In: *Journal of Mathematical Physics*, **15** 1768–1777 (1974).

Proper particle mechanics

David Hestenes

Abstract Spacetime algebra is employed to formulate classical relativistic mechanics without coordinates. Observers are treated on the same footing as other physical systems. The kinematics of a rigid body are expressed in spinor form and the Thomas precession is derived.

Introduction

This paper shows how to formulate conventional relativistic mechanics without referring to observers or coordinates. To emphasize the distinctive features of this formulation, it will be called "*proper* mechanics." The common expression "relativistic mechanics" will be avoided here because, by the most straightforward interpretation of the adjective "relativistic," Einstein's mechanics is less rather than more relativistic than the socalled "nonrelativistic" mechanics of Newton. The equations describing a particle in Newtonian mechanics depend on the motion of the particle *relative* to the observer; in Einsteinian mechanics they do not. Einstein originally formulated his mechanics in terms of "relative variables" (such as the position and velocity of a particle relative to a given observer), but he eliminated dependence of the equations on the observer's motion by the "relativity postulate," which requires that the equations be invariant under a change of relative variables from those of one inertial observer to those of another. Minkowski's covariant formulation of Einstein's theory replaced the explicit use of variables relative to inertial observers by components relative to an arbitrary coordinate system in spacetime. The "proper formulation" used here relates particle motion directly to Minkowski's "absolute spacetime" without the intermediary of a coordinate system.

Minkowski had the great idea of interpreting Einstein's theory of relativity as a prescription for fusing space and time into a single entity "spacetime." The straightforward algebraic characterization of "Minkowski spacetime" by "spacetime algebra" makes a proper formulation of mechanics possible. The spacetime algebra can be regarded as a variant of the Dirac algebra more intimately related to spacetime than the usual matrix version. Proper mechanics shows how generally useful the Dirac algebra is outside its usual domain of "relativistic quantum theory." Besides providing a simple proper formulation of all the usual equations in "classical relativistic mechanics," spacetime algebra brings spinors to bear on the subject; as will be shown, this simplifies many things and brings the subject closer, in its formulation, to quantum theory.

In Sec. 1, the spacetime algebra is introduced along with important notations needed to interpret and apply it efficiently. For later use a number of important algebraic identities are set down and the spinor formulation of Lorentz rotations is discussed.

In Sec. 2, the proper description of a material particle is given. Inertial observers are introduced on the same footing as other physical systems; the distinction between proper and relative vectors is explained, and the reformulation of proper quantities in terms of relative variables is carried out in detail.

In Sec. 3, the relation of the spacetime algebra to the Dirac and Pauli matrix algebras is briefly explained. It is shown how easily the usual covariant equations can be put in proper form and vice versa. This section pertains only to the relation of spacetime algebra to other mathematical systems, and it is not needed in the rest of the paper.

In Sec. 4, a comoving frame is associated with a particle, and its kinematics are completely described in spinor form. This gives immediately a complete and simple formulation of the kinematics of a "rigid point particle" (i.e., a rigid body of negligible dimensions). In particular, the Thomas precession is derived by a new (and hopefully, a clearer and simpler) method, along with a complete treatment of related kinematical results. A great advantage of this approach is that all results can be used directly in an analysis of Thomas precession in the Dirac electron theory, as will be demonstrated elsewhere.

1. Spacetime Algebra

In this paper *spacetime* is understood to be four-dimensional continuum (or manifold) with "Minkowski metric” of signature minus two. Spacetime derives its significance from the facts (or, hypotheses, if you will) that every elementary physical event can be uniquely labeled by a point of spacetime, and that the metric of spacetime determines a unique ordering of physical events.

Spacetime can be given a precise mathematical description by introducing appropriate rules for adding and multiplying points. A vector a is said to be *tangent to a point x in spacetime* if there is a curve $\{x(\alpha); 0 < a < \epsilon\}$ in spacetime extending from the point $x(0) = x$ such that

$$a = \lim_{\alpha \to 0} \alpha^{-1}(x(\alpha) - x)\,. \tag{1.1}$$

The right side of (1.1) is made meaningful by the assumption that the points of spacetime can be added and multiplied by scalars according to the usual rules associated with vectors. However, it should be noted that the validity of (1.1) does not require that the sum of two spacetime points or the scalar multiple of one is again a spacetime point, in short, the spacetime is a linear vector space.

The set of all vectors tangent to a typical spacetime point x is a four-dimensional vector space $T(x)$ called the *tangent space* at x. An element of such a space will sometimes be called a *proper vector* to avoid possible confusion with other uses of the word vector. By multiplication and addition the elements of $T(x)$ generate a noncommutative associative algebra called the *spacetime algebra* (at x). This algebra has been systematically discussed in Ref. 1 and since developed into a more extensive mathematical system especially by Ref. 2. However, the basic multiplication law of spacetime algebra is likely to be familiar to most readers only in the guise of the Dirac matrix algebra, so a sketchy review of the algebra is necessary to establish terminology and a few basic relations. Relation of the spacetime algebra to more familiar formalisms will be discussed in Sec. 3.

The *geometric product* of a generic proper vector a with itself is a scalar quantity describing the metric of spacetime; thus,

$$a^2 > 0 \text{ iff } a \text{ is a } \textit{timelike} \text{ vector;} \tag{1.1a}$$

$$a^2 = 0 \text{ iff } a \text{ is a } \textit{lightlike} \text{ vector;} \tag{1.1b}$$

$$a^2 < 0 \text{ iff } a \text{ is a } \textit{spacelike} \text{ vector;} \tag{1.1c}$$

The term “scalar” here always means “real number.” The geometric product ab of proper vectors a and b can be decomposed into a sum of commuting and anticommuting parts; thus,

$$ab = a \cdot b + a \wedge b\,, \tag{1.2a}$$

where

$$a \cdot b = \tfrac{1}{2}(ab + ba) = b \cdot a\,, \tag{1.2b}$$
$$a \wedge b = \tfrac{1}{2}[a,\, b] = -b \wedge a\,, \tag{1.2c}$$

and $[A, B] \equiv AB - BA$. It follows from (1.1) that $a\cdot$ is a scalar quantity, the usual *inner product* of spacetime vectors. The quantity $a \wedge b$, called the *outer product* of a and b, is a (proper) *bivector* (or 2-vector).

Bivectors are related to vectors by multiplication. A bivector which can be expressed, as in (1.2c), as the outer product of two vectors is said to be simple. A bivector B in the spacetime algebra can be uniquely that every null bivector is simple and, in fact, has a null vector as a factor. Furthermore, every nonnull bivector B in the spacetime algebra can be uniquely expressed as the sum of two simple bivectors or blades; that is, there exist unique blades B_1 and B_2 such that

$$B = B_1 + B_2\,, \tag{1.3}$$

and B_1B_2 is a *pseudoscalar*, or equivalently B_2 is proportional to the dual of B_1. (The meanings of the terms “pseudoscalar” and “dual” will be explained later.)

The *inner* and *outer product* of a vector a with a bivector B can be defined respectively by

$$a \cdot B = \tfrac{1}{2}[a, B] = -B \cdot a\,, \tag{1.4a}$$

and

$$a \wedge B = \tfrac{1}{2}(aB + Ba) = B \wedge a\,, \tag{1.4b}$$

so

$$aB = a \cdot B + a \wedge B\,. \tag{1.4c}$$

Using (1.4a) together with (1.2b) and (1.2c), it is easy to prove that any three vectors a, b, c satisfy the useful identity

$$a \cdot (b \wedge c) = a \cdot b\, c - a \cdot c\, b = -(b \wedge c) \cdot a\,, \tag{1.5}$$

vector. On the other hand, the quantity $a \wedge B$ is a trivector (or 3-vector). Using (1.4c) and (1.2c), one can show that the outer product of vectors is associative, that is

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b \wedge c\,. \tag{1.6}$$

Every trivector in the spacetime algebra can be factored (but not uniquely) into an outer product of three vectors.

It is well at this point to introduce the convention that when parentheses are omitted inner and outer products have priority over the geometric product; for example, for vectors, a, b, c, d,

$$\begin{aligned} (a \cdot b)c &= a \cdot\, bc \neq a \cdot (bc)\,, \\ (a \wedge b)c &= a \wedge\, bc \neq a \wedge (bc)\,, \\ a \cdot b\, c \wedge d &= (a \cdot b)(c \wedge d)\,. \end{aligned}$$

This convention is particularly useful in complicated formulas. It has already been used in (1.5).

