Chapter 2

Geometric Calculus

Geometric calculus is an extension of *geometric algebra* to include the analytic operations of differentiation and integration. It is developed in this book as a computational language for physics. The calculus is designed for efficient computation and representation of geometric relations. This leads to compact formulations for the equations of physics and their solutions as well as elucidation of their geometric contents.

This chapter is concerned primarily with differentiation and integration with respect to vector variables. Of course, vector variables are especially important in physics, because places in Physical Space are represented by vectors. Therefore, the results of this chapter are fundamental to the rest of the book. They make it possible to carry out completely coordinate-free computations with functions of vector variables, one of the major advantages of geometric calculus. Nevertheless, coordinate systems will be introduced here at the beginning for several reasons. Coordinate methods are employed in most of the mathematics and physics literature, so it is necessary to relate them to the coordinate-free methods of geometric calculus. By establishing the relation early, we can refer to standard mathematics texts for the treatment of important points of rigor. Thus we can move along quickly, concentrating attention on the unique advantages of geometric calculus. Finally, we seek to understand precisely when coordinates can be used to advantage and be ready to exploit them. It will be seen that coordinate systems are best regarded as adjuncts of a more fundamental coordinate-free method.

The reader is presumed to be familiar with the standard differential and integral calculus with respect to scalar variables. We will apply it freely to *multivector-valued* functions of scalar variables. Readers who need more background for that are referred to Sections 2–7 and 2–8 of NFCM.

The main results of this chapter are formulated to apply in spaces of arbitrary dimension. However, the examples and applications are limited to three dimensions, since that is the case of greatest interest. Readers interested in greater generality are referred to the more advanced treatment in GC.

2-1 Differentiable Manifolds and Coordinates

Roughly speaking, a *differentiable manifold* (or just *manifold*) is a set on which differential and integral calculus can be carried out. We will be concerned mainly with *vector manifolds* in Euclidean

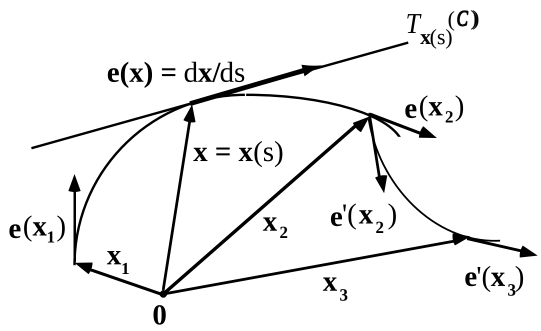

Fig. 1.1. A piecewice differentiable curve $\mathcal{E}$consisting of differentiable curves $\mathcal{C}$ and $\mathcal{C}'$ joined at a corner $\mathbf{x}_2$. The tangent vector field $\mathbf{e}(\mathbf{x}) = d\mathbf{x}/ds$ has a discontinuity at the corner in jumping from $\mathbf{e}(\mathbf{x}_2)$ at the endpoint of $\mathcal{E}$ to $\mathbf{e}'(\mathbf{x}_2)$ at the initial point of $\mathcal{E}'$.

3-space $\mathcal{E}_3$, which is to say that the "points" of the manifolds are vectors in $\mathcal{E}_3$, and the entire geometric algebra $\mathcal{G}_3$ is available for characterizing the manifolds and relations among them. There are various ways of defining manifolds. The usual approach is to define an *m-dimensional manifold* or *m-manifold* as a set of points which can be continuously parametrized locally by a system of m coordinates. The term "locally" here means "in a neighborhood of every point." More than one coordinate system is often needed to "cover" the whole manifold. A completely coordinate-free approach to manifolds is developed in GC, but we begin with coordinates here. Unless otherwise indicated, we tacitly assume that each manifold we work with is *simply-connected*, which means that it consists of a single connected piece.

The Euclidean space $\mathcal{E}_3$ is a 3-dimensional manifold, and it contains 4 kinds of submanifolds: A single point is a zero-dimensional manifold. A *curve* is a 1-manifold. A *surface* is a 2-manifold. And finally, any 3-dimensional *region* in $\mathcal{E}_3$ bounded by a surface is a 3-manifold. The boundary $\partial\mathcal{M}$ of an m-manifold $\mathcal{M}$ is an $(m-1)$-manifold. Thus, the boundary of a 3-dimensional region is a surface, the boundary of a surface is curve, and the boundary of a curve consists of its two endpoints. A manifold without a boundary, such as a circle or sphere, is said to be *closed*. The boundary of a manifold is always a closed manifold. This can be expressed by writing $\partial\partial\mathcal{M} = 0$.

A function $F = F(\mathbf{x})$ defined at each point $\mathbf{x}$ of a manifold $\mathcal{M}$ is called a *field* on $\mathcal{M}$, and $\mathcal{M}$ is said to be the *domain* of F. In general, we allow F to be a multivector-valued function. It is called a *vector field* if it is vector-valued, a *scalar field* if it is scalar-valued, a *spinor field* if it is spinor-valued, etc. Before defining the derivatives and integrals of fields, we need to characterize the manifolds in $\mathcal{E}_3$ in more detail.

Curves

A *curve* $\mathcal{C} = \{\mathbf{x}\}$ is a set of points which can be parametrized by a continuous vector-valued function $\mathbf{x} = \mathbf{x}(s)$ of a scalar variable s defined on a closed interval $s_1 \le s \le s_2$. The endpoints of the curve are thus the vectors $\mathbf{x}_1 = \mathbf{x}(s_1)$ and $\mathbf{x}_2 = \mathbf{x}(s_2)$. If the endpoints coincide the curve is closed, and the parameter range must be changed to $s_1 \le s < s_2$ to avoid double counting of the point $\mathbf{x}_1 = \mathbf{x}_2$. The parameter s is a coordinate for the curve $\mathcal{C}$.

The function $\mathbf{x} = \mathbf{x}(s)$ is often regarded as the curve itself. This has an advantage when the curve is self-intersecting: at an intersection point $\mathbf{x}(s_1) = \mathbf{x}(s_2)$ the parameter values distinguish different branches of the curve. Moreover, on a closed curve the parameter values given above can be extended beyond s_2 to describe a multiple "traversal" of the set $\mathcal{C}$. Confusion on this point can be avoided by referring to the function $\mathbf{x} = \mathbf{x}(s)$ as an *orbit*, or *trajectory* or a *parametrized curve*. It is often convenient to leave the range of the parameter unspecified until it is needed.

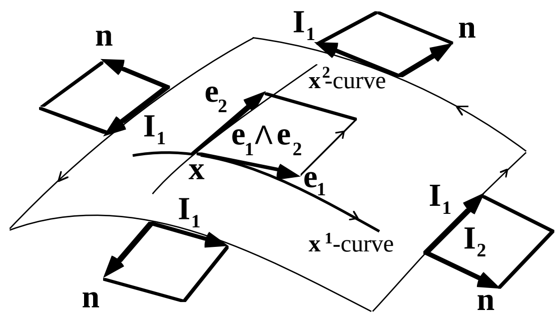

Fig. 1.2. A surface showing coordinate curves and tangent vectors at a point $\mathbf{x}$ as well as boundary normals in relation to the local orientation of the surface and its boundary.

The curve $\mathcal{C}$ is said to be *differentiable* if the derivative $d\mathbf{x}/ds$ exists in the usual mathematical sense. The derivative determines a vector field $\mathbf{e}(\mathbf{x}) = d\mathbf{x}/ds$ on the curve. The value of this function at the point $\mathbf{x}$ is said to be a *tangent vector* (of the curve) at $\mathbf{x}$. A change of parameter specified by a monotonic differentiable function $s = s(\tau)$ determines a new tangent vector

$$\frac{d\mathbf{x}}{d\tau} = \frac{ds}{d\tau}\frac{d\mathbf{x}}{ds} = \lambda(\mathbf{x})\mathbf{e}(\mathbf{x})$$

The 1-dimensional vector space $\mathcal{T}_{\mathbf{x}}(\mathcal{C})$ of all scalar multiples of the vector $\mathbf{e}(\mathbf{x})$ is called the *tangent space* at $\mathbf{x}$ of the curve $\mathcal{C}$. (Fig. 1.1.) The unique parameter s for which $ds = |d\mathbf{x}|$ is called the *arc length* of the curve. For this parameter $\mathbf{e} = d\mathbf{x}/ds$ is a unit vector. This vector is called the *unit tangent* at $\mathbf{x}$ of the manifold $\mathcal{C}$. The unit tangent of $\mathcal{C}$ is a continuous vector field on $\mathcal{C}$, and it specifies an orientation for $\mathcal{C}$ independently of any coordinate system.

If $\mathbf{x} = \mathbf{x}(s)$ has finite derivatives of all orders, the curve $\mathcal{C}$ is said to be *smooth*. The curve is *piecewise smooth* if it consists of a finite number of smooth curves joined at common endpoints forming *corners* in the curve (Fig. 1.1). Note that the unit tangent of $\mathcal{C}$ is discontinuous at corners.

A definite curve is determined by giving $\mathbf{x} = \mathbf{x}(s)$ as a specific function of s. The curve can also be determined by a differential equation for which $\mathbf{x} = \mathbf{x}(s)$ is the solution. Alternatively, a curve can be specified nonparametrically as the intersection of two given surfaces.

Surfaces

A *surface* $\mathcal{S} = \{\mathbf{x}\}$ is a set of points which can be parametrized locally by a continuous vector-valued function $\mathbf{x} = \mathbf{x}(x^1, x^2)$ of two scalar variables x^1, x^2 called *coordinates* of the surface. The surface is said to be *differentiable* if $\mathbf{x}(x^1, x^2)$ is a differentiable function of both coordinates, and it is *smooth* if derivatives of all orders exist. The surface is *piecewise smooth* if it is composed of a finite number of smooth surfaces bounded by piecewise smooth curves. Unless otherwise indicated, we tacitly assume that a given manifold is smooth or piecewise smooth.

When one of the coordinates is kept fixed, the function $\mathbf{x}(x^1, x^2)$ describes a *coordinate curve* with respect to the other coordinate (Fig. 1.2). The derivatives determine vector fields $\mathbf{e}_k = \mathbf{e}_k(\mathbf{x})$ on $\mathcal{S}$ defined by

$$\mathbf{e}_k = \partial_k \mathbf{x} = \frac{\partial \mathbf{x}}{\partial x^k}. \tag{1.1}$$

The coordinate curves "cover" a neighborhood of each point only if they are never tangent to one another; this is best expressed by the condition that $\mathbf{e}_1(\mathbf{x}) \wedge \mathbf{e}_2(\mathbf{x}) \neq 0$ at any point $\mathbf{x}$ where the coordinate system is defined. A vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$ is said to be a *tangent* to $\mathcal{S}$ at $\mathbf{x}$ if

$$\mathbf{v}(\mathbf{x}) \wedge \mathbf{e}_1(\mathbf{x}) \wedge \mathbf{e}_2(\mathbf{x}) = 0 \tag{1.2}$$

The set of all vectors $\mathbf{v}(\mathbf{x})$ satisfying (1.2) is a 2-dimensional vector space $\mathcal{T}_{\mathbf{x}}(\mathcal{S})$ called the *tangent space* of $\mathcal{S}$ at $\mathbf{x}$. If the field $\mathbf{v}$ satisfies (1.2) at every point of $\mathbf{x}$ it is said to be a *tangent vector field.* Any multivector field, such as $\mathbf{e}_1 \wedge \mathbf{e}_2$, which can be generated from vector fields by multiplication and addition of their values at each point $\mathbf{x}$ is said to be a *tangent field.* Note that a single vector $\mathbf{a}$ can be regarded trivially as a constant vector field on $\mathcal{S}$, and it may be a tangent vector at some points of $\mathcal{S}$.

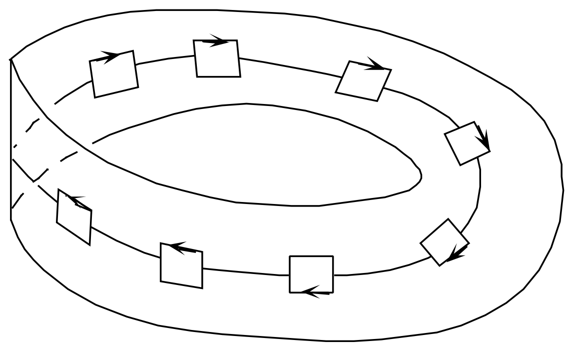

Fig. 1.3. A *Moebius strip* is a one-side surface bounded by a single closed curve. To see that it is *nonorientable*, take a unit tangent 2-blade at one point and slide it "tangentially" along a closed curve around the strip. When the blade returns to the initia point it has the opposite orientation (sign). Therefore, it cannot be defined continuously on the strip with a single orientation.

A bivector-valued *unit tangent* $\mathbf{I}_2 = \mathbf{I}_2(\mathbf{x})$ can be defined on $\mathcal{S}$ and related to a coordinate system by

$$\mathbf{e}_1 \wedge \mathbf{e}_2 = \pm \mathbf{I}_2 |\, \mathbf{e}_1 \wedge \mathbf{e}_2 \,|. \tag{1.3}$$

If the sign here is positive (negative) the coordinate system is said to be *positively* (negatively) *oriented.* A differentiable manifold is said to be *orientable* if a continuous single-valued unit tangent field can be defined on it, and this field is said to be an *orientation* on the manifold. The Moebius strip (Fig. 1.3) is the simplest example of non-orientable manifold. We will be concerned only with orientable manifolds because orientability is necessary for defining integration on a manifold.

