Simplicial Calculus with Geometric Algebra

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ABSTRACT. We construct geometric calculus on an oriented k-surface embedded in Euclidean space by utilizing the notion of an oriented k-surface as the limit set of a sequence of k-chains. This method provides insight into the relationship between the vector derivative, and the Fundamental Theorem of Calculus and Residue Theorem It should be of practical value in numerical finite difference calculations with integral and differential equations in Clifford algebra.

0. Introduction

In 1968, D. Hestenes showed how Geometric Algebra can be used to advantage in reformulating ideas from multivariable calculus [1], [2]. For example, he showed that the proof of Stokes' theorem becomes a one-line identity in geometric algebra, if the integral definition of the vector derivative is adopted. In this paper, we systematically build up calculus on a k-surface in order to more closely examine the content of these theorems.

Section 1 gives a brief introduction to the geometric algebra of Euclidean n-space, including basic definitions and identities which are be used in later sections. The inner, outer, and geometric products of vectors are discussed, as is the notion of a reciprocal frame of vectors. The material in this section is taken from [3], [4], and, primarily [5].

In section 2 the concept of an oriented simplex is introduced and related notions from homology theory are reviewed; more details can be found in [6], [7]. Peculiar to the present approach are the concepts of the *directed content* of a simplex, made possible by the introduction of geometric algebra, and the *simplicial variable* of a k-surface. These concepts are the basic building blocks for our theory of simplicial calculus developed in later sections.

In section 3 a k-surface is defined to be the limit set of an appropriate sequence of chains of simplices. A k-surface φ is said to be *smooth if there exists a smooth* k-vector field, called the *pseudoscalar field*, at each point of φ. Our approach is most closely related to [8].

In section 4 the *directed integral* on a k-surface is defined in terms of the directed content of the limit of the sequence of chains which defines it. The theory of directed integrals is first developed over a k-simplex, and then generalized. The most remarkable theorem of this section, which has no counterpart in the related theory of scalar-valued differential forms, expresses the *directed moment* of the boundary of a k-simplex as the inner product of the directed content of the simplex with the direction of the axis.

In section 5 the vector derivative is defined in terms of the limit of the directed integral over the boundary of the simplicial variable as this variable approaches zero. The vector derivative is shown to be equivalent to the ordinary gradient when applied to scalar functions.

Section 6 presents a simple proof of the *Fundamental Theorem of Calculus* using the simplicial calculus developed in earlier sections. This theorem relates the directed integral of the vector derivative of a function over a k-surface to the directed integral of the function over the boundary of the k-surface.

In section 7 the Dirac delta function is introduced, [9], [10], to quickly obtain a powerful *Residue Theorem*. Cauchy's integral formula is shown to be a special case of this theorem.

1. The Geometric Algebra of Euclidean Space

Let $\mathcal{E}_n$ denote Euclidean n-space represented as an n-dimensional vector space with a positive definite inner product, which we denote by

$$x \cdot y \qquad \text{for} \qquad x, y \in \mathcal{E}\,. \tag{1.1}$$

The elements of $\mathcal{E}_n$ will alternatively be referred to as *points* or *vectors*, depending upon their usage [3,p.15].

The *geometric algebra* $\mathcal{G}_n$ of $\mathcal{E}_n$, is the associative algebra generated by *geometric multiplication* of vectors in $\mathcal{E}_n$. In the literature it is often referred to as the Clifford algebra of the quadratic form x^2, [4]. The *geometric product* of the vectors $x, y \in \mathcal{E}_n$ can be decomposed into

$$xy = x \cdot y + x \wedge y\,, \tag{1.2}$$

where the *inner product*

$$x \cdot y = \tfrac{1}{2}(xy + yx)\,, \tag{1.3}$$

is the symmetric part of the geometric product, and the *outer product*

$$x \wedge y = \tfrac{1}{2}(xy - yx)\,, \tag{1.4}$$

is the antisymmetric part of the geometric product. The quantity $x \wedge y$ is called a 2-*vector*, or *bivector;* it can be interpreted geometrically as a directed area.

Let $x_1, \ldots, x_k$ be k vectors in $\mathcal{E}_n$. The outer product

$$x_1 \wedge \cdots \wedge x_k = \frac{1}{k!} \sum_{\lambda} (-1)^{\lambda} x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_k}\,, \tag{1.5}$$

is defined as the totally antisymmetric part of the geometric product of these vectors. The sum in (1.5) is taken over all permutations λ of the indices λ_i; if λ is an even or odd permutation we set $\lambda = 0$ or 1 respectively. The geometric number $x_1 \wedge \cdots \wedge x_k$ is called a k-vector, it can be interpreted geometrically as a directed k-volume. The *magnitude* or *k-volume* of $x_1 \wedge \cdots \wedge x_k$ will he denoted by $|\, x_1 \wedge \cdots \wedge x_k\,|$. A systematic construction of the geometric algebra $\mathcal{G}$ can be found in [5]. The notation and algebraic identities used here are largely taken from this reference.

We gather here a number of identities and relationships which are indispensable in this work. For vectors b and a_i,

$$
\begin{aligned}
b \cdot (a_1 \wedge a_2 \wedge \cdots \wedge a_k) &= \sum_{i=1}^{k} (-1)^{i+1} (b \cdot a_i)(a_1 \wedge \cdots \wedge \check{a}_i \wedge \cdots \wedge a_k) \\
&= (-1)^{k-1} (a_1 \wedge a_2 \wedge \cdots \wedge a_k) \cdot b
\end{aligned}
\tag{1.6}
$$

The symbol over the a_i in (1.6) means that $\check{a}_i$ is to be deleted from the product. Identity (1.6) explicitly shows that when a k-vector is "dotted" with a vector, the result is a $(k-1)$-vector. Dotting a k-vector with a vector is closely related to contracting an r-form by a vector. For a complete discussion of the relationship between r-forms and r-vectors see [5,p.33].