The product of a vector a with a trivector T is the sum of a bivector $a \cdot T$ and a 4-vector or pseudoscalar $a \wedge T$; thus,

$$aT = a \cdot T + a \wedge T\,, \tag{1.7a}$$

$$a \cdot T \equiv \tfrac{1}{2}(aT + Ta) = T \cdot a\,, \tag{1.7b}$$

$$a \wedge T \equiv \tfrac{1}{2}(aT - Ta) = -T \wedge a\,. \tag{1.7c}$$

From (1.7b), (1.4a), and (1.2) one can establish the useful identity

$$a \cdot (b \wedge B) = a \cdot b\, B = -b \wedge (a \cdot B)\,, \tag{1.8}$$

where a, b are vectors and B is a bivector. Every pseudoscalar is a scalar multiple of a unique unit pseudoscalar which will always be denoted by i. Specification of i assigns an orientation to spacetime. It can be shown that

$$i^2 = -1\,, \tag{1.9a}$$

and the geometric product of i with any vector a is anticommutative; that is,

$$ai = -ia\,. \tag{1.9b}$$

It follows that the outer product $a \wedge i = \frac{1}{2}(ai + ia)$ vanishes, while the inner product $a \cdot i = \frac{1}{2}(ai - ia)$ is a trivector (called the *dual* of a). Every trivector T is the dual of some vector t, that is, $T = ti$. By (1.9a), $Ti = -t$, so the dual Ti of any trivector T is a unique vector. This establishes an isomorphism of the linear space of all trivectors to the space of all vectors. For this reason, trivectors are often called *pseudovectors*.

A generic element of the spacetime algebra will be called a (*proper*) *multivector*. Every proper multivector M can be uniquely expressed as a sum of a 0-vector (or scalar), a 1-vector (or vector), a 2-vector (or bivector), a 3-vector (or pseudovector), and a 4-vector (or pseudoscalar); that is

$$M = [M]_0 + [M]_1 + [M]_2 + [M]_3 + [M]_4\,, \tag{1.10}$$

where $[M]_k$ denotes the k-vector part of M. A multivector M is said to be *even* if $[M]_1 = [M]_3 = 0$. The even multivectors compose an important subalgebra of the full spacetime algebra.

The *reverse* $\widetilde{M}$ of a multivector M can be defined by the equation

$$\widetilde{M} = [M]_0 + [M]_1 - [M]_2 - [M]_3 + [M]_4 \,. \tag{1.11}$$

It can then be shown that the reverse of a product equals the product of reverses, that is, if

$$M = AB, \quad \text{then} \quad \widetilde{M} = \widetilde{B}\widetilde{A} \,.$$

Spacetime algebra makes it possible to describe Lorentz transformations completely, without resorting to coordinates or matrices. Only Lorentz rotations (i.e., Lorentz transformations without time reversal or space inversion) are of interest here. Any *Lorentz rotation* $\mathcal{R}$ which maps a generic proper vector a into the vector $a' = \mathcal{R}(a)$ can be written in the canonical form

$$a' = \mathcal{R}(a) = Ra\widetilde{R} \,; \tag{1.13a}$$

here R is an *even* multivector, unique except for sign, with the property

$$R\widetilde{R} = 1 \,. \tag{1.13b}$$

The multivector R is called a *spinor*. One way to establish (1.13) is to introduce an orthonormal frame of vectors γ_μ and its "reciprocal frame" $\{\gamma^\mu\}$ defined by the equations

$$\gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu \,, \qquad \mu,\, \nu = 0,\, 1,\, 2,\, 3 \tag{1.14}$$

where δ^μ_ν is the "unit matrix." According to (1.13a) the transformation of γ_μ is given by

$$\gamma'_\mu = R\gamma_\mu\widetilde{R} = a^\nu_\mu \gamma_\nu \tag{1.15}$$

(sum over repeated indices), where $a^\nu_\mu = \gamma^\nu \cdot \gamma'_\mu$ is the matrix of the transformation. These equations can be solved for R. One obtains (see Sec. 17 of Ref. 1)

$$R = \pm(\widetilde{A}A)^{-\frac{1}{2}}A \qquad \text{where} \qquad A \equiv \gamma'_\mu\gamma^\mu = a^\nu_\mu \gamma_\nu \gamma^\mu \,. \tag{1.16}$$

This gives R explicitly as a function of the matrix a^ν_μ, but it is of little practical use since in most applications it is easier to determine R directly from the data.

Two special classes of Lorentz rotations are of interest here, boosts and spatial rotations. A Lorentz rotation $\mathcal{L}(a) = La\widetilde{L}$ which takes a unit *timelike* vector u into the vector v is said to be a *boost of u into v* if it leaves vectors orthogonal to the $v \wedge u$-plane invariant. Any vector a can be expressed as the sum of a component $a_\parallel$ in the $v \wedge u$-plane and a component $a_\perp$ orthogonal to it; thus,

$$a = a_\parallel + a_\perp \tag{1.17a}$$

where

$$a_\parallel = a \cdot (v \wedge u)(v \wedge u)^{-1} \,, \tag{1.17b}$$

$$a_\perp = a \wedge (v \wedge u)(v \wedge u)^{-1} \,. \tag{1.17c}$$

By definition

$$La_\perp\widetilde{L} = a_\perp \qquad \text{so} \qquad La_\perp = a_\perp L \,, \tag{1.18a}$$

because $L\widetilde{L} = 1$. It can further be shown that

$$La_\parallel\widetilde{L} = L^2 a_\parallel \qquad \text{or} \qquad La_\parallel = a_\parallel\widetilde{L} \,, \tag{1.18b}$$

in particular,

$$v = L u \widetilde{L} = L^2 u \qquad \text{so} \qquad L^2 = v u\,. \tag{1.18c}$$

The square root in (1.18c) can be taken to give L explicitly in terms of v and u [Eq. (18.14) of Ref. 1], but the result is unduly complicated and can be avoided in applications by using (1.18).

A Lorentz rotation $\mathcal{U}(a) = U a \widetilde{U}$ said to be a *spatial rotation* if it leaves a timelike vector u invariant; that is, if

$$U u \widetilde{U} = u\,, \tag{1.19a}$$

or, equivalently,

$$U U^\dagger = 1 \qquad \text{where} \qquad U^\dagger \equiv u \widetilde{U} u\,. \tag{1.19b}$$

The set of all Lorentz rotations satisfying (1.19) is the group of spatial rotations in the spacelike hypersurface with normal u, called the *little group* of u.

Any Lorentz rotation can be uniquely expressed as a spatial rotation followed by a boost of a given timelike vector u. This decomposition can be completely characterized by factoring the spinor R defined by (1.13) into the form

$$R = LU\,, \tag{1.20}$$

where L and U are defined by (1.18) and (1.19), respectively.

The spacetime algebra associated with a single spacetime point has been discussed. If spacetime is geometrically flat, then, with one point chosen as the zero vector, it is identical with the tangent space at each of its points. In this case there is only one spacetime algebra, and the spacetime points have all the properties of proper vectors mentioned above.

In the rest of this paper spacetime will be assumed geometrically flat. However, the basic ideas and most of the results apply with little or no modification to curved spacetime. To make such applications easier in the future, the definition of proper multivectors has been given in greater generality than is needed in this paper. The mathematical apparatus needed to apply spacetime algebra to curved spacetime is developed in Refs. 1 and 2.

2. The Proper Point of View

The *history* of a material particle is a timelike curve $x = x(\tau)$ in spacetime. Particle conservation is expressed by assuming that the function $x = x(\tau)$ is single-valued and continuous, except at discrete points where particle creation and/or annihilation occurs. Only differentiable particle histories will be considered here, and τ will always refer to the *proper time* (arc length) of a particle history. After a unit of length (say centimeters) has been chosen, the physical significance of the spacetime metric is fixed by the assumption that the proper time of a material particle is equal to the time (in centimeters) recorded on a material clock traveling with the particle.