If a manifold is orientable then its boundary is also orientable. If $\mathbf{I}_2$ is the orientation of a surface, then, a unique orientation $\mathbf{I}_1$ can be assigned to its boundary by the following *convention.* At each point $\mathbf{x}$ on the boundary, $\mathbf{I}_2 = \mathbf{I}_2(\mathbf{x})$ is related to $\mathbf{I}_1 = \mathbf{I}_1(\mathbf{x})$ by

$$\mathbf{I}_2 = \mathbf{I}_1 \mathbf{n} = \mathbf{I}_1 \wedge \mathbf{n}, \tag{1.4}$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the *boundary normal* of the surface (Fig. 1.2). At each point of the boundary, the boundary normal is a unit vector normal to the surface and, by convention, directed *outward* from the interior. Its normality (or orthogonality) is expressed by the condition $\mathbf{I}_1 \cdot \mathbf{n} = 0$, as implied by (1.4). Its outward direction is that of a curve hitting the boundary from the interior of the surface.

Oriented Vector Manifolds

So far, our mathematical characterization of curves and surfaces is completely general, including no assumption that they are submanifolds of $\mathcal{E}_3$. Moreover, it generalizes directly to characterize manifolds of any dimension as follows. A *smooth oriented vector manifold* $\mathcal{M} = \{\mathbf{x}\}$ is a set of vectors (points) which can be parametrized locally by a smooth function

$$\mathbf{x} = \mathbf{x}(x^1, x^2, \ldots, x^m) \tag{1.5a}$$

of m (scalar) coordinates with tangent vector fields

$$\mathbf{e}_k = \mathbf{e}_k(\mathbf{x}) = \frac{\partial \mathbf{x}}{\partial x^k} \tag{1.5b}$$

satisfying

$$\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_m \neq 0. \tag{1.5c}$$

At each point $\mathbf{x}$ the m vectors $\mathbf{e}_k(\mathbf{x})$ constitute a basis or *frame* for the *tangent space* $\mathcal{T}_\mathbf{x}(\mathcal{M})$. The condition (1.5c) assures that they are linearly independent. Any set of m tangent vector fields is said to be a *frame field* on $\mathcal{M}$ if the vectors are linearly independent at each point. The set $\{\mathbf{e}_k\}$ is called a *coordinate frame field*.

An *orientation* $\mathbf{I}_m = \mathbf{I}_m(\mathbf{x})$ of $\mathcal{M}$ is a unit tangent m-blade-valued field related to a positively oriented coordinate system by

$$\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_m = \mathbf{I}_m |\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_m|. \tag{1.6}$$

The orientation determines one *tangent space* $\mathcal{T}_\mathbf{x}(\mathcal{M})$ at each point $\mathbf{x}$ consisting of all vectors $\{\mathbf{v}\}$ satisfying $\mathbf{v} \wedge \mathbf{I}_m(\mathbf{x}) = 0$.

The boundary of $\mathcal{M}$ is an $(m-1)$-manifold $\partial\mathcal{M}$ with orientation $\mathbf{I}_{m-1}$ determined by the *convention*

$$\mathbf{I}_m = \mathbf{I}_{m-1}\mathbf{n} \tag{1.7}$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the (unit outward) *boundary normal* at each point on $\partial\mathcal{M}$.

It is important to note that the relation (1.7) applies even in the case $m = 1$. Then $\mathbf{I}_1$ is a unit tangent vector field on a curve, and $\mathbf{I}_0$ is a scalar-valued orientation assigned to the endpoints. For the curve shown in Fig. 1.1, we have $\mathbf{e} = \mathbf{I}|\,\mathbf{e}\,|$ and

$$\mathbf{I}_0(\mathbf{x}_2) = 1 = -\mathbf{I}_0(\mathbf{x}_1). \tag{1.8}$$

Thus, the endpoints of a curve have opposite orientation.

By the inverse function theorem (proved in GC and many mathematics texts), the condition (1.5c) implies that the function (1.5a) can be inverted to give m scalar-valued *coordinate functions*

$$x^k = x^k(\mathbf{x}). \tag{1.9}$$

Note the ambiguous use of the symbols x^k to represent independent variables in (1.5a) and functions in (1.9). This ambiguity reduces the number of symbols needed and helps keep track of relations. The correct interpretation should be clear from the context. The coordinate functions (1.9) are as important as the inverse relation (1.5a). However, we are not prepared to differentiate $x^k(\mathbf{x})$ on an arbitrary manifold until the "tangential derivative" is defined in a later section.

The manifold $\mathcal{M}$ is said to be *flat* if $\mathbf{I}_m$ is constant on $\mathcal{M}$, that is, if $\mathbf{I}_m(\mathbf{x})$ has the same value at every point $\mathbf{x}$ of $\mathcal{M}$. In that case $\mathcal{M}$ can be identified as an m-dimensional *region* in Euclidean m-space $\mathcal{E}_m$. Note that tangent spaces on $\mathcal{M}$ are all identical, as expressed by $\mathcal{T}_\mathbf{x}(\mathcal{M}) = \mathcal{E}_m$.

For $\mathcal{E}_3$ the standard orientation $\mathbf{I}_3$ is taken to be the dextral unit pseudoscalar, that is, $\mathbf{I}_3 = i$. For a surface $\mathcal{S}$ in $\mathcal{E}_3$ with a specified orientation on its boundary a *surface normal* $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is commonly defined by the right-hand rule (Fig. 1.4), while the orientation $\mathbf{I}_2 = \mathbf{I}_2(\mathbf{x})$ is defined in (1.4) by its relation to the *boundary normal* (not to be confused with the surface normal). Consequently, the surface normal $\mathbf{n}$ is related to the surface orientation $\mathbf{I}_2$ by

$$\mathbf{I}_2 = -i\mathbf{n}\,. \tag{1.10}$$

This implies that $\mathbf{n}\cdot\mathbf{I}_2 = -i\mathbf{n}\wedge\mathbf{n} = \mathbf{n}\times\mathbf{n} = 0$, which means that $\mathbf{n}$ is normal to the surface at every point (Fig. 1.4).

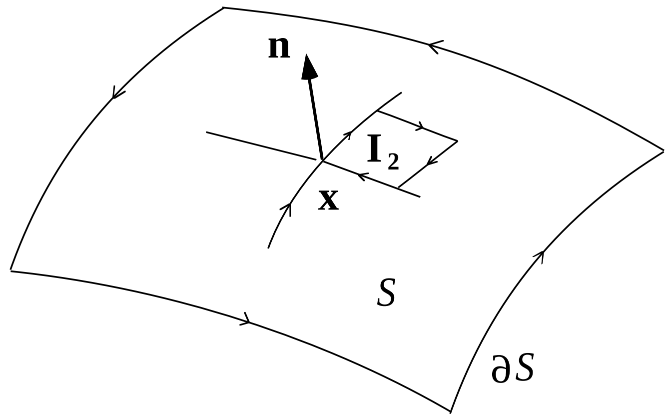

Fig. 1.4. An oriented surface embedded in $\mathcal{E}_3$ has a unique unit *surface normal* $\mathbf{n}=i\mathbf{I}_2$. Note that the direction of $\mathbf{n}$ is related to the orientation of the surface boundary by the "right-hand rule"—fingers aligned with the boundary when the thumb is aligned with the normal. On the other hand, the direction of $\mathbf{n}$ is related to the orientation $\mathbf{I}_2$ by a "left-hand rule." This awkward choice of orientation is dictated by a long-standing convention on the form of Stokes' Theorem (Section 2–4).

Coordinate Systems for $\mathcal{E}_3$

Coordinate systems for the manifold $\mathcal{E}_3$ can be most efficiently characterized by using the vector derivative $\nabla = \nabla_{\mathbf{x}}$ introduced in Section 1–3. Differentiating the coordinate functions $x^k = x^k(\mathbf{x})$, we obtain three vector fields

$$\mathbf{e}^k = \nabla x^k, \tag{1.11}$$

where $k = 1, 2, 3$. These fields have several important properties. From the operator identity $\nabla \wedge \nabla = 0$, we have immediately

$$\nabla \wedge \mathbf{e}^k = i\nabla \times \mathbf{e}^k = 0 \tag{1.12}$$

Also, by the chain rule, we have

$$\mathbf{e}_j \cdot \nabla x^k = \Big(\frac{\partial \mathbf{x}}{\partial x^j}\Big) \cdot \nabla x^k = \frac{\partial x^k}{\partial x^j} = \delta_j^k .$$

Hence

$$\mathbf{e}_j \cdot \mathbf{e}^k = \delta_j^k , \tag{1.13}$$

where $[\delta_j^k]$ is the 3×3 identity matrix. This is the algebraic condition for the frame $\{\mathbf{e}^k\}$ to be *reciprocal* to the frame $\{\mathbf{e}_j\}$. It is easily verified that equations (1.13) have the unique solution

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{e}, \quad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{e}, \quad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{e}, \tag{1.14}$$

where

$$\mathbf{e} = -i(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) = (\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 > 0 \tag{1.15}$$

for a positively oriented (dextral) coordinate system.

The equations (1.14) enable us to compute the $\mathbf{e}^k$ directly from $\mathbf{e}_j$ without first finding the functions $x^k(\mathbf{x})$. But they also have an important geometric meaning. They tell us that the $\mathbf{e}^k$ are everywhere normal to coordinate surfaces. Thus, if the coordinate x^3 is held constant, the function

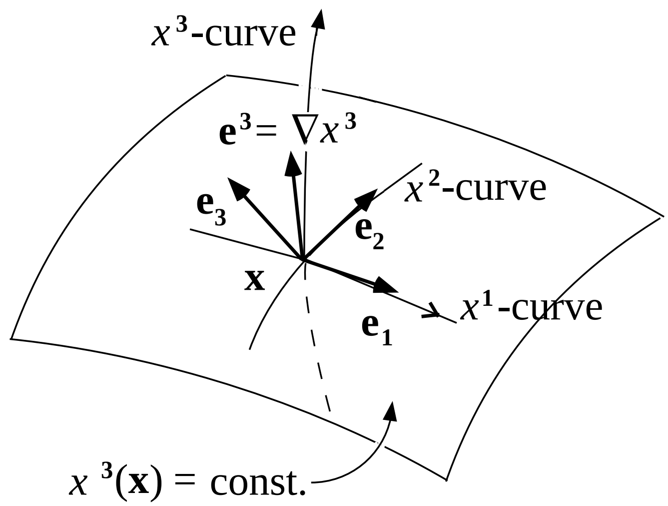

Fig. 1.5. A coordinate surface $x^3 = x^3(\mathbf{x}) = \text{constant}$ has a normal vector field $\mathbf{e}^3 = \nabla x^3$. The coordinate tangent $\mathbf{e}_3$ is not necessarily normal to the surface.

$\mathbf{x} = \mathbf{x}(x^1, x^2, x^3)$ describes a surface with tangent fields $\mathbf{e}_1$ and $\mathbf{e}_2$, and this equation can in principle be inverted to describe the surface by a scalar equation $x^3 = x^3(\mathbf{x})$. (Note that it would be helpful here to have different symbols for the coordinate variable x^3 and the coordinate function $x^3(\mathbf{x})$.) The gradient of the function $x^3(\mathbf{x})$ tells us the direction in which the function changes, and (1.14) tells us that

$$\nabla x^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{e}, \tag{1.16}$$

which is indeed normal to the surface $x^3(\mathbf{x}) = x^3 = \text{constant}$ (Fig. 1.5).

The equations (1.14) can be inverted to give the $\mathbf{e}_k$ as functions of the $\mathbf{e}^k$ if desired. By symmetry we find immediately

$$\mathbf{e}_1 = \frac{\mathbf{e}^2 \times \mathbf{e}^3}{e^{-1}}, \quad \mathbf{e}_2 = \frac{\mathbf{e}^3 \times \mathbf{e}^1}{e^{-1}}, \quad \mathbf{e}_3 = \frac{\mathbf{e}^1 \times \mathbf{e}^2}{e^{-1}}. \tag{1.17}$$

where

$$e^{-1} = \frac{1}{e} = -i(\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3) = (\mathbf{e}^1 \times \mathbf{e}^2) \cdot \mathbf{e}^3 > 0. \tag{1.18}$$

Equation (1.17) also has a geometric interpretation. According to (1.17)

$$\mathbf{e}_3 = e(\nabla x^1) \times (\nabla x^2). \tag{1.19}$$

This implies that $\mathbf{e}_3 \cdot \nabla x^1 = 0 = \mathbf{e}_3 \cdot \nabla x^2$, which tells us that $\mathbf{e}_3$ is tangent to each of the coordinate surfaces $x^1 = $ const. and $x^2 = $ const., so it is directed along the intersection of those surfaces. Therefore, (1.17) tells us that all *the coordinate curves are intersections of coordinate surfaces.*

For efficient algebraic manipulations with coordinates both frames $\{\mathbf{e}_k\}$ and $\{\mathbf{e}^k\}$ are needed. For example, to express a vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$ in the "coordinate form"

$$\mathbf{v} = \textstyle\sum_k v^k \mathbf{e}_k, \tag{1.20a}$$

the $\mathbf{e}^k$ are needed to determine its scalar *components*

$$v^k = v^k(\mathbf{x}) = \mathbf{v} \cdot \mathbf{e}^k. \tag{1.20b}$$

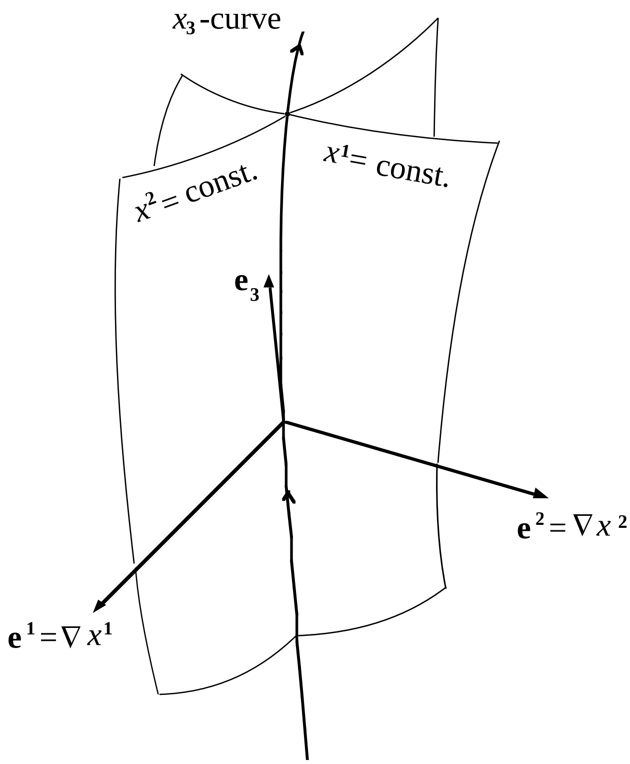

Fig. 1.6. Each coordinate curve is the intersection of two coordinate surfaces.