The following cancellation property will be used in a later section. If A_r and B_r are r-vectors where $r < n$, and

$$
c \wedge A_r = c \wedge B_r \tag{1.7}
$$

for all vectors $c \in \mathcal{E}_n$, then $A_r = B_r$.

Given a set of linear independent vectors, $\{e_i : i = 1, \ldots, k\}$, spanning a k-dimensional subspace of $\mathcal{E}_n$ we can construct a reciprocal frame $\{e^j : j = 1, \ldots, k\}$ spanning the same subspace, and satisfying

$$
e_i \cdot e^j = \delta_i{}^j \quad \text{and} \quad \sum_{i=1}^{k} e^i \wedge e_i = 0\,, \tag{1.8}
$$

where $\delta_i{}^j = 1$ or 0 according to whether $i = j$ or $i \neq j$, respectively. The explicit construction of the reciprocal frame is given in [5,p.28].

We close this section with a useful lemma regarding the volume of a regular k-simplex, and its moment with respect to any of its coordinate axes. The proof, by induction, is omitted.

LEMMA.
$$
\int_0^s \int_0^{s-t_k} \cdots \int_0^{s-t_k-\cdots-t_2} dt_1 dt_2 \cdots dt_k = \frac{s^k}{k!} \tag{1.9}
$$

$$
\int_0^s \int_0^{s-t_k} \cdots \int_0^{s-t_k-\cdots-t_2} t_i \, dt_1 dt_2 \cdots dt_k = \frac{s^{k+1}}{(k+1)!} \tag{1.10}
$$

2. Simplices

Let $\{a_0, a_1, \ldots, a_k\}$ be an ordered set of points in $\mathcal{E}_n$. The *oriented k-simplex* $(a)_{(k)}$ of these points is defined by

$$
(a)_{(k)} \equiv (a_0, a_1, \ldots, a_k) = \big\{ a \big| \, a = \sum_{\mu=0}^{k} t_\mu a_\mu \big\}\,, \tag{2.1}
$$

where $\sum_{\mu=0}^{k} t_\mu = 1$ and $0 \leq t_\mu \leq 1$, are the *barycentric coordinates* of the point a. We say that the simplex $(a)_{(k)}$ is located at the point $a = a_0$. Alternatively, we will use the symbolism $(\Delta_{(k)}a_0)$ to denote a k-simplex at the point a_0. By the *boundary* of $(a)_{(k)}$, we mean the $(k-1)$ chain

$$\partial(a)_{(k)} = \sum_{i=0}^{k}(-1)^{i+1}(\check{a}_{k-i})_{(k-1)} \tag{2.2}$$

where $(\check{a}_{k-i})_{(k-1)}$ is the $(k-1)$-simplex defined by

$$(\check{a}_{k-i})_{(k-1)} \equiv (a_0, a_1, \ldots, \check{a}_{k-i}, \ldots a_k)\,. \tag{2.3}$$

As before, $\check{a}_j$ indicates that this point is omitted. It is not difficult to establish the basic result of homology theory:

$$\partial^2(a)_{(k)} = 0 \tag{2.4}$$

For a discussion of simplices, chains, and their boundaries, see [6,p.57], [7,p.206].

By the *directed content* of the simplex $(a)_{(k)}$, we mean the k-vector (for $k \geq 1$)

$$a_{(k)} \equiv \mathcal{D}[(a)_{(k)}] = \frac{1}{k!}\,(a_1 - a_0) \wedge (a_2 - a_0) \wedge \cdots \wedge (a_k - a_0)\,. \tag{2.5a}$$

With the abbreviated notation $\bar{a}_i = a_i - a_0$, (2.5a) takes the form

$$a_{(k)} = \frac{1}{k!}\,\bar{a}_1 \wedge \cdots \wedge \bar{a}_k\,.$$

The simplex $(a)_{(k)}$ is said to be *non-degenerate* if its directed content $a_{(k)} \neq 0$. In the special case that $k = 0$, we define

$$a_{(0)} \equiv \mathcal{D}[(a)_{(0)}] = 1\,. \tag{2.5b}$$

More generally, it is possible to define $\mathcal{D}[(a_0)] = \rho(a_0)$, where $\rho(a_0)$ is the mass density at the point a_0, but we will not do this here. We have the following useful

LEMMA. $$a_{(k)} = \frac{1}{k!}\,(a_1 - a_0) \wedge (a_2 - a_1) \wedge \cdots \wedge (a_k - a_{k-1}) \tag{2.6}$$

Proof.
$$\begin{aligned}
a_{(k)} &= \frac{1}{k!}\,(a_1 - a_0) \wedge (a_2 - a_0) \wedge \cdots \wedge (a_k - a_0) \\
&= \frac{1}{k!}\,(a_1 - a_0) \wedge \cdots \wedge (a_{k-1} - a_0) \wedge (a_k - a_{k-1} + a_{k-1} - a_0) \\
&= \frac{1}{k!}\,(a_1 - a_0) \wedge \cdots \wedge (a_{k-1} - a_0) \wedge (a_k - a_{k-1}) \\
&\qquad \cdots \\
&= \frac{1}{k!}\,(a_1 - a_0) \wedge (a_2 - a_1) \wedge \cdots \wedge (a_k - a_{k-1})\,.
\end{aligned}$$

□

Lemma (2.6) is needed in establishing that the directed content of the boundary of a simplex vanishes, as proven in the following

THEOREM. $\mathcal{D}[\partial(a)_{(k)}] = 0$ (2.7)

Proof. The method of proof is by induction. For $k = 1$, the boundary of the 1-simplex $(a)_{(1)}$, by definition (2.2), is

$$\partial(a)_{(1)} = -(a_0) + (a_1)\,.$$

With the help of (2.5b), it follows that

$$\mathcal{D}[\partial(a)_{(1)}] = -\mathcal{D}[(a_0)] + \mathcal{D}[(a_1)] = -1 + 1 = 0\,.$$