The unit tangent $v = v(\tau) = dx/d\tau = \dot{x}$ of a particle history will be called the (*proper*) *velocity* of the particle. By the definition of proper time, $d\tau = |\,dx\,| = |\,(dx)^2\,|^{1/2}$, and

$$v^2 = 1\,. \tag{2.1}$$

The term "proper velocity" is to be preferred to the alternative terms "absolute velocity," "world velocity," "invariant velocity," and "four velocity." The adjective "proper" is used to emphasize that the velocity v describes an intrinsic property of the particle, independent of any observer or coordinate system. The adjective "absolute" would do the same, but it may not be free from undesirable connotations. Moreover, the word "proper" is shorter and has already been used in the same sense in the terms "proper mass" and "proper time." The adjective "invariant" is inappropriate, because no transformation group has been introduced. The velocity will not be called a "4-vector" because that term already means pseudoscalar in spacetime algebra; besides, there is no need to refer to any four components of the velocity.

The quantity $dv/d\tau = \dot{v} = \ddot{x}$ will be called the (*proper*) *acceleration* of the particle. The constraint (2.1) implies that $\dot{v}$ is orthogonal to v, that is

$$\dot{v} \cdot v = 0\,. \tag{2.2a}$$

or, equivalently, by virtue of (1.2a),

$$\dot{v}v = \dot{v} \wedge v = -v\dot{v}\,. \tag{2.2b}$$

The motion of a particle is said to be *inertial* if $\dot{v} = 0$.

The physical notion of an *inertial observer* (or system) is fully characterized mathematically by specifying a constant *timelike vector field* u, which, of course, can be constructed from the proper velocity u of a single inertial particle. It is often convenient to regard an inertial observer as an inertial particle with its history passing through the point $x = 0$. The language can be considerably simplified by using the proper velocity of an observer as the *name* of the observer. A description of the motion of a particle according to an observer is, then, just a description of the motion of one particle relative to another.

Let u be an inertial observer and x any spacetime point (labeling some physical event). By virtue of (1.2),

$$xu = x \cdot u + x \wedge u = ct + \mathbf{x}\,, \tag{2.3a}$$

where

$$ct = x \cdot u\,, \tag{2.3b}$$

$$\mathbf{x} = x \wedge u\,. \tag{2.3c}$$

The quantities t and $\mathbf{x}$ are, respectively, the *time* and *position* of the event x according to the observer u. For fixed t and variable x, (2.3b) is an equation for a spacelike hyperplane with normal u, and each point x of the hyperplane is uniquely designated by $\mathbf{x} = x \wedge u$. For variable t, (2.3b) is an equation for a one parameter family of spacelike hyperplanes. The time t designating a hyperplane is the proper time of the observer expressed in convenient units (say seconds); the constant c (with value equal to the speed of light) converts the unit of time into the unit of length.

Note that, by virtue of (1.2), (2.3) gives

$$ux = u \cdot x + u \wedge x = x \cdot u - x \wedge u = ct - \mathbf{x}\,,$$

Using this and $u^2 = 1$, one finds

$$x^2 = (xu)(ux) = (ct + \mathbf{x})(ct - \mathbf{x}) = c^2t^2 - \mathbf{x}^2\,, \tag{2.4}$$

a familiar expression for the "interval" between the event 0 and an event x.

Let $x = x(\tau)$ be the history of a particle with proper velocity $v = dx/d\tau$. Differentiating (2.3a), one finds

$$vu = \frac{d}{d\tau}(xu) = c\frac{dx}{d\tau} = v \cdot u + v \wedge u\,.$$

Introducing the abbreviation $\gamma = v \cdot u = cdt/d\tau$ and defining the relative velocity $\mathbf{v}$ by

$$\mathbf{v} \equiv \frac{d\tau}{dt}\frac{d\mathbf{x}}{d\tau} = c\,\frac{v \wedge u}{v \cdot u}\,, \tag{2.5a}$$

one obtains

$$vu = \gamma(1 + \mathbf{v}/c) = L^2\,, \tag{2.5b}$$

where L is the spinor introduced in (1.18) to describe the boost of u into v. Since both v and u are unit vectors, one obtains from (2.5b)

$$1 = v^2 = (vu)(uv) = \gamma(1 + \mathbf{v}/c)\gamma(1 - \mathbf{v}/c) = \gamma^2(1 - \mathbf{v}^2/c^2)\,.$$

Hence

$$\gamma \equiv v \cdot u = c\frac{dt}{d\tau} = (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}\,. \tag{2.5c}$$

Any proper bivector which can be expressed as the outer product $a \wedge u$ of an observer u with some vector a may be called a *relative vector* (relative to u of course) and denoted by a letter in boldface type, as in (2.3c) and (2.5a). It is not difficult to show that the set of all relative vectors is a three-dimensional linear space, so that *relative position vectors* of the form (2.3c) may serve as labels for (or, indeed, as a definition of) the three-dimensional "physical space" of the observer u. The adjective "relative" serves to distinguish "relative vectors" from "proper vectors" and to emphasize that they describe a particular relation to an observer, but it may be omitted when understood from the context or the use of boldface type. Any proper vector can be re-expressed as an equivalent sum of a relative scalar and a relative vector by multiplying it by u, as has already been shown, for example, by (2.3a) and (2.5b). In this way a proper description of physical events can be reformulated as a *relative description* of events. Several more important examples will be given to show how easily this is accomplished with spacetime algebra.

Let p be the *proper momentum* (i.e., the energy-momentum vector) of a particle. Multiplying by u, one obtains from p the *energy* (or relative mass) E and the *relative momentum* p; thus

$$pu = p\cdot u + p\wedge u = E + c\mathbf{p}\,, \tag{2.6a}$$

$$E \equiv p\cdot u\,, \tag{2.6b}$$

$$\mathbf{p} = c^{-1}p\wedge u\,. \tag{2.6c}$$

For "physical particles" the *proper* (or *rest*) *mass* m is defined by the equation $p^2 = m^2c^4 \geq 0$. The relation of proper mass energy and relative momentum can be obtained from (2.6a); thus

$$p^2 = (pu)(up) = (E + c\mathbf{p})(E - c\mathbf{p}) = E^2 - c^2p^2 = m^2c^4\,. \tag{2.7}$$

For material particles $m \neq 0$, and if the momentum is related to the velocity by the equation

$$p = mc^2v \tag{2.8a}$$

one has, from (2.6c), the famous expressions

$$E = mc^2\gamma = mc^2(1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}\,, \tag{2.8b}$$

$$\mathbf{p} = \frac{E}{c^2}\mathbf{v} = m\gamma\mathbf{v} = \frac{m\mathbf{v}}{(1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}}\,. \tag{2.8c}$$

Like the geometric product of proper vectors in (1.2) the geometric product of relative vectors $\mathbf{a}$ and $\mathbf{b}$ can be decomposed into an inner product $\mathbf{a}\cdot\mathbf{b}$ and an outer product $\mathbf{a}\wedge\mathbf{b}$; thus

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}\,, \tag{2.9a}$$

$$\mathbf{a}\cdot\mathbf{b} = \tfrac{1}{2}(\mathbf{ab} + \mathbf{ba})\,, \tag{2.9b}$$

$$\mathbf{a}\wedge\mathbf{b} = \tfrac{1}{2}[\mathbf{a},\mathbf{b}] = i\mathbf{a}\times\mathbf{b}\,. \tag{2.9c}$$

Equation (2.9c) expresses the relative *bivector* $\mathbf{a}\wedge\mathbf{b}$ as the *dual* of a *relative vector* $\mathbf{a}\times\mathbf{b}$, the i being the unit pseudoscalar already introduced in (1.8) and (1.9). The right side of (2.9c) can be regarded as a definition of the vector *cross product* $\mathbf{a}\times\mathbf{b}$. For further discussion of this relation see Refs. 3 and 1.