This can be proved by using (1.13) to solve (1.20a) for v^k; thus,

$$\mathbf{e}^k \cdot \mathbf{v} = \mathbf{e}^k \cdot (\textstyle\sum_j v^j \mathbf{e}_j) = \sum_j v^j (\mathbf{e}^k \cdot \mathbf{e}_j) = \sum_j v^j \delta_j^k = v^k .$$

For a field $F = F(\mathbf{x}) = F(\mathbf{x}(x^1, x^2, x^3))$, the coordinate derivatives are related to directional derivatives along the coordinate curves by the chain rule; thus,

$$\frac{\partial F}{\partial x^k} = \left(\frac{\partial \mathbf{x}}{\partial x^k}\right) \cdot \nabla = \mathbf{e}_k \cdot \nabla F. \tag{1.21}$$

This formula expresses the coordinate derivative in terms of the vector derivative ∇, and it is useful when the field F is given as a specific function of $\mathbf{x}$ which can be differentiated by ∇ using the results of Section 1–3. However, when the field F is given as an explicit function of the coordinates $F(x^1, x^2, x^3)$, it can be expressed as a function of $\mathbf{x}$ by writing $F(x^1(\mathbf{x}), x^2(\mathbf{x}), x^3(\mathbf{x}))$, and its vector derivative can be computed by using the chain and product rules to obtain

$$\nabla F = (\nabla x^1)\frac{\partial F}{\partial x^1} + (\nabla x^2)\frac{\partial F}{\partial x^2} + (\nabla x^3)\frac{\partial F}{\partial x^2}. \tag{1.22}$$

Thus, the operator equation

$$\nabla = \sum_k (\nabla x^k)\frac{\partial}{\partial x^k} = \sum_k \mathbf{e}^k \partial_k \tag{1.23}$$

expresses the vector derivative in terms of coordinate derivatives for any coordinate system. It is worth noting that (1.22) applies to any scalar-valued functions $x^k(\mathbf{x})$ even if they do not satisfy the condition $(\nabla x^1) \wedge (\nabla x^2) \wedge (\nabla x^3) \neq 0$ which allows them to be regarded as coordinate functions.

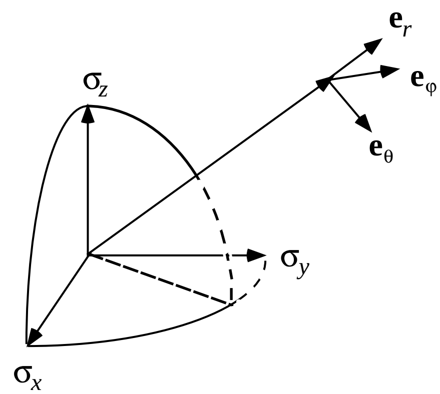

Fig. 1.7. Spherical coordinate system.

Orthogonal Coordinates

Coordinate systems are most useful where they express some symmetry inherent in a given problem, often as a consequence of boundary conditions. Orthogonal coordinate systems are most commonly employed because they express the simplest symmetries. A coordinate system is said to be *orthogonal* if the coordinate vectors $\mathbf{e}_k$ are mutually orthogonal. In that case we can write $\mathbf{e}_k = e_k\hat{\mathbf{e}}_k$, where $\hat{\mathbf{e}}_k^2 = 1$ and the $e_k = |\mathbf{e}_k|$ are *coordinate scale factors*. Then it follows from (1.13) that

$$\mathbf{e}^k = \mathbf{e}_k^{-1} = \frac{\hat{\mathbf{e}}_k}{e_k}. \tag{1.24}$$

Remark on notation: In our general discussion of coordinate systems we have followed the common practice of using a superscript as an index to distinguish the reciprocal vectors $\mathbf{e}^k = \nabla x^k$ from the coordinate vectors $\mathbf{e}_k = \partial_k\mathbf{x}$ indexed with a subscript. Unfortunately, this systematic notation interferes with the more important use of superscripts as exponents, so we will not employ it exclusively. In particular, for orthogonal coordinates a subscript notation is sufficient, since (1.24) tells us that the reciprocal vectors differ from the $\mathbf{e}_k$ only by a scale factor. Thus for orthogonal coordinates x_k we write

$$\mathbf{e}_k = \partial_k\mathbf{x} = \frac{\partial\mathbf{x}}{\partial x_k}, \tag{1.25}$$

and (1.24) combined with (1.11) gives us the reciprocal relation

$$\nabla x_k = \frac{1}{\mathbf{e}_k} = \frac{\hat{\mathbf{e}}_k}{e_k}. \tag{1.26}$$

Now let us consider the two most important examples of orthogonal coordinates.

Rectangular coordinates are characterized by the parametric equation

$$\mathbf{x}(x_1, x_2, x_3) = x_1\boldsymbol{\sigma}_1 + x_2\boldsymbol{\sigma}_2 + x_3\boldsymbol{\sigma}_3, \tag{1.27}$$

where, as always, $\{\boldsymbol{\sigma}_k\}$ is a standard basis. This equation is easily inverted to get the coordinate functions

$$x_k(\mathbf{x}) = \mathbf{x}\cdot\boldsymbol{\sigma}_k\,. \tag{1.28}$$

The coordinate vectors are therefore

$$\partial_k\mathbf{x} = \nabla x_k = \boldsymbol{\sigma}_k. \tag{1.29}$$

The $\boldsymbol{\sigma}_k$ are therefore constant vector fields which are both tangent to coordinate curves and normal to coordinate surfaces.

Spherical coordinates $x_1=\theta,\ x_2=\varphi,\ x_3=r$ are characterized by the parametric equation

$$\mathbf{x}(\theta, \varphi, r) = rR\boldsymbol{\sigma}_3 R^\dagger \tag{1.30a}$$

where

$$R = R(\theta, \varphi) = e^{-\frac{1}{2}i\boldsymbol{\sigma}_3\varphi} e^{-\frac{1}{2}i\boldsymbol{\sigma}_2\theta} \tag{1.30b}$$

with $0 < r < \infty,\ 0 \le \theta < \pi,\ 0 \le \varphi \le 2\pi$ (Fig. 1.7). Differentiating and using each coordinate as an index, we obtain

$$\mathbf{e}_r = \partial_r \mathbf{x} = R\boldsymbol{\sigma}_3 R^\dagger = \hat{\mathbf{x}}, \tag{1.31a}$$

$$\mathbf{e}_\theta = \partial_\theta \mathbf{x} = rR\boldsymbol{\sigma}_1 R^\dagger, \tag{1.31b}$$

$$\mathbf{e}_\varphi = \partial_\varphi \mathbf{x} = r\boldsymbol{\sigma}_3 \times \mathbf{e}_r = \boldsymbol{\sigma}_3 \times \mathbf{x}. \tag{1.31c}$$

Therefore the scale parameters are

$$e_r = 1, \quad e_\theta = r, \quad e_\varphi = r(\boldsymbol{\sigma}_3 \times \mathbf{e}_r) = r \sin\theta, \tag{1.32}$$

and the coordinate directions are

$$\hat{\mathbf{e}}_k = R\boldsymbol{\sigma}_k R^\dagger. \tag{1.33}$$

From (1.26) we obtain

$$\hat{\mathbf{e}}_r = \nabla r, \quad \hat{\mathbf{e}}_\theta = r\nabla\theta, \quad \hat{\mathbf{e}}_\varphi = r\sin\theta\nabla\varphi. \tag{1.34}$$

The radial coordinate function is $r = |\,\mathbf{x}\,|$, but it is not convenient to solve (1.30a,b) for the coordinate functions $\theta(\mathbf{x})$ and $\varphi(\mathbf{x})$, as that would involve inverse trigonometric functions. It is enough to have the gradients of θ and φ given by (1.34). From this we can read off the coordinate surfaces. The point $\mathbf{x}$ lies at the intersection of a sphere of radius r and two orthogonal planes with equations $\mathbf{x}\cdot\nabla\theta = 0$ and $\mathbf{x}\cdot\nabla\varphi = 0$ where $\nabla\theta$ and $\nabla\varphi$ have fixed directions though their magnitutes vary on the planes (Fig. 1.8).

When the radial coordinate r is held fixed, (1.30 a,b) is the parametric equation for a sphere with coordinates θ, φ. The coordinate system has a *singularity* at the *pole* where $\theta = 0$, for (1.31c) implies that $\mathbf{e}_\varphi = 0$ there, so the coordinate direction $\hat{\mathbf{e}}_\varphi$ is not uniquely defined. It is not possible to cover a sphere with a single coordinate system which does not have a singularity somewhere. This follows from the fact that every continuous tangent vector field on a sphere must have a zero value at some point. Note, however, that a tangent bivector field, such as the orientation $\mathbf{I}_2$, can be smoothly defined on the sphere without zeros.

2-1 Exercises

(1.1) A system of *skew coordinates* is defined by the coordinate functions

$$x^k(\mathbf{x}) = \mathbf{x}\cdot\mathbf{a}_k,$$

where the $\mathbf{a}_k$ are constant (non-orthogonal) vectors. For this system find the coordinate frame, reciprocal frame and explicit parametric equation for the point $\mathbf{x}$.

(1.2) Carry out the derivatives of (1.30a,b) to get (1.31a,b,c).

(1.3) The parametric equation $\mathbf{x} = \mathbf{x}(x_1, x_2, x_3)$ is given explicitly below for two orthogonal coordinate systems.

Cylindrical: $\mathbf{x}(\rho, \varphi, z) = \rho\boldsymbol{\sigma}_1 R_\varphi^{-2} + z\boldsymbol{\sigma}_3 = R_\varphi(\rho\boldsymbol{\sigma}_1 + z\boldsymbol{\sigma}_3)R_\varphi^{-1}$

Parabolic: $\mathbf{x}(\xi,\eta,\varphi) = \eta\xi\boldsymbol{\sigma}_1 R_\varphi^{-2} + \frac{1}{2}(\xi^2-\eta^2)\boldsymbol{\sigma}_3$
$$= R_\varphi(\eta\xi\boldsymbol{\sigma}_1 + \tfrac{1}{2}(\xi^2-\eta^2)\boldsymbol{\sigma}_3)R_\varphi^{-1}$$
for $0 \le \theta < \pi,\ 0 \le \varphi < 2\pi$ and $R_\varphi = e^{-\frac{1}{2}i\boldsymbol{\sigma}_3\varphi}$.

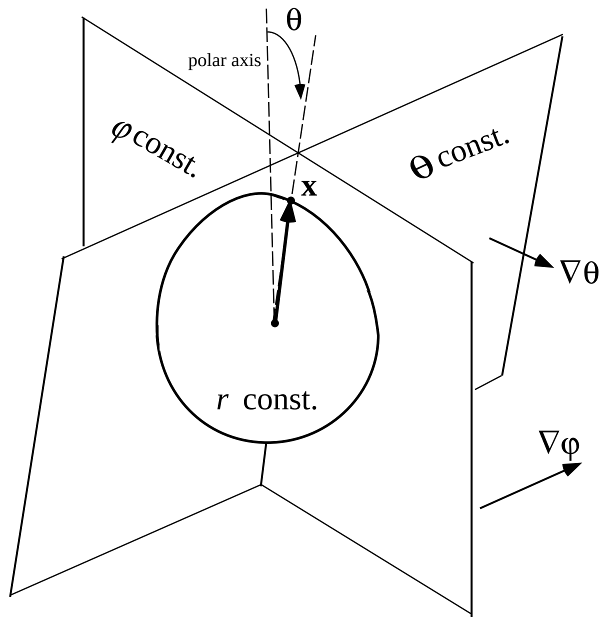

Fig. 1.8. Intersecting coordinate surfaces determine the position vector $\mathbf{x}$.