Now assume true for $k = r$. Using (2.2) and (2.5a), this is equivalent to

$$(a_1 - a_0) \wedge \cdots \wedge (a_{r-1} - a_{r-2}) =$$

$$\sum_{i=1}^{r} (-1)^{i+1} (a_1 - a_0) \wedge \cdots \wedge (a_{r-i-1} - a_{r-i-2}) \wedge (a_{r-i+1} - a_{r-i-1}) \wedge \cdots$$

$$\cdots \wedge (a_r - a_{r-1}) \equiv S$$

Then, for $k = r + 1$, we have

$$\mathcal{D}[\partial(a)_{(r+1)}] = \sum_{i=0}^{r+1} (-1)^{i+1} \mathcal{D}[(\check{a}_{r-i+1})_{(r)}] = -(a_1 - a_0) \wedge \cdots \wedge (a_r - a_{r-1})$$

$$+ (a_1 - a_0) \wedge \cdots \wedge (a_{r-1} - a_{r-2}) \wedge (a_{r+1} - a_{r-1}) - S \wedge (a_{r+1} - a_r)$$

$$= (a_1 - a_0) \wedge \cdots \wedge (a_{r-1} - a_{r-2}) \wedge (a_{r+1} - a_r) - S \wedge (a_{r+1} - a_r) = 0\,,$$

and the proof is complete.

□

Let $(x)_{(k)}$ be a k-simplex in $\mathcal{E}_n$ at the point $x = x_0$. The directed content of $(x)_{(k)}$ is the k-vector

$$x_{(k)} = \mathcal{D}[(x)_{(k)}] = \frac{1}{k!}\, \bar{x}_1 \wedge \cdots \wedge \bar{x}_k\,, \tag{2.8}$$

where again we are employing the notation $\bar{x}_i = x_i - x_0$. By the *mesh* of the simplex $(x)_{(k)}$ we mean

$$\text{mesh}\,[(x)_{(k)}] \equiv \max\{|\, x_i - x_j\,| : \quad \text{for} \quad i, j = 0, 1, 2, \ldots, k\}\,. \tag{2.9a}$$

If $\mathcal{C} = \sum_{i=1}^{r} (\Delta x_i)$ is a chain of simplices (Δx_i) at the points x_i, then

$$\text{mesh}\,[\,\mathcal{C}\,] \equiv \max_i \;\{\text{mesh}\,(\Delta x_i)\}\,. \tag{2.9b}$$

3. Oriented k-Surfaces in $\mathcal{E}_n$

Simplices are the building blocks of oriented surfaces. Just as an oriented curve from a point a to b in $\mathcal{E}_n$ can be considered to be the limit set of a sequence of chords (chains of 1-simplices), a k-surface is defined here to be the limit set of a sequence of chains of

k-simplices. Instead of defining a manifold in terms of charts and atlases, and then proving that such a manifold can be triangulated [8], we take as fundamental the sequence of chains which has as its limit set the points of the surface φ_k. We now give the formal definitions upon which we construct our theory.

DEFINITION. Let $\{\mathcal{C}^j : j = 1, 2, \ldots\}$ be a sequence of k-chains in $\mathcal{E}_n$ with the properties that

i) The vertices of the simplices of $\mathcal{C}^j$ are a subset of the vertices of the simplices of $\mathcal{C}^{j+1}$, for $j = 1, 2, \ldots$.

ii) $\lim_{j\to\infty} \operatorname{mesh}[\,\mathcal{C}^j\,] = 0.$ (3.1)

We say that φ_k is the *k-surface* of this sequence of chains if the limit set of points

$$\varphi_k \equiv \lim_{j\to\infty} \mathcal{C}^j \tag{3.2}$$

is well defined.

Let a point $x \in \varphi_k$ be given. Since x is a limit point of the sequence of chains $\{\mathcal{C}^j\}$, there exist a subsequence of simplices $(\Delta_{(k)}x^j) \in \mathcal{C}^j$ located at the points x^j, such that

$$x = \lim_{j\to\infty} (\Delta_{(k)}x^j)\,.$$

Employing the notation introduced in (2.1), we equivalently write

$$x = \lim_{x_{(k)}\to 0} (x)_{(k)}\,, \tag{3.3}$$

to express x as the limit point of this simplicial variable. Thus, the domain of the simplicial variable $(x)_{(k)}$ is the set of simplices $(\Delta_{(k)}x^j)$ belonging to the chains $\mathcal{C}^j$ which converge to x.

We can now define a smooth, orientable k-surface.

DEFINITION: Let φ_k be the k-surface of the sequence of chains $\{\mathcal{C}^j\}$. φ_k is said to be *smooth* and *orientable* if the unit k-vector $I(x)$, specified by

$$I(x) = \lim_{x_{(k)}\to 0} \frac{x_{(k)}}{|\,x_{(k)}\,|}\,, \tag{3.4}$$

is well defined and smooth for each $x \in \varphi_k$. In this case, $I(x)$ is the *pseudoscalar* field of φ_k.

4. Calculus on a k-Surface

Let φ_k be a k-surface in $\mathcal{E}_n$ and let $F = F(x)$ and $G = G(x)$ be $\mathcal{G}$-valued functions defined on φ_k. By the *directed two sided integral* of F and G over φ_k, we mean

$$\int_{\varphi_k} G\, d_{\bar{k}}x\, F \equiv \lim_{j\to\infty} \sum_{i=1}^{l(j)} G(x_i^j)\, \Delta_{(k)}x_i^j\, F(x_i^j) \tag{4.1}$$

where φ_k is the limit set of the sequence of chains of simplices $\{\mathcal{C}^j\}$ where

$$\mathcal{C}^j = \sum_{i=1}^{l(j)} (\Delta_{(k)} x_i^j) \quad \text{and} \quad \Delta_{(k)} x_i^j \equiv \mathcal{D}[(\Delta_{(k)} x_i^j)] \tag{4.2}$$

provided, of course, that this limit exists. Note that as $j \to \infty$ the simplicial element of directed k-content $\Delta_{(k)} x \to d_{\bar{k}} x$ which is the infinitesimal element of directed k-content at the point x of the surface φ_k. The element of directed k-content $d_{\bar{k}} x$ in general does not commute with F or G; hence the need for the two sided integral (4.1). By the *directed content* of the k-surface φ_k we mean

$$\mathcal{D}[\,\varphi_k\,] \equiv \int_{\varphi_k} d_{\bar{k}} x\,. \tag{4.3}$$