By multiplication and addition the relative vectors generate an algebra which is, in fact, exactly the even subalgebra of the complete spacetime algebra. Indeed, any element E of the even subalgebra can be written in the form

$$[E] = [E]_0 + [E]_2 + [E]_4 = [E]_0 + [E]_1 + [E]_2 + [E]_3\,, \tag{2.10a}$$

where

$$[E]_0 = [E]_0\,, \tag{2.10a}$$

$$[E]_2 = [E]_1 + [E]_2\,, \tag{2.10b}$$

$$\mathbf{a}\wedge\mathbf{b} = \tfrac{1}{2}[\mathbf{a},\mathbf{b}] = i\mathbf{a}\times\mathbf{b}\,. \tag{2.10c}$$

As in (1.10), $[E]_k$ indicates the *proper* k-vector part of E. Similarly, $[E]_{\mathbf{k}}$ indicates the *relative* **k**-vector part. Of the three relations (2.10b)-(2.10d), (2.10c) is of the most interest here. It says that any proper bivector can be expressed as the sum of a relative vector and a relative bivector. To see how this decomposition can be carried out, consider the proper bivector F representing the electromagnetic field at some spacetime point. Note that, by (1.4),

$$F = Fu^2 = (F \cdot u + F \wedge u)\, u\,,$$

so

$$F = \mathbf{E} + i\mathbf{B}\,, \tag{2.11a}$$

where

$$\mathbf{E} \equiv F \cdot u u = (F \cdot u) \wedge u = [F]_1\,, \tag{2.11b}$$

$$i\mathbf{B} \equiv F \wedge u u = (F \wedge u) \cdot u = [F]_2\,. \tag{2.11c}$$

The relative vector $\mathbf{E}$ is the *electric field according to the observer* u. The wedge in (2.11b) can be included or omitted as desired; this follows from (1.2a), since $F \cdot u$ is a proper vector which is orthogonal to u, as shown by $(F \cdot u) \cdot u = F \cdot (u \wedge u) \equiv [Fu \wedge u]_0 = 0$. Similarly, by (1.7a) the dot in (2.11c) can be omitted or included at will because $(F \wedge u) \wedge u = F \wedge (u \wedge u) \equiv 0$. To justify the notation $\mathbf{B}$ indicating a relative vector in (2.11c), note that

$$B = -iF \wedge u u = (-iF) \cdot u u = [(-iF) \cdot u] \wedge u\,, \tag{2.12}$$

showing that the "proper expression" for $\mathbf{B}$ has the same form as the one for $\mathbf{E}$ if only the electromagnetic field F is replaced by its *dual* $-iF$, which is also a proper bivector. The relative vector $\mathbf{B}$ is the *magnetic field according to the observer* u.

In "proper notation" the classical equation of motion for a "test particle" with charge e and mass m takes the form

$$\dot{p} = mc^2 \dot{v} = eF \cdot v\,, \tag{2.13}$$

with all symbols being defined as before, and, of course, $F = F(x(\tau))$. To reexpress (2.13) in "relative notation," it is helpful to note that

$$F^\dagger \equiv u\mathbf{F}u = -uFu = \mathbf{E} - i\mathbf{B}\,. \tag{2.14}$$

So, with the help of (1.4a), (2.5b), (2.11), and (2.9),

$$\begin{aligned} (F \cdot v)u &= \tfrac{1}{2}(Fv - vF)u = \tfrac{1}{2}(Fvu + uvF^\dagger) \\ &= \gamma\tfrac{1}{2}(\mathbf{E}(1 + \mathbf{v}/c) + (1 + \mathbf{v}/c)\mathbf{E} + \gamma\tfrac{1}{2}[i\mathbf{B}, (1 + \mathbf{v}/c)] \\ &= \gamma[\mathbf{E} \cdot \mathbf{v}/c + \mathbf{E} + i\mathbf{B} \wedge \mathbf{v}/c]\,. \end{aligned} \tag{2.15}$$

But (2.13) gives

$$\dot{p}u = \frac{d}{d\tau}(pu) = \frac{\gamma}{c}\frac{d}{dt}(E + c\mathbf{p}) = e(F \cdot v)\, u\,.$$

So

$$c\gamma^{-1}\dot{p} \cdot u = \frac{dE}{dt} = e\mathbf{E} \cdot \mathbf{v}\,. \tag{2.16a}$$

and

$$\gamma^{-1}\dot{p} \wedge u = \frac{d\mathbf{p}}{dt} = e(\mathbf{E} + c^{-1}\mathbf{v} \times \mathbf{B})\,, \tag{2.16b}$$

the usual relative vector form for the Lorentz force.

Obviously the decomposition (2.11) of the electromagnetic field F into electric and magnetic fields depends on the observer. The observer need not be inertial. Thus, the proper velocity $v = v(\tau)$ of a particle in arbitrary motion determines an instantaneous rest frame of the particle in which the electric field is

$$\mathbf{E}_v \equiv F \cdot v v = (F \cdot v) \wedge v \tag{2.17a}$$

and the magnetic field $\mathbf{B}_v$ is given by

$$i\mathbf{B}_v \equiv F \wedge v v = (F \wedge v) \cdot v\,, \tag{2.17b}$$

so that

$$F \equiv \mathbf{E}_v + i\mathbf{B}_v \,. \tag{2.17c}$$

The subscript v indicates the rest system. Some such notation is necessary when relative vectors in more than one rest system are considered. The relative acceleration of the particle itself in its own inertial system is

$$\mathbf{a}_v \equiv c^2 \dot{v} \wedge v = c^2 \ddot{x} \wedge v \,. \tag{2.18}$$

Multiplying (2.13) by v and using (2.26) along with (2.17a) and (2.18), one finds

$$m\mathbf{a}_v = mc^2 \dot{v}\, v = eF \cdot v\, v = e\mathbf{E}_v \,, \tag{2.19}$$

which says that a charge at (relative) rest is accelerated by an electric but not a magnetic field. Indeed, it is by (2.19) that an electric field is defined in the first place.

Now, as one more example and for later use, the proper acceleration $\dot{v}$ will be expressed in relative form. From (2.5b),

$$\dot{v}u = \frac{d}{d\tau}(v\,u) = \dot{\gamma}(1 + \mathbf{v}/c) + \gamma\dot{\mathbf{v}}/c \,. \tag{2.20}$$

Now

$$\dot{\mathbf{v}} = \frac{d\mathbf{v}}{d\tau} = \frac{dt}{d\tau}\frac{d\mathbf{v}}{dt} = c^{-1}\gamma\mathbf{a}$$

where

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{2.21}$$

is the *relative acceleration* of the particle. The quantity $\dot{\gamma}$ can be related to $\mathbf{a}$ by direct differentiation of (2.5c), but it is easier and more instructive to use (2.2). For this reason, consider

$$\begin{aligned} \dot{v}\,v &= (\dot{v}u)(u v) = [\dot{\gamma}(1 + \mathbf{v}/c) + \gamma\dot{\mathbf{v}}/c]\gamma(1 - \mathbf{v}/c) \\ &= \gamma[\dot{\gamma}(1 - \mathbf{v}^2/c^2) + c^{-1}\gamma\dot{\mathbf{v}}(1 - \mathbf{v}/c)] \,. \end{aligned}$$

The scalar part $\dot{v} \cdot v = 0 = \gamma[\dot{\gamma}(1 - \mathbf{v}^2/c^2) - c^{-1}\gamma\dot{\mathbf{v}} \cdot \mathbf{v}/c]$, so, recalling (2.5c), one finds

$$\dot{\gamma} = c^{-2}\gamma^3 \dot{\mathbf{v}} \cdot \mathbf{v} = c^{-3}\gamma^4 \mathbf{v} \cdot \mathbf{a} = \dot{v} \cdot a = \frac{d^2 t}{d\tau^2} \tag{2.22}$$

The bivector part is simply

$$\dot{v}\,v = \dot{v} \wedge v = c^{-1}\gamma^2(\dot{\mathbf{v}} - c^{-1}\dot{\mathbf{v}} \wedge \mathbf{v}) = c^{-2}\gamma^3(\mathbf{a} + c^{-1} i\mathbf{v} \times \mathbf{a}) \,. \tag{2.23}$$

Substitution of (2.22) into the proper bivector part of (2.20) yields

$$\dot{v} \wedge u = c^{-1}(\gamma\dot{\mathbf{v}} + \dot{\gamma}\mathbf{v}) = c^{-2}\gamma^2(\mathbf{a} + c^{-2}\gamma^{-2}\mathbf{v} \cdot \mathbf{a}\,\mathbf{v}) \,.$$

But a more helpful expression can be obtained from (2.23); thus,

$$\dot{v}u = (\dot{v}\,v)(v\,u) = c^{-2}\gamma^3(\mathbf{a} + c^{-1}\mathbf{v} \wedge \mathbf{a})\gamma(1 + c^{-1}\mathbf{v}) \,,$$

the proper bivector part of which is

$$\dot{v} \wedge u = c^{-2}\gamma^4[\mathbf{a} + c^{-2}(\mathbf{v} \wedge \mathbf{a})\mathbf{v}] = c^{-2}\gamma^4[\mathbf{a} + c^{-2}\mathbf{v} \times (\mathbf{v} \times \mathbf{a})] \,. \tag{2.24}$$

3. The Covariant Point of View

Before continuing the proper description of mechanics, a brief discussion of its relation to more conventional formulations may be helpful.