(a) For each coordinate system, calculate the scale factors $\hat{\mathbf{e}}_k = |\hat{\mathbf{e}}_k|$ and the spinor R determining $\hat{\mathbf{e}}_k = R\boldsymbol{\sigma}_k R^{-1}$. Illustrate with a diagram showing coordinate curves for each system.

(b) For a displacement in some direction $\mathbf{a}$, the $\hat{\mathbf{e}}_k$ rotate at a rate
$$\mathbf{a}\cdot\nabla\hat{\mathbf{e}}_k = \boldsymbol{\omega}\times\hat{\mathbf{e}}_k.$$
Calculate the rotational velocity
$$\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{a},\mathbf{x}) = 2i(\mathbf{a}\cdot\nabla R)R^\dagger$$
for each coordinate system.

(1.4) For any system of orthogonal coordinates, show that the divergence and curl of a vector field $\mathbf{v}$ with components $v_k = \hat{\mathbf{e}}_k\cdot\mathbf{v}$ can be expressed in the "coordinate forms"

$$\nabla\cdot\mathbf{v} = \frac{1}{e_1e_2e_3}\Big[\frac{\partial}{\partial x^1}(e_2e_3v_1) + \frac{\partial}{\partial x^2}(e_3e_1v_2) + \frac{\partial}{\partial x^3}(e_1e_2v_3)\Big],$$

$$\begin{aligned}\nabla\times\mathbf{v} = &\frac{1}{e_2e_3}\Big[\frac{\partial}{\partial x^2}(e_3v_3) - \frac{\partial}{\partial^3}(e_2v_2)\Big]\hat{\mathbf{e}}_1\\ &+\frac{1}{e_3e_1}\Big[\frac{\partial}{\partial x^3}(e_1v_1) - \frac{\partial}{\partial x^1}(e_3v_3)\Big]\hat{\mathbf{e}}_2\\ &+\frac{1}{e_1e_2}\Big[\frac{\partial}{\partial x^1}(e_2v_2) - \frac{\partial}{\partial x^2}(e_1v_1)\Big]\hat{\mathbf{e}}_3.\end{aligned}$$

These equations will not be used in this book since coordinate-free methods will be employed.

2-2 Directed and Iterated Integrals

In this section we develop the concept of directed integral, a generalization of the standard Riemann integral in which geometric algebra plays an essential role. In subsequent sections we shall see that it leads to some of the most powerful formulas in all of mathematics. To motivate the general definition of directed integral, we first consider the special cases of curves and surfaces.

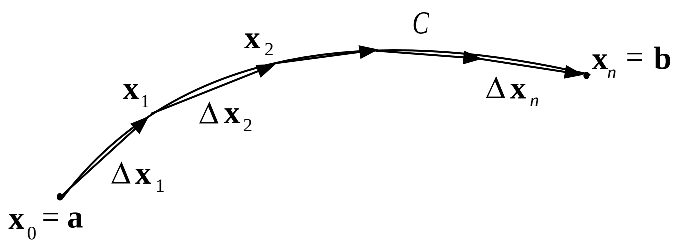

Fig. 2.1. Approximation of a curve $\mathcal{C}$ by directed line segments representing vectors.

Line Integrals

Line integrals have been discussed at some length in NFCM so this will be a brief review to get at the key idea to be generalized. The *line integral* of a field $F = F(\mathbf{x})$ over a curve $\mathcal{C} = \{\mathbf{x}\}$ is defined by

$$\int_{\mathcal{C}} F\,d\mathbf{x} = \int_{\mathcal{C}} F(\mathbf{x}\,d\mathbf{x}) = \lim_{\substack{\boldsymbol{\Delta}\mathbf{x}_i \to 0\\ n\to\infty}} \sum_{i=1}^{n} F(\mathbf{x}_i)\Delta\,\mathbf{x}_i. \tag{2.1}$$

The limit process should be understood as formally the same as the one defining a scalar integral in elementary calculus. As indicated in Fig. 2.1 the curve is subdivided into shorter (and straighter!) arcs by selecting a sequence of points $\mathbf{x}_i$, and to each arc assigning a "measure" $\Delta\mathbf{x}_i = \mathbf{x}_i - \mathbf{x}_{i-1}$ of its length and direction. This differs from the standard definition of a scalar integral only in using vector-valued measure of "directed length" instead of a scalar measure of length. All details of the limit process are essentially the same.

Since F is generally multivector-valued it need not commute with the vector $d\mathbf{x}$, so $\int d\mathbf{x}F$ is different than $\int Fd\mathbf{x}$, and it is defined by the obvious reversal of factors in (2.1).

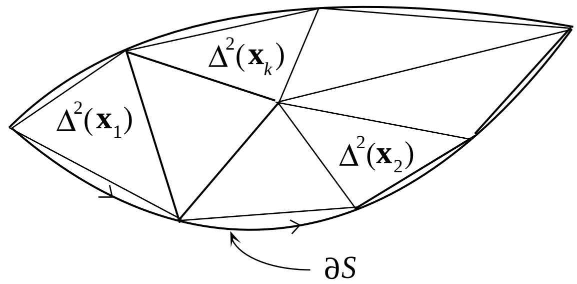

Fig. 2.2. *Triangulation* of an orientable surface $\mathcal{S}$.

If the curve $\mathcal{C}$ has a parametric representation $\mathbf{x} = \mathbf{x}(s)$ with endpoints $\mathbf{a} = \mathbf{x}(\alpha)$, $\mathbf{b} = \mathbf{x}(\beta)$, then the line integral can alternatively be defined in terms of the "scalar integral" by

$$\int_{\mathcal{C}} F\, d\mathbf{x} = \int_{\alpha}^{\beta} F\, \frac{d\mathbf{x}}{ds}\, ds = \int_{\alpha}^{\beta} F(\mathbf{x}(s))\, \frac{d\mathbf{x}}{ds}\,(s)\, ds. \tag{2.2}$$

The definition of line integral given here is more general than the usual one. To see the difference, note that the integral of a vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$ can be separated into scalar and bivector parts, thus

$$\int \mathbf{v}\, d\mathbf{x} = \int \mathbf{v} \cdot d\mathbf{x} + \int \mathbf{v} \wedge d\mathbf{x}. \tag{2.3}$$

The scalar (-valued) part is

$$\int \mathbf{v} \cdot d\mathbf{x} = \frac{1}{2} \int \mathbf{v}\, d\mathbf{x} + \frac{1}{2} \int d\mathbf{x}\, \mathbf{v} \tag{2.4}$$

and the bivector part is

$$\int \mathbf{v} \wedge d\mathbf{x} = \frac{1}{2} \int (\mathbf{v}\, d\mathbf{x} - d\mathbf{x}\, \mathbf{v}). \tag{2.5}$$

Both parts are called line integrals, and it is often of interest to consider them separately. The ordinary concept of line integral is limited to the scalar-valued integral $\int \mathbf{v} \cdot d\mathbf{x}$. However, it will be shown in Chapter 3 that the complete integral (2.3) is needed to obtain the powerful results of complex variable theory.

As a specific example of a line integral, let $\mathbf{v} = d\mathbf{x}/|d\mathbf{x}|$ be the unit tangent on the curve, then

$$\int \mathbf{v}\, d\mathbf{x} = \int \mathbf{v} \cdot d\mathbf{x} = \int |\, d\mathbf{x}\,|$$

is the *arc length* of the curve. It should be noted that,

$$\left| \int d\mathbf{x} \right| \;\le\; \int |\, d\mathbf{x}\,| \tag{2.6}$$

with equality only for a straight line (see Exercise 2.1).

Surface Integrals

The *directed surface integral* of a field $F = F(\mathbf{x})$ over an orientable surface $\mathcal{S} = \{\mathbf{x}\}$ is defined by

$$\int_{\mathcal{S}} F\, d^2\mathbf{x} = \int_{\mathcal{S}} F(\mathbf{x})\, d^2\mathbf{x} = \lim_{n \to \infty} \sum_{k=1}^{n} F(\mathbf{x}_k)\, \Delta^2(\mathbf{x}_k). \tag{2.7}$$

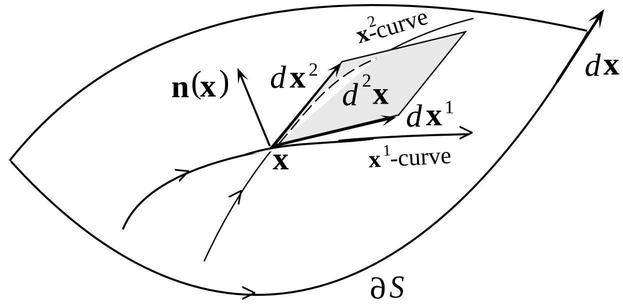

Fig. 2.3. Parametric representation of the area element $d^2\mathbf{x}{=}d\mathbf{x}^1 \wedge d\mathbf{x}^2{=}{-}i\mathbf{n}(\mathbf{x})|\, d^2\mathbf{x}\,|$ on a surface $\mathcal{S}$ in $\mathcal{E}_3$.

The meaning of this formula is to be understood as follows. The surface can be approximated to any desired accuracy by a connected system of n triangles as illustrated in Fig. 2.2. Each triangle can be associated with a point $\mathbf{x}_k$ on $\mathcal{S}$ near its center. The *directed area* $\Delta^2(x_k)$ of the triangle is a bivector with direction approximating the direction of the tangent plane at $\mathbf{x}_k$ and magnitude $|\,\Delta^2(\mathbf{x}_k)\,|$ approximating a scalar area element. In the limit $\Delta^2(\mathbf{x}_k)$ becomes the *directed area element*

$$d^2\mathbf{x} = \mathbf{I}_2\, d^2x, \tag{2.8}$$

where $\mathbf{I}_2 = \mathbf{I}_2(x)$ is the bivector-valued orientation of $\mathcal{S}$ defined earlier, and

$$d^2x = |\, d^2\mathbf{x}\,| = \mathbf{I}_2^\dagger\, d^2\mathbf{x} \tag{2.9}$$

is the ordinary *Riemannian measure* of area. As in the standard theory of the Riemann integral, the limit requires that the subdivision of $\mathcal{S}$ becomes increasingly fine as new triangulation points are added.

With standard mathematical arguments it can be proved that the limit is independent of the way $\mathcal{S}$ is triangulated provided $\Delta^2(\mathbf{x}_k) \to 0$ for each k as $n \to \infty$. The surface integral is said to exist if the limiting value of the sum is a multivector with a finite magnitude. The integral over a piecewise smooth surface is defined as the sum of integrals over each smooth piece.

The directed integral can be reduced to a standard Riemann integral simply by writing

$$\int F\, d^2\mathbf{x} = \int G\, d^2x, \tag{2.10}$$

where $G = F\mathbf{I}_2$. However, we shall see that the directed measure $d^2\mathbf{x}$ should be regarded as more fundamental than the scalar Riemann measure d^2x, because it is essential for the general form of the fundamental theorem of calculus.

If the surface $\mathcal{S}$ is described by a coordinate function $\mathbf{x} = \mathbf{x}(x^1, x^2)$, as explained in the preceding section, then the *directed line element* for an x^1-curve is given by

$$d\mathbf{x}^1 = dx^1 \frac{\partial \mathbf{x}}{\partial x^1} = dx^1 \mathbf{e}_1. \tag{2.11}$$

Similarly, $d\mathbf{x}^2 = dx^2\mathbf{e}_2$ for an x^2-curve. Consequently the *directed area element* for the surface can be given the coordinate form

$$d^2\mathbf{x} = d\mathbf{x}^1 \wedge d\mathbf{x}^2 = \mathbf{e}_1 \wedge \mathbf{e}_2\, dx^1\, dx^2 \tag{2.12}$$

(see Fig. 2.3). Now the surface integral of $F = F(\mathbf{x}(x^1, x^2))$ can be expressed as an *iterated integral*

$$\int_{\mathcal{S}} F\, d^2\mathbf{x} = \int_{\mathcal{S}} F\, d\mathbf{x}^1 \wedge d\mathbf{x}^2 = \int_{\alpha_1}^{\beta_1} dx^1 \int_{\alpha_2(x^1)}^{\beta_2(x^1)} dx^2 F\, \mathbf{e}_1 \wedge \mathbf{e}_2. \tag{2.13}$$

This reduces the surface integral to a succession of integrals with respect to scalar variables, thus enabling us to use results from elementary scalar calculus for evaluating surface integrals. Equivalence of the iterated integral (2.13) to the coordinate-free directed integral (2.7) is an important mathematical result which will be taken for granted here.

As an important example of a surface integral, we take $F = \mathbf{I}_2^\dagger$ in (2.13) to get the *surface area*

$$A = \int d^2x = \int |\, d^2\mathbf{x}\,| = \int |\, \mathbf{e}_1 \wedge \mathbf{e}_2\,|\, dx^1\, dx^2. \tag{2.14}$$

Note that

$$\left| \int d^2\mathbf{x} \right| \;\le\; \int |\, d^2\mathbf{x}\,|, \tag{2.15}$$

with equality only for a flat surface (i.e. a surface lying in a plane). In fact, *for any closed surface*, such as a sphere,

$$\oint d^2\mathbf{x} = 0, \tag{2.16}$$

where $\oint$ indicates integration over a closed manifold. Note also that the surface index on the integral sign is often dropped when the domain of integration is clear from the context. By the way, the result (2.16) is an easy consequence of the fundamental theorem of calculus.