Before we develop the theory of directed integrals over a k-surface φ_k, we will prove a number of theorems regarding directed integrals over a k-simplex $(a)_{(k)}$

THEOREM. $$\int_{(a)_{(k)}} d_{\bar{k}} x = a_{(k)} \tag{4.4}$$

Proof. In terms of the barycentric coordinates of a point $x \in (a)_{(k)}$

$$x = a_0 + t_1(a_1 - a_0) + \cdots + t_k(a_k - a_0)\,.$$

From this expression we define the differentials

$$d_i x \equiv dt_i\,(a_i - a_0) = dt_i\,\bar{a}_i\,,$$

and construct the element of content

$$d_{\bar{k}} x \equiv d_1 x \wedge \cdots \wedge d_k x = k!\, a_{(k)} dt_1 \cdots dt_k\,.$$

With the help of lemma (1.9), we integrate to get

$$\int_{(a)_{(k)}} d_{\bar{k}} x = k!\, a_{(k)} \int_0^1 \int_0^{1-t_k} \cdots \int_0^{1-t_k-\cdots-t_2} dt_1 dt_2 \cdots dt_k = k!\,\frac{a_{(k)}}{k!} = a_{(k)}\,,$$

and the proof is complete.

□

Theorem (4.4) shows that (4.3) is consistent with (2.5).

The next theorem finds the moment of the simplex $(a)_{(k)}$ with respect to an axis having the direction of the vector b.

THEOREM. $$\int_{(a)_{(k)}} d_{\bar{k}} x\,(\bar{x} \cdot b) = \frac{a_{(k)}}{k+1} \sum_{i=1}^{k} \bar{a}_i \cdot b\,, \quad \text{where} \quad \bar{x} = x - a_0\,. \tag{4.5a}$$

Proof. Using barycentric coordinates, we write

$$\bar{x} = \sum_{i=1}^{k} t_i \bar{a}_i \,,$$

where $\bar{x} = x - a_0$, and $\bar{a}_i = a_i - a_0$. Using the element of content found in the proof of theorem (4.4), and lemma (1.10), we carry out the integration as follows:

$$\begin{aligned}\int_{(a)_{(k)}} d_{\bar{k}} x \, (\bar{x} \cdot b) &= \sum_{i=1}^{k} \int_{(a)_{(k)}} d_{\bar{k}} x \, t_i \, (\bar{a}_i \cdot b) \\ &= k! \, a_{(k)} \sum_{i=1}^{k} \int_0^1 \int_0^{1-t_k} \cdots \int_0^{1-t_k-\cdots-t_2} t_i \, dt_1 dt_2 \cdots dt_k (\bar{a}_i \cdot b) \\ &= \frac{a_{(k)}}{k+1} \sum_{i=1}^{k} \bar{a}_i \cdot b \,.\end{aligned}$$

□

We have the following easy

COROLLARY. $$\int_{(a)_{(k)}} d_{\bar{k}} x \, \bar{x} = \frac{a_{(k)}}{k+1} \sum_{i=1}^{k} \bar{a}_i \,. \tag{4.5b}$$

The next theorem shows that the directed content of the boundary of a k-simplex is zero.

THEOREM. $$\int_{\partial (a)_{(k)}} d_{\overline{k-1}} x = 0 \,. \tag{4.6}$$

Proof. With the help of theorems (4.3) and (2.7), we get

$$\int_{\partial (a)_{(k)}} d_{\overline{k-1}} x = \mathcal{D}[\, \partial (a)_{(k)} \,] = 0 \,.$$

We close this section with a basic theorem which directly relates the moment of the boundary of a simplex, with respect to the vector b, to the dot product of its directed content with b. This theorem has no direct counterpart in the related theory of differential forms. It provides the key to the proof of the generalized *fundamental theorem of calculus* in section 6.

THEOREM. $$\int_{\partial (a)_{(k)}} d_{\overline{k-1}} x \, (x \cdot b) = a_{(k)} \cdot b \,. \tag{4.7}$$

Proof. Using (2.2) and (4.5a),

$$\int_{\partial (a)_{(k)}} d_{\overline{k-1}} x \, (x \cdot b) = \frac{1}{k} \sum_{i=0}^{k} (-1)^{i+1} \mathcal{D}[(\check{a}_{k-i})_{(k-1)}](a_0 + \cdots \check{a}_{k-i} \cdots + a_k) \cdot b \qquad (*)$$

so we need only show that the starred expression is equal to $a_{(k)} \cdot b$. We do this by using the cancellation property (1.7). Wedging $a_{(k)} \cdot b$ with $a_i - a_{i-1}$ for $i = 1, \ldots, k$, gives, with the help of (1.6),

$$(a_{(k)} \cdot b) \wedge (a_i - a_{i-1}) = a_{(k)}(a_i - a_{i-1}) \cdot b$$

Wedging the expression $(*)$ with $(a_i - a_{i-1})$ gives only the two surviving terms

$$(-1)^{k-i} \Big\{ \mathcal{D}[(\check{a}_{i-1})_{k-1}] \wedge (a_i - a_{i-1})(a_0 + \cdots \check{a}_{k-i} \cdots + a_k) \cdot b$$

$$- \mathcal{D}[(\check{a}_i)_{k-1}] \wedge (a_i - a_{i-1})(a_0 + \cdots \check{a}_i \cdots + a_k) \cdot b \Big\} = a_{(k)}(a_i - a_{i-1}) \cdot b\,.$$

Since the two expressions are the same when wedged with $(a_i - a_{i-1})$, the proof is complete. □

It is easy to prove the following, so its proof is omitted.