Given an orthonormal frame $\{\gamma_\mu\}$, the coefficients $g_{\mu\nu}$ of the metric tensor (relative to that frame) are determined by the equation

$$g_{\mu\nu} = \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \gamma_\mu \cdot \gamma_\nu \,. \tag{3.1}$$

This equation will appear familiar to anyone acquainted with the Dirac matrix algebra. Indeed, the spacetime algebra used here is algebraically isomorphic to the algebra generated by the Dirac matrices over the real numbers (a subalgebra of the full Dirac algebra over the complex numbers). It is important to understand the differences between these algebras. The γ_μ in (3.1) are regarded as vectors, whereas the corresponding Dirac matrices are ordinarily related to vectors only indirectly with the help of spinors. The Dirac matrices are hardly used except in connection with spin-$\frac{1}{2}$ particles, so one gets the impression that the Dirac algebra merely describes some property of spin. On the contrary, (3.1) is here a direct expression of the metric of spacetime as a rule for multiplying vectors, from which it follows that the full spacetime algebra directly expresses basic geometrical properties of spacetime. It is as applicable to any classical theory as it is to the quantum theory of spin-$\frac{1}{2}$ particles. The fact that the γ_μ can be represented by 4×4 matrices is irrelevant to any geometrical or physical application of spacetime algebra. Indeed, matrices introduce unnecessary mathematical complications and obscure interpretations even in the Dirac electron theory. This has been established in Refs. 4 and 5 and will be discussed more fully in a forthcoming paper.

With $u = \gamma_0$ being the proper velocity of an inertial observer, the relative vectors

$$\boldsymbol{\sigma}_i \equiv \gamma_i\gamma_0 = \gamma_i \wedge \gamma_0 \qquad (i = 1, 2, 3) \tag{3.2}$$

compose a basis for the space of all relative vectors. The $\boldsymbol{\sigma}_i$ can be represented by the 2×2 Pauli matrices, from which it follows that the even subalgebra of the spacetime algebra is isomorphic to the Pauli matrix algebra. But again, matrices are of negative value. For example, from (3.2) one obtains

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i = \gamma_0\gamma_1\gamma_2\gamma_3 \,, \tag{3.3}$$

where i is the unit pseudoscalar, a fundamental geometrical quantity; on the other hand, no geometrical significance is ordinarily attributed to the corresponding matrix equation. Moreover, the simple relations (3.2) and (3.3) between the γ_μ and the $\boldsymbol{\sigma}_i$ do not obtain if the γ_μ are to be represented by 4×4 matrices while the $\boldsymbol{\sigma}_i$ are represented by 2×2 matrices. For purposes of comparison with matrix representations of the Lorentz group, it should be noted that (3.2) enables one to write $A = a_\mu^\nu \gamma_\nu\gamma_0\gamma_0\gamma^\mu = a_0^0 + (a_i^0 + a_0^i)\boldsymbol{\sigma}_i + a_j^i \boldsymbol{\sigma}_i\boldsymbol{\sigma}_j$ in (1.16). So (1.16) can be represented as a 2×2 matrix, which the work of MacFarlane[6] shows immediately to be the representation of a Lorentz transformation in $\mathrm{SL}_2(C)$.

To transcribe proper equations into covariant tensor form, it is necessary to introduce a set of coordinates $\{x^\mu = x^\mu(x);\ u = 0, 1, 2, 3\}$. It suffices to consider a set of "Cartesian coordinates," which can always be written in the form

$$x^\mu = x^\mu(x) = x \cdot \gamma^\mu \,, \tag{3.4a}$$

where $\{\gamma^\mu\}$ is an orthonormal frame of constant vectors with reciprocal frame $\{\gamma_\mu\}$. Equation (3.4a) expresses the coordinates as a function of the point x. The inverse function expressing the point x as a function of the coordinates $\{\gamma^\mu\}$ is

$$x = x(x^0,\, x^1,\, x^2,\, x^3) = x^\mu\gamma_\mu \,. \tag{3.4b}$$

One readily verifies that

$$\Box x^\mu = \gamma^\mu \,, \tag{3.5a}$$

$$\partial_\mu x = \gamma_\mu \,, \tag{3.5b}$$

where

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \gamma_\mu \cdot \Box \qquad \text{and} \qquad \Box = \gamma^\mu \partial_\mu \,. \tag{3.5c}$$

Indeed, the relations of the form (3.5a, b, c) are completely general, obtaining for any set of coordinates.

As an example, the classical "Lorentz equation" (2.13) will be put in covariant form. The components of the velocity are

$$v^\mu = v \cdot \gamma^\mu \qquad \text{and} \qquad v_\mu = v \cdot \gamma_\mu \,,$$

Since the γ^μ are constant,

$$\dot{v}\cdot\gamma^\mu = \frac{d}{d\tau}(v\cdot\gamma^\mu) = \frac{dv^\mu}{d\tau}\,. \tag{3.5c}$$

The tensor components of the electromagnetic field F are

$$F^{\mu\nu} = \gamma^\mu\cdot F\cdot\gamma^\nu = F\cdot(\gamma^\mu\wedge\gamma^\nu) = -F^{\nu\mu},$$

and the expression for F in terms of the $F^{\mu\nu}$ is

$$F = \tfrac{1}{2}F^{\mu\nu}\gamma_\mu\wedge\gamma_\nu\,.$$

So, with the help of (1.5),

$$\begin{aligned}
&F\cdot v = \tfrac{1}{2}F^{\mu\nu}(\gamma_\mu\wedge\gamma_\nu)\cdot v = \tfrac{1}{2}F^{\mu\nu}(\gamma_\mu v_\nu - v_\mu\gamma_\nu) = F^{\mu\nu}\gamma_\mu v_\mu\,,\\
&\gamma^\mu\cdot F\cdot v = F^{\mu\nu}v_\nu\,.
\end{aligned}$$

Thus, if (2.13) is "dotted" with γ^μ, it can be given the familiar covariant tensor form

$$mc^2\frac{dv^\mu}{d\tau} = eF^{\mu\nu}v_\nu\,. \tag{3.6}$$

The covariant equation (3.6) describes the motion of a "test charge" relative to an arbitrarily chosen set of (Cartesian) coordinates. In contrast, the proper equation (2.13) is simpler because it is formulated and, as will be shown in a subsequent paper, can be solved without reference to any set of scalar coordinates.

It is a simple matter to reexpress any covariant tensor equation in proper form. But the converse is not true; for example, the important spinor representation (1.13) of a Lorentz rotation has no simple tensor form, nor, of course, does the Dirac equation. Therefore, the spacetime algebra is a more powerful mathematical tool than conventional tensor analysis.

4. Proper Kinematics of a Rigid Point Particle

As before, let v and $\dot{v}$ be, respectively, the proper velocity and the proper acceleration of a material point particle. From the fact that $\dot{v}\cdot v = 0$, it follows that it is always possible to find a bivector valued function $\Omega = \Omega(\tau)$ such that

$$\dot{v} = \Omega\cdot v\,. \tag{4.1}$$

Indeed, as shown by (2.17) and (2.11), Ω submits to the decomposition

$$\Omega = \alpha_\nu + i\beta = \dot{v}v + B\,, \tag{4.2a}$$

where

$$\begin{aligned}
\alpha_\nu &\equiv \Omega\cdot v\,v = \dot{v}v = \dot{v}\wedge v\,,\\
i\beta_\nu &\equiv \Omega\wedge v\,v = (\Omega\wedge v)\cdot v \equiv B\,.
\end{aligned}$$

Noting that

$$B\cdot v = 0\,, \tag{4.2b}$$

and using the identity (1.5), one shows easily that (4.2a) satisfies (4.1). So any choice of B in (4.2a) will satisfy (4.1) provided only that $B\cdot v = 0$.

A *comoving frame* of vectors $e_\mu = e_\mu(\tau)$ $(\mu = 0,1,2,3)$ can be introduced by the equations

$$e_\mu = R\gamma_\mu\widetilde{R} \qquad \text{with} \qquad e_0 = v = \dot{x}\,, \tag{4.3a}$$

where $\{\gamma_\mu\}$ is a fixed orthonormal frame of vectors, and $R = R(\tau)$ is a unimodular spinor, i.e.,

$$R\widetilde{R} = 1\,, \tag{4.3b}$$

with the equation of motion

$$\dot{R} \equiv \frac{dR}{d\tau} = \tfrac{1}{2}\Omega R\,. \tag{4.3c}$$

Frequently, it is convenient to adopt the initial condition

$$R(0) = 1\,, \qquad \text{or equivalently} \qquad e_\mu(0) = \gamma_\mu\,, \tag{4.4}$$

but this will not be required in this section.