For a surface in $\mathcal{E}_3$, according to (1.10) we can write the directed area element in the form

$$d^2\mathbf{x} = -i\mathbf{n}\, d^2x, \tag{2.17}$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the unit surface normal shown in Figs. 1.4 and 2.3. Since i is constant and commutes with F the directed integral can then be written

$$\int F\, d^2\mathbf{x} = -i \int F\, \mathbf{n}\, d^2x. \tag{2.18}$$

For a vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$, this gives

$$i \int \mathbf{v}\, d^2\mathbf{x} = \int \mathbf{v}\,\mathbf{n}\, d^2x = \int \mathbf{v}\cdot\mathbf{n}\, d^2x + \int \mathbf{v} \wedge \mathbf{n}\, d^2x. \tag{2.19}$$

The usual definition of surface integral is limited to the scalar part

$$\int \mathbf{v}\cdot\mathbf{n}\, d^2x = \int \mathbf{v}\cdot\mathbf{n}\, |\, \mathbf{e}_1 \times \mathbf{e}_2\,|\, dx^1 dx^2. \tag{2.20}$$

Our more general approach shows that the directed measure is the crucial concept underlying this definition.

Integrals of Differential Forms

The generalization of our definition of surface integral to manifolds of any dimension is straightforward. On an oriented m-manifold $\mathcal{M}$ we can define an m-vector-valued directed measure

$$d^m\mathbf{x} = \mathbf{I}_m d^m x, \tag{2.21}$$

where $\mathbf{I}_m = \mathbf{I}_m(\mathbf{x})$ is the orientation of $\mathcal{M}$ defined in the previous section, and $d^m x = |d^m \mathbf{x}|$. The *directed integral* of a field $F = F(\mathbf{x})$ over $\mathcal{M}$ can be defined by reducing it to a standard Riemann integral with

$$\int_{\mathcal{M}} F\, d^m\mathbf{x} = \int_{\mathcal{M}} F\, \mathbf{I}_m d^m x. \tag{2.22}$$

If $\mathcal{M}$ is described by a coordinate function $\mathbf{x} = \mathbf{x}(x^1, x^2, \ldots, x^m)$, the directed measure can be put in the "coordinate form"

$$\begin{aligned} d^m\mathbf{x} &= d\mathbf{x}^1 \wedge d\mathbf{x}^2 \wedge \ldots \wedge d\mathbf{x}^m \\ &= \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_m\, dx^1 dx^2 \ldots dx^m, \end{aligned} \tag{2.23}$$

where

$$d\mathbf{x}^k = dx^k \frac{\partial \mathbf{x}}{\partial x^k} = dx^k \mathbf{e}_k. \tag{2.24}$$

Also,

$$d^m x = |\, d^m\mathbf{x}\,| = |\, \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_m \,|\, dx^1 dx^2 \ldots dx^m \tag{2.25}$$

By substitution of (2.23) or (2.25) into (2.22) the directed integral can be converted to an *iterated integral* with respect to m successive scalar variables.

For a 3-manifold $\mathcal{M}$ in $\mathcal{E}_3$, the directed measure is

$$d^3\mathbf{x} = i d^3 x, \tag{2.26}$$

where i is the unit pseudoscalar. The directed integral can therefore be put in the form

$$\begin{aligned} \int_{\mathcal{M}} F\, d^3\mathbf{x} &= i \int_{\mathcal{M}} F\, d^3 x \\ &= i \int_{\alpha_1}^{\beta_1} dx^1 \int_{\alpha_2}^{\beta_2} dx^2 \int_{\alpha_3}^{\beta_3} dx^3 |\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3|\, F \end{aligned} \tag{2.27}$$

Of course, for $F = 1$ this reduces to an integral for the *volume* of $\mathcal{M}$:

$$V = \int d^3 x = \int dx^1 dx^2 dx^3\, |\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \,|. \tag{2.28}$$

We have not yet arrived at the most general definition of a directed integral. To achieve that we introduce the general concept of differential form. A *differential form of degree m* (or *m-form*) L on an m-dimensional manifold $\mathcal{M} = \{\mathbf{x}\}$ is a multivector-valued field which is also a linear function of the directed measure, that is, $L = L(\mathbf{x}, d^m\mathbf{x})$. With the $\mathbf{x}$-dependence taken for granted we write

$$L = L(d^m\mathbf{x}). \tag{2.29}$$

By (2.21) and (2.23), the linearity of L implies that

$$L(d^m\mathbf{x}) = d^m x\, L(\mathbf{I}_m) = dx^1 \ldots dx^m L(\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_m). \tag{2.30}$$

Therefore the integral of a differential form over $\mathcal{M}$ can be defined or reduced to the Riemann integral of a field $F(\mathbf{x}) = L(\mathbf{x}, I_m(\mathbf{x}))$ as expressed by

$$\int_{\mathcal{M}} L = \int_{\mathcal{M}} L(d^m\mathbf{x}) = \int_{\mathcal{M}} F\, d^m x. \tag{2.31}$$

By the very definition of a differential form its integral is a *directed integral*, that is, an integral based on the directed measure $d^m\mathbf{x}$.

It can be proved that any differential m-form can be expressed in the form

$$L(d^m\mathbf{x}) = \sum_k F_k d^m\mathbf{x}\, G_k, \tag{2.32}$$

where the $F_k = F_k(\mathbf{x})$ and $G_k = G_k(\mathbf{x})$ are multivector-valued fields. Of particular importance is the special case

$$\int_{\mathcal{M}} L(d^m\mathbf{x}) = \int_{\mathcal{M}} F\, d^m\mathbf{x}\, G. \tag{2.33}$$

Obviously, this reduces to (2.22) when $G = 1$.

Another important special case is the scalar-valued differential form. Taking the scalar part of (2.32), with the help of (1–1.37) we find

$$\langle L\rangle = \textstyle\sum_k \langle F_k d^m\mathbf{x}\, G_k\rangle = \langle d^m\mathbf{x}\, F\rangle = d^m\mathbf{x}\cdot F_{\overline{m}}, \tag{2.34}$$

where F is defined by $F = \sum_k G_k F_k$, and we note that only the m-vector part of F contributes, because $d^m\mathbf{x}$ is an m-vector. The scalar-valued integral

$$\langle \int L\,\rangle \;= \int \langle L\,\rangle = \int d^m\mathbf{x}\cdot F_{\overline{m}} \tag{2.35}$$

is obviously just what we would have obtained by taking the scalar part of (2.22).

The differential forms ordinarily employed in mathematics and physics are scalar-valued, and they are equivalent to m-forms defined by (2.34). It will be seen in subsequent sections that the more general concept of multivector-valued differential form introduced here is much more powerful. The most important thing to understand is that the concept of differential form is subsidiary to the more fundamental concepts of directed measure and directed integrals. Mathematicians have been slow to recognize this fact.

2-2 Exercises

(2.1) For a curve $\mathcal{C}$ with endpoints $\mathbf{a}, \mathbf{b}$, evaluate the following integrals by using definitions (2.1) and (2.2) respectively.

$$\int_e d\mathbf{x} = \mathbf{b} - \mathbf{a}$$

$$\int_e \mathbf{x}\cdot d\mathbf{x} = \tfrac{1}{2}(\mathbf{b}^2 - \mathbf{a}^2)$$

Note that the result is the same for all curves with the same endpoints.

(2.2) For spherical coordinates evaluate $|\,\mathbf{e}_1\wedge\mathbf{e}_2\,|$ and $|\,\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3\,|$ as functions of the coordinates. For a sphere of radius r, compute the surface area $A = 4\pi r^2$ and the volume enclosed $v = 4\pi r^3/3$.

(2.3) For more exercises, get some good coordinate-surface integrals from calculus texts.

2-3 Tangential Derivatives

We are now prepared to formulate the general concept of differentiation with respect to a point on a vector manifold. The *tangential derivative* $\partial F = \partial_{\mathbf{x}} F(\mathbf{x})$ of a field $F = F(\mathbf{x})$ on an m-manifold $\mathcal{M}$ is defined by

$$\partial F(\mathbf{x}) = \mathbf{I}_m^{-1}(\mathbf{x}) \lim_{N\to 0} \frac{1}{N} \oint_{\partial\mathcal{N}} d^{m-1}\mathbf{x}' F(\mathbf{x}'), \tag{3.1}$$

which is to be understood as follows:

(1) A neighborhood $\mathcal{N} = \mathcal{N}(\mathbf{x})$ of the point $\mathbf{x}$ in $\mathcal{M}$ is an oriented m-dimensional submanifold of $\mathcal{M}$ with $\mathbf{x}$ as an interior point. A scalar measure N is defined by

$$N = \int_{\mathcal{N}} d^m x \tag{3.2}$$

The orientation at $\mathbf{x}$ is specified by the unit tangent $\mathbf{I}_m(\mathbf{x})$.

(2) The neighborhood boundary $\partial\mathcal{N}$ has a directed measure

$$d^{m-1}\mathbf{x}' = \mathbf{I}_{m-1}(\mathbf{x}')|\, d^{m-1}\mathbf{x}' \,|$$

with the standard orientation

$$\mathbf{I}_m = \mathbf{I}_{m-1}\mathbf{n}, \tag{3.3}$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x}')$ is the boundary normal (as explained in Section 2–1).

(3) The limit is taken by shrinking $\mathcal{N}(\mathbf{x})$ to the point $\mathbf{x}$, and this entails $N\to 0$. By standard mathematical argument it can be proved that the limit is independent of the choice of successively smaller neighborhoods provided their boundaries are not allowed to have "pathological shapes."

(4) Ordinarily the derivative $\partial F(\mathbf{x})$ is said to *exist* at $\mathbf{x}$ if the limit converges to a multivector of finite magnitude. However, we shall find it convenient later to allow infinite limits in certain circumstances.

The *tangential derivative* should be regarded as *the derivative with respect to a point* on a vector manifold. The adjective "tangential" is hardly necessary here, and it will be employed mainly for emphasis, marking the difference from other concepts of derivative. The term "tangential" is meant to indicate that the *domain* of the vector variable is restricted to a given manifold as assumed in the definition (3.1). However, the domain of the variable is taken for granted in other definitions of derivatives.

There are good reasons to regard the tangential derivative as the most fundamental of all concepts of derivative. The best reason is the central role it plays in the generalized Fundamental Theorem of Calculus, developed in Section 2–4. We show below that tangential derivative is a natural generalization of the conventional derivative in "scalar calculus." Moreover, it generalizes the "vector derivative" introduced in Section 1–3 and can be regarded as a generalized partial derivative. To establish these points, we examine the limit in the tangential derivative definition (3.1) for each of the three cases $m = 1, 2, 3$.

On the manifold $\mathcal{E}_n$, the tangential derivative is identical to the vector derivative ∇ defined in terms of rectangular coordinates in Section 1–3. Furthermore, (3.1) supplies us with a *coordinate-free definition* of ∇. It can be related to coordinate partial derivatives, however, by expressing the integral in (3.1) in terms of coordinates before taking the limit.

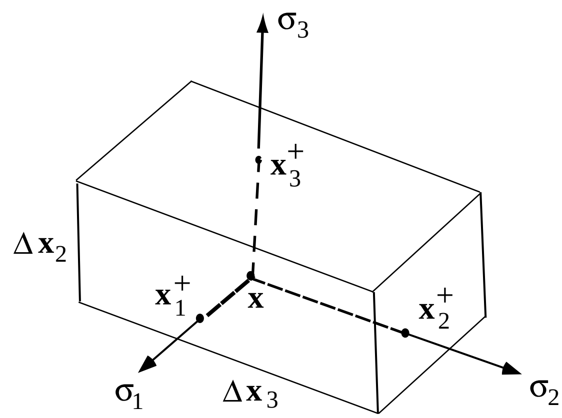

Fig. 3.1. Features of rectangular neighborhood of a point $\mathbf{x}$ in $\mathcal{E}_3$.

For $\mathcal{E}_3$ we recognize (3.2) as the volume integral $V = \int |\, d^3\mathbf{x}\,|$, and since $\mathbf{I}_3 = i$ is constant, (3.3) allows us to write $i^{-1}d^2\mathbf{x} = \mathbf{n}d^2x$, so the definition (3.1) can be put in the form

$$\partial F(\mathbf{x}) = \lim_{V\to 0} \frac{1}{V} \oint_{\partial\mathcal{N}} d^2x\, \mathbf{n}\, F. \tag{3.4}$$

For the neighborhood $\mathcal{N}$, we choose a parallelepiped parametrized by rectangular coordinates x_1, x_2, x_3. The boundary consists of six faces $\mathcal{S}_1^{\pm}, \mathcal{S}_2^{\pm}, \mathcal{S}_3^{\pm}$ centered at the points $\mathbf{x}_1^{\pm}, \mathbf{x}_2^{\pm}, \mathbf{x}_3^{\pm}$ with constant outer normals $\pm\boldsymbol{\sigma}_1, \pm\boldsymbol{\sigma}_2, \pm\boldsymbol{\sigma}_3$, as indicated in Fig. 3.1. The boundary integral is the sum of the integrals over the six faces as indicated by

$$\oint_{\partial N} = \int_{S_1^+} + \int_{S_1^-} + \int_{S_2^+} + \int_{S_1^-} + \int_{S_3^+} + \int_{S_3^-} .$$

We take the limit in (3.4) by first shrinking the area integral over each face to zero and then evaluating the limit as opposing faces are brought together at $\mathbf{x}$. Thus, we put (3.4) in the form

$$\begin{aligned}\partial F &= \lim_{V\to 0} \frac{1}{V} \sum_{k=1}^{3} \left\{ \int_{\mathcal{S}_k^+} d^2x\, \boldsymbol{\sigma}_k\, F + \int_{\mathcal{S}_k^-} d^2x\, (-\boldsymbol{\sigma}_{\mathbf{k}})F \right\} \\ &= \sum_{k=1}^{3} \boldsymbol{\sigma}_k \lim_{\Delta x_k \to 0} \frac{1}{\Delta x_k} \left\{ F(\mathbf{x}_k^+) - F(\mathbf{x}_k^-) \right\} = \sum_k \boldsymbol{\sigma}_k \frac{\partial F}{\partial \mathbf{x}_k}.\end{aligned}$$

This establishes the equivalence

$$\partial = \nabla \tag{3.5}$$

on $\mathcal{E}_3$ and any 3-dimensional submanifold within $\mathcal{E}_3$.

