol{\sigma}_i \lambda_k(\boldsymbol{\sigma}_j))_C \tag{40}$$

where the subscript $C$ means "complex" (scalar + pseudoscalar) part.

The above remarks suggest that we associate quark states with even multivectors. This might be done by representing the basic quark multiplet by a pair of even spinor fields $\Psi$ and $\Phi$ so that

$$M_\mu \equiv \gamma_0 \widetilde{\Phi} \gamma_\mu \Psi = \Phi^\dagger \gamma_0 \gamma_\mu \Psi = F_\mu + \phi_\mu$$

is a set of quark currents in isospace. A theory developed along these lines will not be attempted here. These remarks are intended only to indicate the possibility of handling strong interactions within the present geometric framework.

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