Equation (4.3a) is a Lorentz rotation of the frame $\{\gamma_\mu\}$ into the frame $\{e_\mu(\tau)\}$ determined by the spinor $R(\tau)$. From (4.3a,b) one shows easily that $e_\mu \cdot e_\nu = \gamma_\mu \cdot \gamma_\nu$, so the e_μ, are orthonormal. The quantity Ω is the angular velocity of the spinor-valued function $R = R(\tau)$. Solving (4.3c) for Ω and differentiating (4.3b), one finds

$$\Omega = 2\dot{R}\widetilde{R} = -2R\dot{\widetilde{R}} = -2R\widetilde{\dot{R}} = -\widetilde{\Omega}\,. \tag{4.5}$$

Since R is an even multivector, so is Ω; more particularly, by virtue of (1.11), (4.5) implies that Ω is a bivector.

By differentiating (4.3a) one can obtain, with the help of (4.5), a set of differential equations for the $\{e_\mu\}$ which is equivalent to the single spinor equation (4.3c); thus

$$\begin{aligned}\dot{e}_\mu &= \dot{R}\gamma_\mu\widetilde{R} + R\gamma_\mu\dot{\widetilde{R}}\\ &= \tfrac{1}{2}(2\dot{R}\widetilde{R})R\gamma_\mu\widetilde{R} + \tfrac{1}{2}R\gamma_\mu\widetilde{R}(2R\dot{\widetilde{R}})\,.\end{aligned}$$

or

$$\dot{e}_\mu = \tfrac{1}{2}[\Omega, e_\mu] = \Omega \cdot e_\mu\,. \tag{4.6}$$

This displays Ω as the *angular velocity* of the comoving frame, and for $\mu = 0$ it is seen to be identical to (4.1).

The arbitrariness in Ω which exists when (4.1) is considered alone obviously does not exist when the complete equations (4.6) for a comoving frame are given. But this is worth proving by solving (4.6) explicitly for Ω. Introducing the reciprocal frame $\{e^\mu\}$ defined by the equations

$$e^\mu \cdot e_\nu = \gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu \qquad (\mu,\, \nu = 0,\, 1,\, 2,\, 3)\,, \tag{4.7}$$

one can prove the identities

$$e_\mu e^\mu = e_\mu \cdot e^\mu = 4\,, \tag{4.8}$$

$$e_\mu \Omega e^\mu = 0 \tag{4.9}$$

(sum over repeated indices). Identity (4.9) requires that Ω be a bivector. Multiplying (4.6) by e^μ and summing, one gets

$$\dot{e}_\mu e^\mu = \dot{e}_\mu \wedge e^\mu = (\Omega \cdot e_\mu)\, e^\mu = \tfrac{1}{2}(\Omega\, e_\mu - e_\mu \Omega)e^\mu = \tfrac{1}{2}\Omega 4\,.$$

So

$$\Omega = \tfrac{1}{2}\dot{e}_\mu e^\mu = \tfrac{1}{2}\dot{e}_\mu \wedge e^\mu\,. \tag{4.10}$$

Clearly, there are many comoving frames which can be associated with the history of a particle, since with only the conditions set down so far the history $x = x(\tau)$ itself determines only one of the e_μ, the velocity $e_0 = v = \dot{x}$. A frame more intimately related to the history is easy to construct. Suppose the angular velocity Ω has the form

$$\Omega = \kappa_1 e_1 e_0 + \kappa_2 e_1 e_2 + \kappa_3 e_2 e_3 \tag{4.11}$$

where the κ_i ($i = 1,\, 2,\, 3$) are scalar quantities. Substitution of (4.11) into (4.6) yields, with the help of identity (1.5),

$$
\begin{aligned}
\dot{e}_0 &= \kappa_1 e_1 \,, \\
\dot{e}_1 &= \kappa_1 e_0 \ + \kappa_2 e_2 \,, \\
\dot{e}_2 &= -\kappa_1 e_1 + \kappa_3 e_3 \,, \\
\dot{e}_3 &= -\kappa_3 e_2 \,.
\end{aligned} \tag{4.12}
$$

These are the so-called Frenet-Serret equations for the particle history. It follows that the ith *curvature* κ_i of the history satisfies

$$\kappa_i = e_i \cdot \dot{e}_{i-1} = e_i \cdot \Omega \cdot e_{i-1} = \Omega \cdot (e_{i-i} \wedge e_i) \,. \tag{4.10}$$

The angular velocity Ω of the *Frenet frame* $\{e_\mu\}$ satisfying (4.12) is called the Darboux bivector, because it generalizes the *Darboux vector* of classical differential geometry. It is not difficult to show that the Frenet frame determines all the derivatives of $x = x(\tau)$, and conversely, if none of the κ_i vanish the Frenet frame is uniquely determined by the derivatives of the history. One important feature of the formulation given here is that the single spinor equation (4.3c) with Ω related to the e_μ by (4.11) may be easier to solve than the simultaneous set of equations (4.12).

In spite of the geometrical significance of Frenet frames, other choices of a comoving frame are more important physically. Every material particle has some structure, usually because it approximates some extended body. A comoving frame can be used as a basic description of such structure. In particular, the comoving vectors e_1, e_2, e_3 may be used to specify a frame fixed in a rigid body (of negligible dimensions) moving with the particle; then Ω is the *proper angular velocity* of the rigid body and the spinor R completely describes any changes in orientation of the body. With this interpretation Eqs. (4.3) will be said to describe a *rigid* (*point*) *particle*. The dynamics of a rigid particle can be described by relating Ω to the motion of other physical system. But to facilitate the analysis of dynamics, it is worthwhile first to study the general kinematics of comoving frames in more detail.

The Lorentz rotation (4.3a) can be decomposed into a spatial rotation and a boost in the manner described in Section 1. Take $u = \gamma_0$ and write as before

$$R = LU \,, \tag{4.14}$$

where L satisfies (1.18) and U satisfies (1.19). By substituting (4.14) into (4.3c), an equation of motion for the spinor U can be obtained; thus

$$R = \dot{L}U + \dot{U}L = \tfrac{1}{2}LU \,.$$

So, since $\widetilde{L}L = 1$,

$$\dot{U} = \tfrac{1}{2}\omega U \,, \tag{4.15a}$$

where

$$\omega = \widetilde{L}\Omega L - 2\widetilde{L}\dot{L} \,. \tag{4.15b}$$

The angular velocity ω can be separated into two parts

$$\omega = \omega_T + \omega_L \,, \tag{4.16a}$$

where

$$\omega_T \equiv \widetilde{L}\dot{v}vL - 2\widetilde{L}\dot{L} \,, \tag{4.16b}$$

$$\omega_L \equiv \widetilde{L}BL = [\widetilde{L}\Omega L]_2 \,. \tag{4.16c}$$

To prove this, note that since $v = Ru\widetilde{R} = Lu\widetilde{L}$,

$$\widetilde{L}(\Omega \cdot v + \Omega \wedge v)L = \widetilde{L}\Omega v L = \widetilde{L}\Omega L u = (\widetilde{L}\Omega L) \cdot u + (\widetilde{L}\Omega L) \wedge u \,.$$

Since a Lorentz rotation does not mix multivectors of different degree, one gets by separately equating vector and trivector parts

$$\widetilde{L}\Omega \cdot vL = (\widetilde{L}\Omega L) \cdot u \qquad \text{and} \qquad \widetilde{L}\Omega \wedge vL = (\widetilde{L}\Omega L) \wedge v \,,$$

which on multiplication by u gives, as in (2.11),

$$\widetilde{L}\Omega \cdot vL = (\widetilde{L}\Omega L) \cdot v\,u = [\widetilde{L}\Omega L]_1\,, \tag{4.17a}$$

and

$$\widetilde{L}\Omega \wedge v\,vL = (\widetilde{L}\Omega L) \wedge u\,u = [\widetilde{L}\Omega L]_2\,. \tag{4.17b}$$

Recalling (4.2), one obtains (4.16) immediately by using (4.17a,b) in (4.15b).