Let us next examine the definition of the tangential derivative (3.1) for the case $m = 1$. In this case, the neighborhood $\mathcal{N}$ is a curve through $\mathbf{x}$ with endpoints $\mathbf{x}_{\pm}$. The integral (3.2) gives the arc length $\Delta s = \int |\, d\mathbf{x}\,|$, and $\mathbf{I}_1(\mathbf{x}) = \mathbf{e}$ is the *unit tangent*. Since the boundary of $\mathcal{N}$ consists only of the two points $\mathbf{x}_{\pm}$, the integral over the boundary is evaluated by a trivial appeal to the definition of the an integral as the limit of a sum. The sum is not affected by the limit in this case, so we obtain

$$\oint d^0\mathbf{x}' F(\mathbf{x}') = \Delta^0(\mathbf{x}_+)F(\mathbf{x}_+) + \Delta^0(x_-)F(\mathbf{x}_-) = F(\mathbf{x}_+) - F(\mathbf{x}_-)\,, \tag{3.6}$$

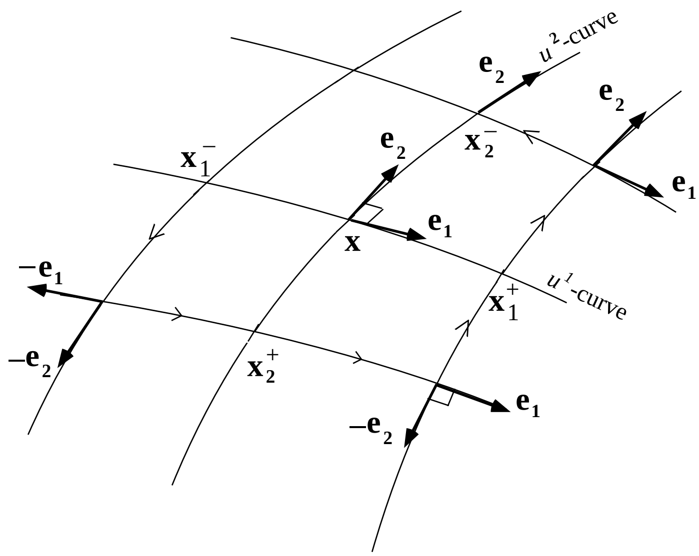

Fig. 3.2. A neighborhood of a point $\mathbf{x}$ on a surface parametrized by orthogonal curvilinear coordinates. Note that the outer normal is $\mathbf{e}_1$ on the curve $\mathcal{C}_1^+$ and $-\mathbf{e}_1$ on the curve $\mathcal{C}_1^-$.

where the orientations $\Delta^0(\mathbf{x}_+) = \mathbf{I}_0(\mathbf{x}_+) = 1$ and $\Delta^0(\mathbf{x}_-) = \mathbf{I}_0(\mathbf{x}_-) = -1$ have been assigned to the endpoints in accordance with the condition (3.3). Consequently, for the case $m = 1$ the definition (3.1) for the tangential derivative reduces to

$$\partial F(\mathbf{x}) = \mathbf{e}(\mathbf{x}) \lim_{\Delta s \to 0} \frac{F(\mathbf{x}_+) - F(\mathbf{x}_-)}{\Delta s} = \mathbf{e}\frac{dF}{ds}. \tag{3.7}$$

Thus, *on a curve the tangential derivative is essentially equivalent to the derivative with respect to arc length.*

There is an important lesson to be learned from this result. In elementary calculus differentiation is introduced before integration and the derivative is defined as the limit of a "difference quotient," just as in (3.6) with $\Delta s \sim |\mathbf{x}_+ - \mathbf{x}_-|$. Many attempts to generalize calculus beyond scalar variables by the same approach have proved unsatisfactory, and now we can see why. The emphasis on difference quotients was misguided. According to (3.1) and (3.2) *the derivative should be based on the quotient of two integrals*, and by (3.5) this just happens to reduce to a difference quotient in the 1-dimensional case. Of course, this means that integrals should be regarded as more fundamental than derivatives—not unreasonable considering that sums are more fundamental than quotients.

Finally, let us examine the definition of tangential derivative for the case $m = 2$. In this case the point $\mathbf{x}$ lies on a surface and we can enclose it with four coordinate curves $\mathcal{C}_1^+, \mathcal{C}_1^-, \mathcal{C}_2^+, \mathcal{C}_2^-$, as shown in Fig. 3.2, thus determining a neighborhood $\mathcal{N}$ of the point. Employing the notations introduced earlier for coordinates, we use (3.3) to relate the orientation $\mathbf{I}_2$ to the coordinate frame by $\mathbf{I}_2^\dagger = \mathbf{e}_1 \wedge \mathbf{e}_2 / |\,\mathbf{e}_1 \wedge \mathbf{e}_2\,|$, and we note that the reciprocal frame is given by

$$\mathbf{e}^1 = -(\mathbf{e}_1 \wedge \mathbf{e}_2)^{-1}\mathbf{e}_2, \quad \mathbf{e}^2 = (\mathbf{e}_1 \wedge \mathbf{e}_2)^{-1}\mathbf{e}_1$$

For the purpose of investigating the limit, we may assume that the neighborhood $\mathcal{N}$ is small enough so the $\mathbf{e}_k$ can be regarded as constant on $\mathcal{N}$ while F is regarded as constant on each of the bounding curves. Then (3.2) gives

$$N = \int d^2x = \int |\,\mathbf{e}_1 \wedge \mathbf{e}_2\,|\, dx^1 dx^2 \sim |\,\mathbf{e}_1 \wedge \mathbf{e}_2\,|\, \Delta x^1 \Delta x^2$$

Also, with reference to Fig. 3.2,

$$\oint_{\partial\mathcal{N}} d\mathbf{x}'F' = \int_{e_1^+}(dx^2\mathbf{e}_2)F + \int_{e_1^-}(-dx^2\mathbf{e}_2)F + \int_{e_2^+}(-dx^1\mathbf{e}_1)F + \int_{e_2^-}(dx^1\mathbf{e}_1)F$$
$$\sim \Delta x^2\mathbf{e}_2(F(\mathbf{x}_1^+) - F(\mathbf{x}_1^-)) - \Delta x^1\mathbf{e}_1(F(\mathbf{x}_2^+) - F(\mathbf{x}_2^+)).$$

Accordingly, we can put (3.1) in the form

$$\begin{aligned}\partial F &= \lim_{\Delta x^k\to 0} \frac{(\mathbf{e}_2\wedge\mathbf{e}_1)^{-1}}{\Delta x^1\Delta x^2}\Big\{\Delta x^2\mathbf{e}_2(F(\mathbf{x}_1^+) - F(\mathbf{x}_1^-)) \\ &\qquad\qquad - \Delta x^1\mathbf{e}_1(F(\mathbf{x}_2^+) - F(\mathbf{x}_2^-))\Big\} \\ &= \sum_{k=1}^{2}\mathbf{e}^k \lim_{\Delta x^k\to 0}\frac{F(\mathbf{x}_k^+) - F(\mathbf{x}_k^-)}{\Delta x^k}.\end{aligned}$$

Hence

$$\partial F = \sum_{k=1}^{2}\mathbf{e}^k\frac{\partial F}{\partial x^k}. \tag{3.8}$$

This is the desired result relating the tangential derivative to partial derivatives for *arbitrary* surface coordinates.

Recall that for the flat manifold $\mathcal{E}_3$ we proved the result (1.23)

$$\nabla = \mathbf{e}^1\frac{\partial}{\partial x^1} + \mathbf{e}^2\frac{\partial}{\partial x^2} + \mathbf{e}^3\frac{\partial}{\partial x^3}. \tag{3.9}$$

For any surface embedded in $\mathcal{E}_3$ we can take x^1 and x^2 as coordinates so $\mathbf{n} = \hat{\mathbf{e}}^3$ is a unit normal to the surface. Consequently, (3.8) is related to (3.9) by

$$\partial = \mathbf{n}\mathbf{n}\wedge\nabla\,. \tag{3.10}$$

This is a *coordinate-free* relation of the tangential derivative to the vector derivative for any surface in $\mathcal{E}_3$ with unit normal $\mathbf{n}$.

Similarly, for any curve in $\mathcal{E}_3$ we can identify a parametrization locally with one coordinate of a coordinate system for $\mathcal{E}_3$. With the choice x^1 for parameter, the unit tangent for the curve is $\mathbf{e} = \hat{\mathbf{e}}_1$, so (3.9) yields

$$\partial = \mathbf{e}\,\mathbf{e}\cdot\nabla. \tag{3.11}$$

For any curve in $\mathcal{E}_3$ with unit tangent $\mathbf{e}$, this is a coordinate-free relation of the tangential derivative (3.7) to the vector derivative ∇.

Equations (3.10), (3.11) and (3.5) can be combined in a single formula

$$\partial = P(\nabla) \tag{3.12}$$

expressing the tangential derivative for any manifold in $\mathcal{E}_3$ as a projection of ∇ onto the manifold. The projection operator P simply annihilates the component of ∇ which differentiates in a direction normal to the manifold. Note that for any unit vector field $\mathbf{n} = \mathbf{n}(\mathbf{x})$ on $\mathcal{E}_3$ we can write

$$\nabla = \mathbf{n}^2\nabla = \mathbf{n}\mathbf{n}\cdot\nabla + \mathbf{n}\mathbf{n}\wedge\nabla. \tag{3.13}$$

According to (3.10) and (3.11), this can be interpreted as a decomposition of ∇ into the sum of a tangential derivative and a “normal derivative” with respect to a surface (if $\mathbf{n}$ is its normal) or a curve (if $\mathbf{n}$ is its tangent).

Our derivation of the result (3.7) did not assume that the surface is embedded in $\mathcal{E}_3$. Moreover, its generalization to vector manifolds of any dimension is so obvious that we can simply write down the result with complete confidence. Thus, for any oriented vector m-manifold with coordinate function $\mathbf{x} = \mathbf{x}(x^1, \ldots, x^m)$, the tangential derivative (3.1) is related to the coordinate (partial) derivatives by

$$\partial = \sum_{k=1}^{m} \mathbf{e}^k \frac{\partial}{\partial x^k} . \tag{3.14}$$

It follows immediately that, for coordinate functions $x^k = x^k(\mathbf{x})$,

$$\mathbf{e}^k = \partial x^k . \tag{3.15}$$

This is the promised generalization of (1.11) which applies to any manifold, flat or not. Also, from (3.14) we have

$$\partial \mathbf{x} = \sum_k \mathbf{e}^k \frac{\partial \mathbf{x}}{\partial x^k} = \sum_k \mathbf{e}^k \mathbf{e}_k .$$

Thus, we obtain the basic formula for coordinate-free differentiation on manifolds:

$$\partial \mathbf{x} = m . \tag{3.16}$$

Since ∂ is a vector operator we also have $\partial \mathbf{x} = \partial \cdot \mathbf{x} + \partial \wedge \mathbf{x}$; hence

$$\partial \cdot \mathbf{x} = m , \tag{3.17a}$$

$$\partial \wedge \mathbf{x} = 0 . \tag{3.17b}$$

This generalizes the result $\nabla \mathbf{x} = m$ which we found for the flat manifold $\mathcal{E}_m$. One other general property of ∂ should be noted here. Since the $\mathbf{e}^k$ are all tangent vectors on the manifold, they satisfy $\mathbf{I}_m \wedge \mathbf{e}^k = 0$ and $\mathbf{I}_m \mathbf{e}^k = \mathbf{I}_m \cdot \mathbf{e}^k$. Therefore, (3.14) implies

$$\mathbf{I}_m \wedge \partial = 0 \tag{3.18a}$$

and

$$\mathbf{I}_m \partial = \mathbf{I}_m \cdot \partial, \tag{3.18b}$$

where $\mathbf{I}_m = \mathbf{I}_m(\mathbf{x})$ is the unit tangent (or orientation) of the manifold.

As a final generalization of the tangential derivative, we need to define it for differential forms. For the differential form $L(d^{m-1}\mathbf{x}) = L(\mathbf{x}, d^{m-1}\mathbf{x})$ of degree $(m-1)$ defined on an m-manifold with orientation $\mathbf{I}_m$ the appropriate definition of tangential derivative is

$$\grave{L}(\mathbf{I}_m \grave{\partial}) = \lim_{N \to 0} \frac{1}{N} \oint L(d^{m-1}\mathbf{x}'), \tag{3.19}$$

where the limit is defined in the same way as for (3.1). The notation

$$\grave{L}(\mathbf{I}_m \grave{\partial}) = L(\grave{\mathbf{x}}, \mathbf{I}_m \grave{\partial}) = \grave{L}(\mathbf{I}_m \cdot \grave{\partial}) \tag{3.20}$$

is meant to indicate that the $\mathbf{x}$-dependence of L is differentiated while $\mathbf{I}_m = \mathbf{I}_m(\mathbf{x})$ is not, although it is proved in GC that $\partial \cdot \mathbf{I}_m = 0$. Note that the last equality in (3.20) is a consequence of (3.18b).