COROLLARY. $$\int_{\partial(a)_{(k)}} d_{\overline{k-1}}x\, x = k\, a_{(k)}\,. \tag{4.8}$$

5. The Vector Derivative

Let φ_k be a smooth, oriented k-surface. In (4.1) we defined the two sided directed integral of functions F and G over φ_k. Now we define the (two-sided) *vector derivative* of F and G in terms of a directed integral over the boundary of a k-simplex. First, given a $\mathcal{G}$-valued function H on φ_k, and a simplicial variable $(a)_{(k)}$ of φ_k, we introduce an auxiliary function $h(x)$ called the *affine mapping* of H on the simplex $(a)_{(k)}$.

DEFINITION. $$h(x) \equiv H_0 + \sum_{i=1}^{k} (\bar{x} \cdot \bar{a}^i)\, \overline{H}_i\,, \tag{5.1}$$

where $\bar{x} = x - a_0$, and $\overline{H}_i \equiv H(a_i) - H(a_0)$ are the differences of the values of H on the vertices of the simplex $(a)_{(k)}$, and $\{\bar{a}^i\}$ is the reciprocal frame to $\{\bar{a}_i\}$ satisfying (1.8).

It follows from (5.1) and (1.8) that the finite differences

$$\bar{h}_i = \overline{H}_i, \quad \text{for} \quad i = 1, \ldots, k\,. \tag{5.2}$$

The affine mapping $h(x)$ is the linear approximation to $H(x)$ on $(a)_{(k)}$ which agrees with $H(x)$ on the vertices of this simplex. Note that the domain of h is the points of the simplex $(a)_{(k)}$ and that this is a subset of φ_k only when φ_k is flat.

If $\{\mathcal{C}^j\}$ is a sequence of chains converging to the surface φ_k, then we may define a corresponding sequence of functions $\{h^j\}$, where each h^j is the piecewise affine approximation to H on the chain of simplices $\mathcal{C}^j$. If H is continuous on φ_k, then the sequence $\{h^j\}$ converges to H at each point $x \in \varphi_k$. We have the following important

THEOREM. If φ_k is compact, then

$$\int_{\varphi_k} G(x)\, d_{\bar{k}}x\, F(x) = \lim_{j\to\infty} \int_{\mathcal{C}^j} g^j(x)\, d_{\bar{k}}x\, f^j(x), \qquad \text{and} \tag{5.3}$$

$$\int_{\partial\varphi_k} G(x)\, d_{\overline{k-1}}x\, F(x) = \lim_{j\to\infty} \int_{\partial\mathcal{C}^j} g^j(x)\, d_{\overline{k-1}}x\, f^j(x), \tag{5.4}$$

where f^j and g^j are the respective *piecewise* affine approximations to F and G on the chain of simplices $\mathcal{C}^j$.

Proof. We will only prove (5.3) since the proof of (5.4) is essentially the same. Let $\{\mathcal{C}^j\}$ be a sequence of chains converging to φ_k. By definitions (4.1) and (5.1), we have

$$\int_{\varphi_k} G\, d_{\bar{k}}x\, F \equiv \lim_{j\to\infty} \sum_{i=1}^{l(j)} G(x)\Delta_{(k)}x\, F(x) = \lim_{j\to\infty} \sum_{i=1}^{n(j)} {g_i}^j\, \Delta_{(k)}{x_i}^j\, {f_i}^j$$

$$= \lim_{j\to\infty} \int_{\mathcal{C}^j} g^j\, d_{\bar{k}}x\, f^j\,,$$

since ${f_i}^j = F({x_i}^j)$ at each of the vertices of the simplices of the chain $\mathcal{C}^j$. $\square$

We are now ready to define the *two-sided vector derivative* $G\partial F$ of the functions F and G at the point $x \in \varphi_k$. Recalling (3.3), let $(a)_{(k)}$ be a simplicial variable of the surface φ_k at the point $x = a_0$.

$$\text{DEFINITION.}\quad G\partial F = \lim_{a_{(k)}\to 0} \int_{\partial(a)_{(k)}} g(x)\, {a_{(k)}}^{-1}\, d_{\overline{k-1}}\, x\, f(x)\,, \tag{5.5}$$

where ${a_{(k)}}^{-1} \equiv 1/a_{(k)}$, and f and g are the affine mappings of F and G on $a_{(k)}$.

The integral over the boundary of the simplex in (5.5) can be evaluated to identify $G\partial F$ as the limit of a generalized difference quotient. We have the following

$$\text{THEOREM.}\quad G\partial F = \lim_{a_{(k)}\to 0} \sum_{i=1}^{k} [\, \bar{G}_i \bar{a}^i F_0 + G_0 \bar{a}^i \bar{F}_i\,]\,. \tag{5.6}$$

Proof. Using (5.5), (4.6), (4.7), we have

$$\begin{aligned}
G\partial F &= \lim_{a_{(k)}\to 0} \int_{\partial(a)_{(k)}} g(x)\, {a_{(k)}}^{-1}\, d_{\overline{k-1}}\, x\, f(x) \\
&= \lim_{a_{(k)}\to 0} \int_{\partial(a)_{(k)}} [\, G_0 + \bar{x}\cdot\bar{a}^j\, \bar{G}_j\,]\, {a_{(k)}}^{-1} d_{\overline{k-1}}\, x\, [\, F_0 + \bar{x}\cdot\bar{a}^i \bar{F}_i\,] \\
&= \lim_{a_{(k)}\to 0} \bar{G}_j\, {a_{(k)}}^{-1} \int_{\partial(a)_{(k)}} \bar{x}\cdot\bar{a}^j\, d_{\overline{k-1}}x\, F_0 \\
&\qquad\qquad + \lim_{a_{(k)}\to 0} G_0\, {a_{(k)}}^{-1} \int_{\partial(a)_{(k)}} d_{\overline{k-1}}\, x\, \bar{x}\cdot\bar{a}^i\, \bar{F}_i \\
&\qquad\qquad\qquad + \lim_{a_{(k)}\to 0} \bar{G}_j \int_{\partial(a)_{(k)}} \bar{x}\cdot\bar{a}^j\, \bar{x}\cdot\bar{a}^i\, {a_{(k)}}^{-1}\, d_{\overline{k-1}}\, x\, \bar{F}_i \\
&= \lim_{a_{(k)}\to 0} \sum_{i=1}^{k} [\, \bar{G}_i \bar{a}^i F_0 + G_0 \bar{a}^i \bar{F}_i\,] + \lim_{a_{(k)}\to 0} \bar{G}_j\, {a_{(k)}}^{-1} \int_{\partial(a)_{(k)}} \bar{x}\cdot\bar{a}^j\, \bar{x}\cdot\bar{a}^i\, d_{\overline{k-1}}\, x\, \bar{F}_i
\end{aligned}$$