The rigid frame $\{e_i = R\gamma_i\widetilde{R}; i = 1,2,3\}$ describes the orientation of a rigid body in the *instantaneous rest system* of the particle. The rigid frame provides an equivalent description of the rigid body in the *inertial system* U obtained by a (de)-boost from v. Alternatively, in the inertial system it is convenient to use the frame of relative vectors

$$\mathbf{e}_i \equiv U\boldsymbol{\sigma}_i U^\dagger = U\gamma_i\widetilde{U}\gamma_0 = \widetilde{L}\,e_i vL \tag{4.18}$$

where, as before, $\boldsymbol{\sigma}_i = \gamma_i\gamma_0$ and $U^\dagger = \gamma_0\widetilde{U}\gamma_0$. Differentiating (4.18) and using (4.15a) as well as $U^\dagger U = 1$, one finds the equation of motion for the $\mathbf{e}_i$;

$$\dot{e}_i = \omega \cdot \mathbf{e}_i = \omega_T \cdot \mathbf{e}_i + \omega_L \cdot \mathbf{e}_i\,. \tag{4.19}$$

These equations describe a precession of the rigid body which according to (4.16) can be separated into two parts, the *Thomas precession* with angular velocity ω_T which is due to the acceleration of the particle, and the (generalized) *Larmor precession* with angular velocity ω_L of a nonaccelerated body.

The Thomas precession can be expressed in terms of u, v, and $\dot{v}$. Introducing the symbol w for the angular velocity of the boost, one has

$$\dot{L} = \tfrac{1}{2}wL \qquad \text{or} \qquad w = 2\dot{L}L = -2L\dot{\widetilde{L}}\,. \tag{4.20}$$

Differentiating $L^2 = v\,u$,

$$\dot{v}u = \frac{dL^2}{d\tau} = \dot{L}L + L\dot{L} = \tfrac{1}{2}(wL^2 + LwL) = \tfrac{1}{2}(w + Lw\widetilde{L})L^2\,,$$

then dividing by $\frac{1}{2}u$ and using $\widetilde{L}v = u\widetilde{L}$, one gets

$$2\dot{v} = wv + Lwu\widetilde{L}\,. \tag{4.21}$$

Now since L is a function of u and v only, the bivector w is a function of the vectors u, v, and $\dot{v}$ only hence the trivector $w \wedge u$ must be proportional to $\dot{v} \wedge v \wedge u$. It follows, then, from (1.18) that $Lw \wedge u\widetilde{L} = w \wedge u$. So the trivector part of (4.21) yields the equation

$$w \wedge (v + u) = 0\,. \tag{4.22}$$

This can be solved for w by dotting with v and using (1.8), thus

$$[w \wedge (v+u)] \cdot v = w(v+u) \cdot v - (w \cdot v) \wedge (v+u) = 0\,,$$

and since $\dot{v} = w \cdot v$, which is easily established by differentiating $v = Ru\widetilde{R} = Lu\widetilde{L}$, one obtains

$$w = 2\dot{L}\widetilde{L} = \frac{\dot{v} \wedge (v+u)}{v \cdot (v+u)} = \frac{\dot{v}\,v + \dot{v} \wedge u}{1 + v \cdot u}\,. \tag{4.23}$$

Now from the vector part of (4.21) one finds, again using $\dot{v} = w \cdot v$

$$\dot{v} = Lw \cdot u\widetilde{L} = w \cdot v\,, \tag{4.24}$$

from which one easily obtains the following expression for the relative vector part of w

$$[w]_1 = (w \cdot u)u = \widetilde{L}\dot{v}vL\,. \tag{4.25}$$

This result can also be obtained directly from (4.16b) by using the fact that $\omega_T = [w_T]_2$ which can be proved from (4.15a) and (4.16c).

From (4.23) and (4.25) one obtains an expression for the relative bivector part of w:

$$[w]_2 = (w \wedge u)u = \frac{(\dot{v} \wedge v \wedge u)u}{1 + v \cdot u} = 2\dot{L}\widetilde{L} - \widetilde{L}\dot{v}vL\,. \tag{4.26}$$

This is, in fact, identical to the Thomas expression (4.16b). To show this, recall from (1.18b) that $\widetilde{L} = uLu$; so, by (4.20),

$$2\widetilde{L}\dot{L} = u(2Lu\dot{L}) = u(2L\dot{\widetilde{L}})u = -uwu = [w]_1 - [w]_2\,.$$

the last step being the same as in (2.14)

To sum up, the Thomas angular velocity ω_T can be written in the several different forms:

$$\begin{aligned}\omega_T = -[2\widetilde{L}\dot{L}]_2 = [2\dot{L}\widetilde{L}]_2 &= \frac{(\dot{v} \wedge v \wedge u)u}{1 + v \cdot u} \\ = \frac{[\dot{v}v]_2}{1 + v \cdot u} &= \frac{\gamma^3}{c^3(1+\gamma)}\, i\mathbf{v} \times \mathbf{a}\,,\end{aligned} \tag{4.27}$$

the last expression as a relative bivector being obtained directly from (2.23); it is identical to that obtained by Thomas[7] and again by Bacry[8] in a review of Thomas' work.

The problem remains to express the Larmor bivector ω_L in terms of relative vectors. First express Ω in "relative form"

$$\Omega = \boldsymbol{\alpha} + i\boldsymbol{\beta}\,, \tag{4.28a}$$

$$\boldsymbol{\alpha} = \Omega \cdot u u\,, \tag{4.28b}$$

$$i\boldsymbol{\beta} = \Omega \wedge u u\,. \tag{4.28c}$$

Then, write Ω in the form

$$\Omega = \Omega_\parallel + \Omega_\perp\,, \tag{4.29a}$$

$$\Omega_\parallel \equiv \tfrac{1}{2}(\Omega\hat{\mathbf{v}} + \hat{\mathbf{v}}\Omega)\hat{\mathbf{v}} = \boldsymbol{\alpha}_\parallel + i\boldsymbol{\beta}_\parallel\,, \tag{4.29b}$$

$$\Omega_\perp \equiv \tfrac{1}{2}[\Omega, \hat{\mathbf{v}}]\hat{\mathbf{v}} = \boldsymbol{\alpha}_\perp + i\boldsymbol{\beta}_\perp\,, \tag{4.29c}$$

where

$$\hat{\mathbf{v}} \equiv v \wedge u/|\,v \wedge u\,| = \mathbf{v}/|\,\mathbf{v}\,|$$

is the unit relative velocity of the particle, and

$$\boldsymbol{\alpha}_\perp \equiv \tfrac{1}{2}[\boldsymbol{\alpha}, \hat{\mathbf{v}}]\hat{\mathbf{v}} = \boldsymbol{\alpha} \wedge \hat{\mathbf{v}}\hat{\mathbf{v}} = -(\boldsymbol{\alpha} \times \hat{\mathbf{v}}) \times \hat{\mathbf{v}}\,, \tag{4.30a}$$

$$\boldsymbol{\alpha}_\parallel = \boldsymbol{\alpha} - \boldsymbol{\alpha}_\perp = \boldsymbol{\alpha} \cdot \hat{\mathbf{v}}\,\hat{\mathbf{v}}\,, \tag{4.30b}$$

and similar relations for $\boldsymbol{\beta}$. The significance of (4.29) lies in the fact that $\Omega_\parallel$ commutes with $\hat{\mathbf{v}}$ while $\Omega_\perp$ anticommutes with $\hat{\mathbf{v}}$ and since the bivector part of L is proportional to $\hat{\mathbf{v}}$, one has the relations

$$\widetilde{L}\Omega_\parallel L = \widetilde{L}L\Omega_\parallel = \Omega_\parallel\,, \tag{4.31a}$$

$$\widetilde{L}\Omega_\perp L = \widetilde{L}^2 L\Omega_\perp = \Omega_\perp L^2\,. \tag{4.31b}$$

Hence, using $L^2 = vu = \gamma(1 + c^{-1}\mathbf{v})$, one gets

$$\widetilde{L}\Omega L = \Omega_\parallel + \Omega_\perp L^2 = \Omega + (\gamma - 1)\Omega_\perp + c^{-1}\gamma\Omega_\perp\mathbf{v}\,. \tag{4.32}$$