2-3 Exercises

(3.1) Let $\mathbf{x}$ be a point on a manifold $\mathcal{M}$ in $\mathcal{E}_n$ with tangential derivative $\partial = \partial_{\mathbf{x}}$. For $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ where $\mathbf{x}'$ is a constant vector, establish the basic derivatives

$$\partial \mathbf{r} = m,$$
$$\partial \mathbf{r}\cdot\mathbf{a} = \mathbf{a}\cdot\partial\mathbf{r} = P(\mathbf{a}),$$
$$\partial|\mathbf{r}| = P(\hat{\mathbf{r}}),$$

where $\mathbf{a}$ is a constant vector and P is a projection into the tangent space $\mathcal{T}_{\mathbf{x}}(\mathcal{M})$.

2-4 The Fundamental Theorem of Calculus

In this Section we establish one of the most important theorems in all of mathematics. The theorem has many different versions with different names attached to them. In its most elementary form it is called "the fundamental theorem of integral calculus." This well-established title is commensurate with the theorem's importance, so let us adopt it here. However, the term "integral" in the title should be dropped as redundant, a relic of times when differential calculus was developed apart from integral calculus. Indeed, we have seen that on manifolds of more than one dimension the "differential calculus" should be based on "integral calculus." The Fundamental Theorem is often called "Stokes' Theorem" or "the Generalized Stokes' Theorem," but without much historical justice.

Geometric algebra makes it possible to simplify the formulation and applications of the fundamental theorem significantly. The theorem is presented below in two versions which apply with equal efficiency to all special cases. It generalizes the ordinary formulation in terms of differential forms and yields the special formulations most widely used in physics by simple algebraic manipulations.

The Fundamental Theorem of Calculus (Basic Version):

Let $F = F(\mathbf{x})$ be a field with tangential derivative ∂F defined on a piecewise smooth oriented m-manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$. Then

$$\int_{\mathcal{M}} d^m\mathbf{x}\,\partial F = \oint_{\partial\mathcal{M}} d^{m-1}\mathbf{x}\,F, \tag{4.1}$$

where the orientations of $\mathcal{M}$ and $\partial\mathcal{M}$ have the standard relation on $\partial\mathcal{M}$

$$\mathbf{I}_m = \mathbf{I}_{m-1}\mathbf{n}, \tag{4.2}$$

with $d^m\mathbf{x} = \mathbf{I}_m d^m x$ and boundary normal $\mathbf{n}$.

The fundamental theorem can be proved by establishing the following sequence of equations

$$\begin{aligned}\int_{\mathcal{M}} d^m\mathbf{x}\,\partial F &= \lim_{n\to\infty}\sum_{k=1}^{n} \Delta^m\mathbf{x}_k \partial F(\mathbf{x}_k)\\ &= \lim_{n\to\infty}\sum_{k=1}^{n} \Delta^m\mathbf{x}_k \Big\{\frac{1}{\Delta^m\mathbf{x}_k}\oint_{\partial\mathcal{M}_k} d\mathbf{x}^{m-1}F\Big\}\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\oint_{\partial\mathcal{M}_k} d\mathbf{x}^{m-1}F = \oint_{\partial\mathcal{M}} d\mathbf{x}^{m-1}F.\end{aligned}$$

In the first step the integral is expressed as the limit of a sum. This involves partitioning $\mathcal{M}$ into n pieces $\mathcal{M}_k$. The piece $\mathcal{M}_k$ contains the point $\mathbf{x}_k$, and it has a directed measure $\Delta^m\mathbf{x}_k = \mathbf{I}_m(\mathbf{x}_k)|\,\boldsymbol{\Delta}^m\mathbf{x}_k\,|$ where

$$|\,\Delta^m\mathbf{x}_k\,| = \int_{\mathcal{M}_k} d^m x.$$

In the second step the definition (3.1) is used to approximate the tangential derivative at each $\mathbf{x}_k$ by an integral over $\partial\mathcal{M}_k$. The third step is just a simple cancellation of factors. The key point in the final step is that adjacent regions $\mathcal{M}_k$ share a common boundary, and their integrals over this boundary exactly cancel because they have opposite orientation. This follows from (4.2) since the outer normals for adjacent regions are oppositely directed. With complete cancellation for all internal boundaries, only integrals over those pieces of $\partial\mathcal{M}_k$ which are on $\partial\mathcal{M}$ remain to produce the final result.

The reason that this proof is so simple is that the tangential derivative has been defined to make the formulation and proof of the fundamental theorem as simple as possible. Note that if $\mathcal{M}$ is sufficiently small so that $\partial F = \partial F(\mathbf{x})$ is approximatly constant in $\mathcal{M}$, then (4.1) yields

$$\Big(\int_{\mathcal{M}} d^m\mathbf{x}'\Big)\partial F(\mathbf{x}) \sim \oint_{\partial\mathcal{M}} d^{m-1}\mathbf{x}' F(\mathbf{x}'),$$

and this approximation becomes exact as $\mathcal{M}$ shrinks to the point $\mathbf{x}$. Thus we recover the definition (3.1) of the tangential derivative from the fundamental theorem.

Note that the fundamental theorem relates the derivative ∂F inside $\mathcal{M}$ to F itself on $\partial\mathcal{M}$. Thus the derivative is a kind of "boundary operator" for functions. Moreover, this holds for vector manifolds of any dimension.

Special Cases in $\mathcal{E}_3$

Special cases of the fundamental theorem have many different forms which we now consider.

If $\mathcal{M}$ is a 3-manifold in $\mathcal{E}_3$, we can write

$$d^3\mathbf{x} = i d^3x, \quad \partial = \nabla, \quad d^2\mathbf{x} = i\mathbf{n}\, d^2x \tag{4.3}$$

So, after factoring out the pseudoscalar i the fundamental theorem (4.1) gives us *Gauss' Theorem* (Basic version):

$$\int d^3x\, \nabla F = \oint d^2x\, \mathbf{n}\, F. \tag{4.4}$$

For a scalar field $\varphi = \varphi(\mathbf{x})$, this gives us a formula for the integral of a gradient,

$$\int d^3x\, \nabla\varphi = \oint d^2x\, \mathbf{n}\, \varphi. \tag{4.5}$$

If $F = \mathbf{v} = \mathbf{v}(\mathbf{x})$ is a vector field, we can use $\nabla\mathbf{v} = \nabla\cdot\mathbf{v} + i\nabla\times\mathbf{v}$ in (4.4) and separate scalar and bivector parts to get the *divergence theorem*

$$\int d^3x \nabla\cdot\mathbf{v} = \oint d^2x\, \mathbf{n}\cdot\mathbf{v}, \tag{4.6}$$

as well as a *curl theorem*

$$\int d^3x \nabla\times\mathbf{v} = \oint d^2x\, \mathbf{n}\times\mathbf{v}. \tag{4.7}$$

The divergence theorem (4.6) is also called Gauss' theorem, but is seems better to retain the latter name for the more general result (4.4).

For the case $m = 2$, $\mathcal{M}$ is a surface in $\mathcal{E}_3$, and according to (2.17) we can write

$$d^2\mathbf{x} = -i\mathbf{n}\, d^2x, \quad d^1\mathbf{x} = d\mathbf{x}, \tag{4.8}$$

where $\mathbf{n}$ is the *surface normal* shown in Figs. 1.4 and 2.3 (not to be confused with the boundary normal in (4.2)). As noted in (3.10), we can also write

$$\partial = \mathbf{n}\mathbf{n}\wedge\nabla = (i\mathbf{n})\,\mathbf{n}\times\nabla \tag{4.9}$$

Hence

$$d^2x\,\partial = d^2x\,\mathbf{n}\times\nabla, \tag{4.10}$$

and the fundamental theorem (4.1) for a surface in $\mathcal{E}_3$ takes the form of *Stokes' theorem*:

$$\int d^2x\,\mathbf{n}\times\nabla F = \oint d\mathbf{x}\,F. \tag{4.11}$$

For a vector field $F = \mathbf{v}$, we can separate (4.11) into scalar and bivector parts to get the well-known *scalar Stokes' Theorem*:

$$\int d^2x\,\mathbf{n}\cdot(\nabla\times\mathbf{v}) = \oint d\mathbf{x}\cdot\mathbf{v}, \tag{4.12}$$

and a less well-known *vector Stokes' Theorem*:

$$\int d^2x\,(\mathbf{n}\times\nabla)\times\mathbf{v} = \oint d\mathbf{x}\times\mathbf{v}. \tag{4.13}$$

In most of the physics literature only (4.12) is known as Stokes' theorem. However, the theorem was originally due to Kelvin who told Stokes about it in a letter; Stokes' only contribution was to set the proof of the theorem as an examination question.

Finally, if $\mathcal{M}$ is a curve in $\mathcal{E}_3$ with endpoints $\mathbf{a}$ and $\mathbf{b}$, we can write

$$d^1\mathbf{x} = d\mathbf{x} = \mathbf{e}|\,d\mathbf{x}\,| \quad \text{and} \quad \partial = \mathbf{e}\mathbf{e}\cdot\nabla$$

Hence,

$$d\mathbf{x}\partial = d\mathbf{x}\cdot\nabla, \tag{4.14}$$

and the fundamental theorem (4.1) reduces to the elementary result

$$\int d\mathbf{x}\cdot\nabla F = \int dF = F(\mathbf{b}) - F(\mathbf{a}) \tag{4.15}$$

Generalizations

We noted in Section 2-2 that the most general integrand for a directed integral is a differential form. Therefore the most general version of the fundamental theorem must involve differential forms as well. To give it a compact formulation we introduce the concepts of exterior differential. The *exterior differential* of an $(m-1)$-form $L = L(d^{m-1}\mathbf{x})$ is an m-form $dL = dL(d^m\mathbf{x})$ defined by

$$dL = \grave{L}(d^m\mathbf{x}\,\grave{\partial}), \tag{4.16}$$

where $d^m\mathbf{x} = \mathbf{I}_m d^m x$ and the accent indicates the tangential derivative defined by (3.19). Now we can state

The Fundamental Theorem of Calculus (General Version):

If L is a differential $(m-1)$-form with exterior derivative dL defined on an oriented m-manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, then

$$\int_{\mathcal{M}} dL = \oint_{\partial\mathcal{M}} L. \tag{4.17}$$

More details of the theorem included in the basic version are regarded as understood here. The proof of this general version of the theorem does not differ from the proof of the basic version in any essential way, so it need not be discussed here.

The compact form (4.17) for the fundamental theorem is deceptively simple. In particular, it suppresses the dependence on the directed measure, so it might be desirable to make that explicit by writing

$$\int dL(d^m\mathbf{x}) = \oint L(d^{m-1}\mathbf{x})\,.$$

To get the conventional formulation of the fundamental theorem in terms of scalar-valued differential forms, we simply take the scalar part of (4.16). According to (2.34) we can write

$$\langle L\rangle = (d^{m-1}\mathbf{x})\cdot\mathbf{F} \tag{4.18a}$$

where $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is an $(m-1)$-vector-valued field. Whence

$$d\langle L\rangle = \langle dL\rangle = \langle d^m\mathbf{x}\,\partial\mathbf{F}\rangle = (d^m\mathbf{x})\cdot(\partial\wedge\mathbf{F}), \tag{4.18b}$$

where the last form follows from $\partial F = \partial\cdot\mathbf{F} + \partial\wedge\mathbf{F}$ and the fact that $\partial\wedge\mathbf{F}$ is an m-vector while $\partial\cdot\mathbf{F}$ is an $(m-2)$-vector.

By using (4.18 a,b) we obtain from (4.16) the *scalar Stokes' Theorem* (General Form):

$$\int (d^m\mathbf{x})\cdot(\partial\wedge\mathbf{F}) = \oint (d^{m-1}\mathbf{x})\cdot\mathbf{F} \tag{4.19}$$

Obviously, this contains (4.12) as a special case.