Note that in using (5.5), in the above steps, the summation convention over upper and lower indicies i and j has been utilized. The proof is completed by using lemma (6.2) from the next section, and corollary (5.7) below, to show that

$$\lim_{a_{(k)}\to 0} \overline{G}_j \, a_{(k)}{}^{-1} \int_{\partial(a)_{(k)}} \bar{x}\cdot\bar{a}^j \, \bar{x}\cdot\bar{a}^i \, d_{\overline{k-1}}\, x \, \overline{F}_i = \lim_{a_{(k)}\to 0} \overline{G}_j \, a_{(k)}{}^{-1} \, [\, a_{(k)}(\bar{a}^j + \bar{a}^i)/k \,] \, \overline{F}_i$$

$$= (1/k) \lim_{a_{(k)}\to 0} \overline{G}_j (\bar{a}^j + \bar{a}^i) \, \overline{F}_i = 0 \, . \qquad \square$$

Although the proof of (5.6) *depends* in the last step upon the proof of corollary (5.7) below, this last step is not needed in the special case when either F or G are constant functions, in which case $\overline{G}_i \equiv 0$ or $\overline{F}_i \equiv 0$, giving the following

COROLLARY. If $G(x) \equiv 1$ then $\partial F = \lim_{a_{(k)}\to 0} \sum_{i=1}^{k} \bar{a}^i \overline{F}_i \, .$ (5.7)

An immediate consequence of Theorem (5.6) and Corollary (5.7) is the Leibniz product rule for the vector derivative:

COROLLARY. $G\partial F = G\dot{\partial}\dot{F} + \dot{G}\dot{\partial}F,$ (5.8)

where the dots indicate the direction of differentiation and what is being differentiated. The vector derivative ∂ is necessarily a two-sided operator because of the lack of universal commutivity in the geometric algebra $\mathcal{G}$.

In doing calculations with finite differences it is sometimes helpful to use the following corollary whose proof is omitted.

COROLLARY. $G\partial F = \lim_{a_{(k)}\to 0} \sum_{i=1}^{k} [\, G_i \bar{a}^i F_i - G_0 \bar{a}^i F_0 \,] \, .$ (5.9)

Now let $x^i(x)$ be a set of coordinates on φ_k, and suppose that the vertices of the simplicial variable $(a)_{(k)}$ contract to zero along the axes defined by these coordinates. We have the following

THEOREM. $\partial F = \sum_{i=1}^{k} e^i \partial_i F(x) \, ,$ (5.10)

where $\partial_i F(x)$ are the partial derivatives of F with respect to these coordinates, and $e_i = \partial_i \, x$ are the basis vectors along the coordinate axes x_i at $x = x(x^1, \ldots, x^k)$.

Proof. From corollary (5.7),

$$
\begin{aligned}
\partial F &= \lim_{a_{(k)} \to 0} \sum_{i=1}^{k} \bar{a}^i (F_i - F_0) \\
&= \lim_{a_{(k)} \to 0} \sum_{i=1}^{k} \bar{a}^i \, \Delta x^i \, \frac{F(x^1, \ldots, x^i + \Delta x^i, \ldots, x^k) - F(x^1, \ldots, x^k)}{\Delta x^i} \\
&= \sum_{i=1}^{k} e^i \partial_i \, F(x) \,,
\end{aligned}
$$

where

$$
e^i \equiv \lim_{a_{(k)} \to 0} \sum_{i=1}^{k} \bar{a}^i \, \Delta x^i \,.
$$

□

Theorem (5.10) identifies the vector derivative ∂ as the ordinary gradient when applied to a scalar-valued function. Basic properties and differentiation formulas for the vector derivative as applied to more general functions can be found in [5;51].

6. The Fundamental Theorem of Calculus

The most important theorem in calculus is the *Fundamental Theorem of Calculus*. Generalized to k-dimensions, this theorem relates the integral of the vector derivative of a function over a k-surface φ_k to the integral of the function over the $(k-1)$-surface $\partial \varphi_k$ which is the boundary of φ_k.

THEOREM (Fundamental Theorem of Calculus). Let φ_k be a compact k-surface, and let $F = F(x)$ be a function differentiable almost everywhere on φ_k. Then

$$
\int_{\varphi_k} d_{\bar{k}} x \, \partial F = \int_{\partial \varphi_k} d_{\overline{k-1}} x \, F \,. \tag{6.1a}
$$

$$
\int_{\varphi_k} G \, d_{\bar{k}} x \, \overline{\partial} \, F = \int_{\partial \varphi_k} G \, d_{\overline{k-1}} x \, F \,, \tag{6.1b}
$$

where $\overline{\partial}$ is understood to differentiate both to the left and right, but does not differentiate the k-surface element $d_{\bar{k}} x$.

Proof. The proof of part a) is direct. Using (4.1), (5.5), and (5.4):

$$
\begin{aligned}
\int_{\varphi_k} d_{\bar{k}} x \, \partial F &= \lim_{j \to \infty} \sum^{j} \Delta_{(k)} x \, \frac{1}{\Delta_{(k)} x} \int_{\partial(\Delta_{(k)} x)} d_{\overline{k-1}} x \, f^j = \lim_{j \to \infty} \sum^{j} \int_{\partial(\Delta_{(k)} x)} d_{\overline{k-1}} x \, f^j \\
&= \lim_{j \to \infty} \sum^{j} \int_{\partial \mathcal{C}^j} d_{\overline{k-1}} x \, f^j = \int_{\partial \varphi_k} d_{\overline{k-1}} x \, F \,,
\end{aligned}
$$

where $\mathcal{C}^j = \sum^j \Delta_{(k)}x$. The proof of part b) is similar to part a) and is omitted.