Now (4.30a) shows that $\boldsymbol{\beta}_\perp \mathbf{v} = \boldsymbol{\beta} \wedge \mathbf{v} = i\boldsymbol{\beta} \times \mathbf{v}$, so

$$\Omega_\perp \mathbf{v} = \boldsymbol{\alpha} \wedge \mathbf{v} + i\boldsymbol{\beta} \wedge \mathbf{v} = -\boldsymbol{\beta} \times \mathbf{v} + i\,\boldsymbol{\alpha} \times \mathbf{v}\,. \tag{4.33}$$

Decomposing (4.30) into relative vector and bivector parts, one gets

$$\widetilde{L}\Omega L = \boldsymbol{\alpha} + (\gamma-1)\boldsymbol{\alpha}_\perp - c^{-1}\gamma\boldsymbol{\beta} \times \mathbf{v} + i\big(\boldsymbol{\beta} + (\gamma-1)\boldsymbol{\beta}_\perp + c^{-1}\gamma\boldsymbol{\alpha} \times \mathbf{v}\big)\,. \tag{4.34}$$

The relative bivector part of (4.34) gives the desired expression for the Larmor bivector:

$$\begin{aligned}\omega_L = [\widetilde{L}\Omega L]_2 &= i(\boldsymbol{\beta} + (\gamma-1)\boldsymbol{\beta}_\perp + c^{-1}\gamma\,\boldsymbol{\alpha} \times \mathbf{v}) \\ &= i\left(\boldsymbol{\beta} - \frac{c^{-2}\gamma^2}{(\gamma-1)}(\boldsymbol{\beta} \times \mathbf{v}) \times \mathbf{v} + c^{-1}\gamma\,\boldsymbol{\alpha} \times \mathbf{v}\right)\,.\end{aligned} \tag{4.35}$$

The Thomas bivector can also be expressed in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Replacing F by Ω in (2.15), one finds

$$\begin{aligned}\dot{v}v &= (\Omega \cdot v)v = (\Omega \cdot vu)uv \\ &= \gamma(c^{-1}\boldsymbol{\alpha} \cdot \mathbf{v} + c^{-1}\mathbf{v} \times \boldsymbol{\beta})\gamma(1 - c^{-1}\mathbf{v}) \\ &= \gamma^2(\boldsymbol{\alpha} + c^{-2}\boldsymbol{\alpha} \cdot \mathbf{v}\,\mathbf{v} + c^{-1}\mathbf{v} \times \boldsymbol{\beta}) + i\gamma^2(c^{-1}\mathbf{v} \times \boldsymbol{\alpha} + c^{-2}(\boldsymbol{\beta} \times \mathbf{v}) \times \mathbf{v})\,.\end{aligned} \tag{4.36}$$

Using this in (4.27), one obtains

$$\omega_T = \frac{[\dot{v}v]_2}{1 + v \cdot u} = \frac{i\gamma^2}{c^2(1+\gamma)}\,(\boldsymbol{\beta} \times \mathbf{v}) \times \mathbf{v} + \frac{i\gamma^2}{c^2(1+\gamma)}\,\mathbf{v} \times \boldsymbol{\alpha}\,. \tag{4.37}$$

Finally, adding (4.35) and (4.37) one gets for the total angular velocity

$$\omega = \omega_T + \omega_L = i\left(\boldsymbol{\beta} + \frac{\gamma}{c(1+\gamma)}\,\boldsymbol{\alpha} \times \mathbf{v}\right) \equiv -i\omega\,, \tag{4.38}$$

and substituting this into (4.19), one gets for the equations of motion of the rigid body in the inertial system

$$\begin{aligned}\dot{\mathbf{e}}_i &= \omega \cdot \mathbf{e}_i = -i\omega \wedge \mathbf{e}_i = \omega \times \mathbf{e}_i \\ &= \frac{\gamma}{c}\frac{d\mathbf{e}_i}{dt} = \left(-\boldsymbol{\beta} + \frac{\gamma}{c(1+\gamma)}\,\mathbf{v} \times \boldsymbol{\alpha}\right) \times \mathbf{e}_i\,.\end{aligned} \tag{4.39}$$

This result agrees with Thomas,[7] though it may be more general than he realized. It applies to any motion whatever of a rigid point particle. All dynamics lie in the specification of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, or equivalently of Ω.

The precession of a rigid body can be described either by equations (4.6) or by (4.37) (or better by their corresponding spinor equations). Failure to distinguish between these two different modes of description can cause confusion. The former describes the precession in the instantaneous rest frame of the rigid body, while the latter describes an equivalent motion of a rigid body in some arbitrarily chosen inertial frame. It is worthwhile to work out the relation of the (actual) axes $\mathbf{e}_i$ of the body in the instantaneous rest frame to the equivalent axes $\mathbf{e}_i$ in the inertial frame. Using (4.18) and the decomposition (4.30) with $\boldsymbol{\alpha}$ replaced by $\mathbf{e}_i$, one finds

$$\begin{aligned}e_i u = e_i v v u &= L\mathbf{e}_i\widetilde{L}L^2 = L\mathbf{e}_i L \\ &= L^2\mathbf{e}_i^{\parallel} + \mathbf{e}_i^{\perp} \\ &= \gamma(1 - c^{-1}\mathbf{v})\mathbf{e}_i^{\parallel} + \mathbf{e}_i - \mathbf{e}_i^{\parallel} \\ &= c^{-1}\mathbf{e}_i \cdot \mathbf{v} + \mathbf{e}_i + (\gamma-1)\mathbf{e}_i \cdot \mathbf{v}\,\mathbf{v}\,.\end{aligned}$$

Hence,

$$e_i \cdot u = c^{-1}\gamma\mathbf{v} \cdot \mathbf{e}_i\,, \tag{4.40a}$$

and the relative vector part is

$$e_i \wedge u = \mathbf{e}_i + \frac{(\gamma - 1)}{\mathbf{v}^2}\, \mathbf{e}_i \cdot \mathbf{v}\,\mathbf{v} = \mathbf{e}_i + \frac{\gamma^2}{c^2(\gamma+1)}\, \mathbf{e}_i \cdot \mathbf{v}\,\mathbf{v}\,. \tag{4.40b}$$

[1]D. Hestenes, *Space-Time Algebra* (Gordon and Breach, New York, 1966).
[2]D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus* (Kluwer, 1984).
[3]D. Hestenes, *Am. J. Phys.* **39**, 1013 (1971).
[4]D. Hestenes, *J. Math. Phys.* **8**, 748 (1967).
[5]D. Hestenes, *J. Math. Phys.* **14**, 893 (1973).
[6]A. J. MacFarlane, *J. Math. Phys.* **3**, 1116 (1962).
[7]L. Thomas, *Phil. Mag.* **3**, 1 (1927).
[8]H. Bacry, *Nuovo Cimento* **26**, 1164 (1962).

APPENDIX: ERRATA TO REFERENCE 1

Since this paper elaborates certain parts of Ref. 1, it is appropriate to include here the following list of errata to that monograph: The last line of Eq. (3.12) should read

$$+(-1)^{s-r}(a_1 \wedge \ldots \wedge a_r) \cdot (b_{s-r+1} \wedge \ldots \wedge b_s)\, b_1 \wedge b_2 \wedge \ldots \wedge b_{s-r}\,.$$

Delete the last minus sign on the right-hand side of Eq. (6.16). Equation (19.22) should read

$$\mathbf{E}' + i\mathbf{B}' = \mathbf{E}_{\parallel} + \beta(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}) + i[\mathbf{B}_{\parallel} + \beta(\mathbf{B}_{\perp} - \mathbf{v} \times \mathbf{E})]\,.$$

Equation (20.2) should read $\gamma^i \cdot \gamma_j = \delta^i_j$. Insert a factor of $\frac{1}{2}$ on the right-hand sides of Eq. (21.5) and (21.20) and in front of $R^{\mu}_{\alpha\beta\sigma}$ in Eq. (21.7). Delete the explicit factors of $\frac{1}{2}$ from Eq. (21.10). Dispense with the pseudoscalar part of (22.3) and delete Eq. (22.5b). Equation (23.15) should read $C_{ijk} = -C_{ikj}$. Equation (24.14) should read $C \equiv \gamma^k C_k$. The sentence following Eq. (A7) should read "where the *signature* s is the maximum number of linearly independent vectors . . . ." Replace the subscript i in (A12) by 1. Six lines after Eq. (B1), the sentence should begin "If $T \neq 0$, ..."