□

The two-sided form of the fundamental theorem given in part b) is useful in proving the general residue theorem in the next section. Further discussion of the fundamental theorem and its special cases can be found in [2] and [5;256].

We can now prove the lemma needed in the proof of theorem (5.6).

LEMMA.
$$\int_{\partial(a)_{(k)}} d_{\overline{k-1}}x\,(\bar{x}\cdot\bar{a}^i)(\bar{x}\cdot\bar{a}^j) = \frac{1}{k+1}\,a_{(k)}[\,\bar{a}^i+\bar{a}^j\,] \tag{6.2}$$

Proof. The vector derivative ∂F for $F(\bar{x}) = (\bar{x}\cdot\bar{a}^i)(\bar{x}\cdot\bar{a}^j)$ is

$$\partial F = (\bar{x}\cdot\bar{a}^i)\,\bar{a}^j + (\bar{x}\cdot\bar{a}^j)\,\bar{a}^i\,;$$

this can be either calculated directly from Corollary (5.7), or can be found in [5,Chp. 2]. Using this result, the Fundamental Theorem (6.1a), and (4.5a), we then find

$$\int_{\partial(a)_{(k)}} d_{\overline{k-1}}x\,(\bar{x}\cdot\bar{a}^i)(\bar{x}\cdot\bar{a}^j) = \int_{(a)_{(k)}} d_{\bar{k}}x\,[\,(\bar{x}\cdot\bar{a}^i)\,\bar{a}^j + (\bar{x}\cdot\bar{a}^j)\,\bar{a}^i\,]$$
$$= \frac{a_{(k)}}{k+1}\sum_{l=1}^{k}[\,(\bar{a}_l\cdot\bar{a}^i)\,\bar{a}^j + (\bar{a}_l\cdot\bar{a}^j)\,\bar{a}^i\,] = \frac{a_{(k)}}{k+1}\,[\,\bar{a}^j+\bar{a}^i\,]\,.$$

□

7. Residue Theorem

Introduction of the Dirac delta function into the class of integrable functions on a k-surface allows an approach to the powerful residue theorems which is intuitive and at the same level of rigor as the more standard approaches, [9], [10]. We begin with the definition of the Green's function.

DEFINITION. Let φ_k be a smooth, compact k-surface, and let x, $x_0 \in \varphi_k$ and $\partial = \partial_x$ be the vector derivative at x. A function $G(\bar{x})$, where $\bar{x} = x - x_0$, is said to be a *Green's function* for φ_k provided
$$\partial G(\bar{x}) = \delta(\bar{x}) = G(\bar{x})\,\overline{\partial} \tag{7.1a}$$
where $\delta(\bar{x})$ is the Dirac delta function having the property

$$\int_{\mathcal{R}} d_{\bar{k}}x\,\delta(\bar{x})F(x) = I(x_0)F(x_0)\,, \tag{7.1b}$$

where $\mathcal{R}$ is any region in φ_k containing the point x_0, $I(x_0)$ is the unit pseudoscalar (3.4) at x_0, and $F(x)$ is any function which is continuous in $\mathcal{R}$.

We will restrict our discussion to k-planes in which case our candidate for the Green's function is
$$G(\bar{x}) = \frac{\Omega\bar{x}}{|\,\bar{x}\,|^k}\,, \tag{7.2}$$

where Ω is a constant scalar to be determined. With the help of the vector differentiation formula [5;2-(1.36)], we obtain the result

$$\partial G(\bar{x}) = 0 \quad \text{for all } \bar{x} \neq 0\,, \tag{7.3}$$

as is required.

The functions $G(\bar{x})$ and $\delta(\bar{x}) = \partial G$ are singular at $\bar{x} = 0$, i. e., when $x = x_0$, so we must exercise caution when integrating over any region $\mathcal{R}$ containing the point x_0.

To define the meaning of integrals containing $G(\bar{x})$ and/or $\delta(\bar{x})$, we introduce a sequence of functions $g_m(\bar{x})$ by

$$g_m(\bar{x}) \equiv \left\{ \begin{array}{ll} G(\bar{x}) & \text{for } |\,\bar{x}\,| > 1/m \\ \Omega\, m^k \bar{x} & \text{for } |\,\bar{x}\,| \leq 1/m \end{array} \right\}. \tag{7.4}$$

Differentiating $g_m(\bar{x})$, we get

$$\partial g_m(\bar{x}) \equiv \left\{ \begin{array}{ll} \partial G(\bar{x}) = 0 & \text{for } |\,\bar{x}\,| > 1/m \\ k\,\Omega\, m^k & \text{for } |\,\bar{x}\,| < 1/m \end{array} \right\}. \tag{7.5}$$

The function $g_m(\bar{x})$ and $\partial g_m(\bar{x})$ are differentiable almost everywhere, except on a set of k-content 0, and it is easy to see that

$$G(\bar{x}) = \lim_{m\to\infty} g_m(\bar{x}) \quad \text{for all } \bar{x} \neq 0\,. \tag{7.6}$$

Improper integrals involving G and $\delta = \partial G$, can now be defined:

DEFINITION. $$\int_{\mathcal{R}} d_{\bar{k}} x\, \partial G = \lim_{m\to\infty} \int_{\mathcal{R}} d_{\bar{k}} x\, \partial g_m\,. \tag{7.7}$$

We now choose the constant $\Omega = 1/\odot$ where $\odot$ is the surface area of the unit k-dimensional ball. Let $\mathcal{R}$ be a k-region containing the point x_0. We have the following

THEOREM. $$i) \quad \int_{\mathcal{R}} d_{\bar{k}} x\, \partial G = I(x_0)\,, \quad ii) \quad \int_{\mathcal{R}} d_{\bar{k}} x\, (\partial G) F(x) = I(x_0) F(x_0)\,. \tag{7.8}$$

Proof. *i)* Using (7.7) and (6.1a), we find

$$\begin{aligned} \int_{\mathcal{R}} d_{\bar{k}} x\, \partial G &= \lim_{m\to\infty} \int_{\mathcal{R}} d_{\bar{k}} x\, \partial g_m = \lim_{m\to\infty} \int_{\mathcal{B}_m} d_{\overline{k-1}} x\, k \Omega m^k \\ &= \lim_{m\to\infty} \frac{\odot (1/m)^k I(x_0) m^k}{\odot} = I(x_0)\,, \end{aligned}$$

where $\mathcal{B}_m$ is the k-ball of *radius* $1/m$ centered at x_0.

ii) In light of *i)*, it is sufficient to show that

$$\Big|\int_{\mathcal{R}} d_{\bar{k}} x\, \partial G\, [\, F(x) - F(x_0)\,]\,\Big| = 0\,.$$

By definition (7.7)

$$\int_{\mathcal{R}} d_{\bar{k}}x\,\partial G\,[\,F(x) - F(x_0)\,] = \lim_{m\to\infty} \int_{\mathcal{R}} d_{\bar{k}}x\,(\partial g_m)\,[\,F(x) - F(x_0)\,]$$
$$= \lim_{m\to\infty} k(1/\odot)\,m^k \int_{\mathcal{B}_m} d_{\bar{k}}x\,[\,F(x) - F(x_0)\,]\,.$$

But

$$\Big|\int_{\mathcal{B}_m} d_{\bar{k}}x\,[\,F(x) - F(x_0)\,]\Big| \le (\odot/m^k)\epsilon\,,$$

where $\epsilon = \max\limits_{x\in\mathcal{B}_m}\big\{\,|\,F(x) - F(x_0)\,|\big\}$. It follows that

$$\Big|\int_{\mathcal{B}_m} d_{\bar{k}}x\,(\partial G)\,[\,F(x) - F(x_0)\,]\Big| \le \lim_{m\to\infty} \frac{km^k}{\odot}\,(\odot/m^k)\epsilon = \lim_{m\to\infty} k\epsilon\,.$$

Since $F(x)$ is continuous, as $m \to \infty$, $\epsilon \to 0$, so the integral goes to zero as claimed.

□

We can now prove the following

RESIDUE THEOREM.

$$F(x_0) = \frac{(-1)^{k-1}}{\odot\,I} \int_{\partial\mathcal{R}} \frac{\bar{x}}{|\,\bar{x}\,|^k}\,d_{\overline{k-1}}x\,F + \frac{(-1)^k}{\odot\,I} \int_{\mathcal{R}} \frac{\bar{x}}{|\,\bar{x}\,|^k}\,d_{\bar{k}}x\,\partial F\,. \tag{7.9}$$

Proof. We have

$$\int_{\partial\mathcal{R}} G\,d_{\bar{k}}x\,F = \int_{\mathcal{R}} G\,d_{\bar{k}}x\,\overline{\partial}\,F = \int_{\mathcal{R}} G\,d_{\bar{k}}x\,(\partial F) + (-1)^{k-1}\int_{\mathcal{R}} (\dot{G}\dot{\partial})\,d_{\bar{k}}x\,F$$
$$= \int_{\mathcal{R}} G\,d_{\bar{k}}x\,(\partial F) + (-1)^{k-1} I(x_0)F(x_0)\,.$$

The first equality in the steps above is a consequence of (6.1b), the second equality is a consequence of the Leibniz product rule and (1.6), and the third is a consequence of (7.8*ii*). The remainder of the proof is a straight forward substitution for G, using (7.2).

□

Let us examine theorem (7.9) in the case when $k = 2$. In this case, $\odot = 2\pi$, $i = e_2e_1$, and we have

$$F(x_0) = \frac{1}{2\pi i} \int_{\mathcal{R}} \frac{1}{x - x_0}\,d_{\overline{2}}x\,\partial F - \frac{1}{2\pi i} \int_{\partial\mathcal{R}} \frac{1}{x - x_0}\,dx\,F\,. \tag{7.10}$$

We can translate (7.10) into an equivalent statement in the complex number plane of the bivector $i = e_2e_1$ by writing

$$dz = dx\,e_1, \quad \frac{d}{d\bar{z}} = \tfrac{1}{2}e_1\partial_x, \quad \frac{1}{x - x_0} = e_1\,\frac{1}{z - z_0}, \quad \text{and} \quad idxdy = d_{\overline{2}}x\,.$$

Letting

$$f(z) = e_1 F(x) = \overline{F(x)e_1}\,,$$

we have,

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial\mathcal{R}} \frac{f(z)}{z - z_0}\,dz - \frac{1}{\pi}\iint_{\mathcal{R}} \frac{(d/d\bar{z}\,)f(z)}{z - z_0}\,dxdy \tag{7.11}$$

which is valid over any region $\mathcal{R}$ for which $f(z)$ is differentiable. When $f(z)$ is analytic in $\mathcal{R}$,

$$(d/d\bar{z}\,)f(z) = 0\,,$$

and (7.11) reduces to Cauchy's Integral Formula,

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial\mathcal{R}} \frac{f(z)}{z - z_0}\,dz\,. \tag{7.12}$$

8. Concluding Remarks

There are many advantages to the simplicial calculus on an oriented k-surface advocated in this work. Whereas it is coordinate free in its development, relevant coordinates are easily introduced whenever specific calculations are called for, such as in (4.4) and (5.10). Introducing geometric algebra into n-dimensional analysis makes possible the concept of a directed measure (2.5), whose fundamental significance is revealed in the one-line proof of the Fundamental Theorem of Calculus (6.1). A more comprehensive treatment of the theory of directed integration as well as the development of geometric algebra as a unified language for mathematics and physics can be found in [5].

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