For a flat manifold the fundamental theorem has an important alternative formulation. Since the orientation $\mathbf{I}_m$ is constant on a flat manifold, we can write

$$dL = \grave{L}(\mathbf{I}_m\grave{\partial})d^m x = \mathbf{I}_m\grave{T}(\grave{\partial})d^m x\,, \tag{4.20a}$$

$$L = L(\mathbf{I}_m\mathbf{n})d^{m-1}x = \mathbf{I}_m T(\mathbf{n})d^{m-1}x\,, \tag{4.20b}$$

where $T(\mathbf{n}) = T(\mathbf{x},\mathbf{n})$ is a *tensor field* defined by

$$T(\mathbf{n}) = \mathbf{I}_m^{-1}L(\mathbf{I}_m\mathbf{n}), \tag{4.21}$$

and its tangential derivative is

$$\grave{T}(\grave{\partial}) = T(\grave{\mathbf{x}},\grave{\partial})\,. \tag{4.22}$$

Any linear function $T(\mathbf{n})$ of a vector variable $\mathbf{n}$ is called a *tensor*, and it is called a *tensor field* when it is also a field on a manifold, that is, when $T(\mathbf{n}) = T(\mathbf{x},\mathbf{n}) = T(\mathbf{x},\mathbf{n}(\mathbf{x}))$. Now by substituting (4.20 a,b) into (4.17) and factoring out the $\mathbf{I}_m$, we obtain

Gauss' Theorem (General form):

If $T(\mathbf{n})$ is a tensor field on a flat manifold $\mathcal{M}$, then

$$\int_{\mathcal{M}} d^m x\,\grave{T}(\grave{\nabla}) = \oint_{\partial\mathcal{M}} d^{m-1}x\,T(\mathbf{n}). \tag{4.23}$$

For *rectangular coordinates* on a flat manifold we can write $\partial = \sum_k \boldsymbol{\sigma}_k\partial_k$ and

$$\grave{T}(\grave{\partial}) = \sum_{k=1}^{m}\partial_k T_k\,, \tag{4.24}$$

where $T_k \equiv T(\boldsymbol{\sigma}_k)$. The derivative (4.24) is often called the *divergence* of the tensor field T because of its similarity to the divergence of a vector field $\mathbf{v}$ in the coordinate form $\nabla\cdot\mathbf{v} = \partial_k v_k$. Since $T(\mathbf{n}) = \sum_k n_k T_k$ where $n_k = \mathbf{n}\cdot\boldsymbol{\sigma}_k$, (4.23) can be put in the alternative form

$$\int d^m x \sum\nolimits_k \partial_n T_k = \oint d^{m-1} x \sum\nolimits_k n_k T_k. \tag{4.25}$$

We will not be using this form, however.

For the important case of a tensor field with the form

$$T(\mathbf{n}) = G\mathbf{n}F = G(\mathbf{x})\mathbf{n}F(\mathbf{x}), \tag{4.26}$$

the derivative can be expanded by the product rule,

$$\grave{T}(\grave{\partial}) = \grave{G}\grave{\partial}\grave{F} = \grave{G}\grave{\partial}F + G\partial F. \tag{4.27}$$

As asserted before for ∇, we understand that ∂ differentiates to the right only, while $\grave{\partial}$ differentiates all functions or variables with accents on both the left and the right. Applying Gauss' theorem to (4.26) we obtain for any flat manifold

$$\int d^m\mathbf{x}\,\grave{G}\grave{\partial}\grave{F} = \int d^m\ x\,\grave{G}\grave{\partial}F + \int d^m x\, G\partial F = \oint d^{m-1}x\, G\mathbf{n}F. \tag{4.28}$$

This result applies only to a flat manifold, since it depends on requiring that the orientation $\mathbf{I}_m$ constant so it can be removed from under the integral sign. The appropriate generalization to nonflat manifolds is obtained by applying (4.17) to $L = Gd\mathbf{x}^{m-1}F$, with the result

$$\begin{aligned} \int \grave{G}\, d^m\mathbf{x}\,\grave{\partial}\grave{F} &= (-1)^{m-1}\int \grave{G}\,\grave{\partial}\, d^m\mathbf{x}\, F + \int G d^m\mathbf{x}\,\partial F \\ &= \oint G\, d^{m-1}\mathbf{x}\, F, \end{aligned} \tag{4.29}$$

where (3.18b) has been used to deduce that

$$d^m\mathbf{x}\grave{\partial} = d^m\mathbf{x}\cdot\grave{\partial} = (-1)^{m+1}\grave{\partial}d^m\mathbf{x}, \tag{4.30}$$

so factors can be reordered.

The formula (4.29) has very important applications in the next chapter. Here we apply it to the case $G = \mathbf{I}_m^\dagger$ with $d^m\mathbf{x} = \mathbf{I}_m d^m x$ as before. It is proved in GC that

$$\partial\mathbf{I}_m = \mathbf{N}\mathbf{I}_m, \tag{4.31a}$$

where $\mathbf{N} = \mathbf{N}(\mathbf{x})$ is a vector field normal to the manifold as expressed by

$$\partial\cdot\mathbf{I}_m = \mathbf{N}\cdot\mathbf{I}_m = 0. \tag{4.31b}$$

The proof involves differential geometry of vector manifolds which we cannot go into here. Accepting the result as given and using it in (4.29), we obtain

$$\int d^m\, x\,(\partial F + \mathbf{N}F) = \oint d^{m-1}x\,\mathbf{n}F \tag{4.32}$$

This is a generalization to arbitrary manifolds of Gauss' theorem (4.4) for flat manifolds. For the important special case of a vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$, the scalar part of (4.32) yields

$$\int d^m x\,(\partial\cdot\mathbf{v} + \mathbf{N}\cdot\mathbf{v}) = \oint d^{m-1}x\,\mathbf{n}\cdot\mathbf{v}. \tag{4.33}$$

Note that (4.31b) implies $\mathbf{N}\cdot\mathbf{v} = 0$ if $\mathbf{v}$ is a tangent field on the manifold.

Implications of the Fundamental Theorem

The Fundamental Theorem, with all its many forms, is the single most powerful tool for investigating the properties of fields on a manifold, because it completely specifies the relation between differentiation and integration. Derivatives describe *local properties* of a field, that is, behavior of the field in the immediate neighborhood of a point. On the other hand, integrals describe *global properties* of the field over the whole manifold. Therefore, the Fundamental Theorem interrelates local and global properties. Let us consider some simple but important examples to illustrate this.

Suppose that a manifold $\mathcal{M}$ is the boundary of some other manifold $\mathcal{V}$, as expressed by $\mathcal{M} = \partial\mathcal{V}$. Then $\mathcal{M}$ itself has no boundary, as can be expressed by the equation $\partial\partial\mathcal{V} = 0$, which applies to *any* manifold. If we apply the Fundamental Theorem (4.1) to a closed manifold $\mathcal{M}$, the right side of (4.1) vanishes automatically, so we have

$$\oint d^m\mathbf{x}\,\partial F = 0. \tag{4.34}$$

for any closed manifold. More specially, if $\mathcal{M}$ is a surface bounding a region $\mathcal{V}$ in $\mathcal{E}_3$, we can rewrite this in the form (4.11) and use Gauss' Thm. to get

$$\int_{\mathcal{V}} d^3x\,\nabla\times\nabla F = \oint_{\partial\mathcal{V}} d^2x\,\mathbf{n}\times\nabla F = 0. \tag{4.35}$$

This must hold for any region $\mathcal{V}$ and any differentiable function F, so we must have $\nabla\times\nabla F = 0$ at every point, a result which we found by different means in Section 1–3. Now we see that the operator equation $\nabla\wedge\nabla = i\nabla\times\nabla = 0$ corresponds to the "topological equation" $\partial\partial\mathcal{V} = 0$; it can be regarded as a *local* expression of the *global* topological property that a closed manifold has no boundary. Sometimes this is expressed by calling $\nabla\wedge\nabla = 0$ "the integrability condition."

As another example, if we put $F = 1$ in the Fundamental Theorem (4.1), we immediately get *for any closed m-manifold*

$$\oint d^m\mathbf{x} = 0 \tag{4.36}$$

This generalizes the result $\oint d\mathbf{x} = 0$ for a closed curve, which we noted in Section 2–2. It tells us that the directed area integral over *any* closed surface is zero.

If we take $F = \mathbf{x}$ in (4.1) and use $\partial x = m$, we get

$$\int d^{m+1}\mathbf{x} = \frac{1}{m}\oint d^m\mathbf{x}\,\mathbf{x} = \frac{1}{m}\oint d^m\mathbf{x}\wedge\mathbf{x}, \tag{4.37}$$

with $\oint d^m\mathbf{x}\cdot\mathbf{x} = 0$ as a corollary. For the case $m = 2$, this becomes a general formula for the *directed area* $\mathbf{A}$ of a surface,

$$\mathbf{A} \equiv \int d^2\mathbf{x} = \frac{1}{2}\oint d\mathbf{x}\wedge\mathbf{x}. \tag{4.38}$$

For the case $m = 3$, in accordance with the notation (4.3), we have

$$V = \int d^3x = \frac{1}{3}\oint d^2x\,\mathbf{n}\,\mathbf{x} = \frac{1}{3}\oint d^2x\,\mathbf{n}\cdot\mathbf{x} \tag{4.39}$$

for any closed surface bounding a region with volume V.

The above examples are special cases of the obvious general theorem that if a field G can be expressed as the derivative ∂F of some other function F on a manifold $\mathcal{M}$, then the integral of G over $\mathcal{M}$ can be expressed as the integral of F over $\partial\mathcal{M}$, that is, if $G = \partial F$ on $\mathcal{M}$, then

$$\int_{\mathcal{M}} d^m\mathbf{x}\, G = \oint_{\partial\mathcal{M}} d^{m-1}\mathbf{x}\, F \tag{4.40}$$

This result is more general than one might imagine at first. In the next chapter we shall prove that if the derivative of G is continuous on a flat space (or even singular in certain ways), then there exists an F such that $G = \partial F$. Therefore, it is generally possible to express a volume integral as an equivalent surface integral. A couple more examples are given in the exercises.

The Fundamental Theorem provides us with general conditions for the *path independence* of a line integral. Suppose we have two oriented curves $\mathcal{C}_1$ and $\mathcal{C}_2$ with common endpoints. If the orientation of one of them is reversed, the two curves can be combined into a single oriented closed curve $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$. The line integrals along these curves are related by

$$\oint_{\mathcal{C}} L(d\mathbf{x}) = \int_{\mathcal{C}_1} L(d\mathbf{x}) - \int_{\mathcal{C}_2} L(d\mathbf{x}). \tag{4.41}$$

So if $\oint L(d\mathbf{x}) = 0$ for all closed curves in some *region* $\mathcal{R}$, then the line integrals $\int L(d\mathbf{x})$ over all paths in $\mathcal{R}$ with common endpoints have the same value. In this case, we say that the line integral $\int L(d\mathbf{x})$ is *path independent* in $\mathcal{R}$. It follows, then, from the fundamental theorem (4.17) that $dL = 0$ at every point in $calR$ is a necessary and sufficient condition for this path independence. For the particular integrand $L(d\mathbf{x}) = \mathbf{v}\cdot d\mathbf{x}$, Stokes' Theorem (4.12) implies that if $\nabla\times\mathbf{v} = 0$ in a *region* $\mathcal{R}$ in $\mathcal{E}_3$, then $\oint \mathbf{v}\cdot d\mathbf{x} = 0$ for all closed curves in $\mathcal{R}$. The condition that $\mathcal{R}$ be simply-connected is necessary to ensure that each closed curve is the complete boundary of a surface on which $\nabla\times\mathbf{v} = 0$ so Stokes' Theorem applies. A region is *simply-connected* if any closed curve in the region can be shrunk continuously to a point without leaving the region. Since the surface bounded by a given closed curve in $\mathcal{R}$ can be chosen arbitrarily, Stokes' Theorem also implies that if $\oint \mathbf{v}\cdot d\mathbf{x} = 0$ then $\nabla\times\mathbf{v} = 0$. Thus, we have proved that $\nabla\times\mathbf{v} = 0$ *in a simply-connected region* $\mathcal{R}$ iff $\oint \mathbf{v}\cdot d\mathbf{x} = 0$ *for all closed curves in* $\mathcal{R}$. Therefore, $\nabla\times\mathbf{v} = 0$ is a necessary and sufficient condition for path independence of the line integral $\int \mathbf{v}\cdot d\mathbf{x}$ in $\mathcal{R}$. Note that this is a relation between local and global properties of the field $\mathbf{v}$ in the region $\mathcal{R}$. Tese theorems on path independence of line integrals have counterparts in theorems about surface integrals, which follow in a similar way from the Fundamental Theorem (see Ex. 3.3).

2-4 Exercises

(4.1) Use the definition (3.1) to prove $\nabla x = 3$ in $\mathcal{E}_3$ by evaluating the integral on a sphere surrounding the point before taking the limit. Show that the result is independent of the choice of origin.

(4.2) Use Eq. (4.1) to prove that the surface area A for a sphere of radius r in $\mathcal{E}_n$ is related to the volume V enclosed by $rA = mV$. Note that it is unnecessary to use coordinates in the proof.

(4.3) For multivector fields $F = F(\mathbf{x})$ and $G = G(\mathbf{x})$ on $\mathcal{E}_3$, derive *Green's Theorem*

$$\int d^3x\, [(\nabla^2 G)F - G\nabla^2 F] = \oint d^2x\, [(\mathbf{n}\cdot\nabla G)F - G\mathbf{n}\cdot\nabla F]$$

from Gauss' theorem (4.23) by determining T.

(4.4) Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are surfaces in $\mathcal{E}_3$ with a common oriented boundary. Prove that if $\nabla F = 0$ in a simply-connected region enclosed by these surfaces, then

$$\int_{\mathcal{S}_1} d^2\mathbf{x}\, F = \int_{\mathcal{S}_2} d^2\mathbf{x}\, F$$

Thus, state and prove a theorem about the independence of directed surface integral on the choice of surface spanning a given closed curve.

(4.5) Prove that

$$\int_{\mathcal{S}} d^2x\, \mathbf{n}\cdot\nabla F = -\oint_{\partial\mathcal{S}} d\mathbf{x}\, F$$

for any surface $\mathcal{S}$ with surface normal $\mathbf{n}$ in a region where $\nabla F = 0$.

(4.6) Let $\mathcal{B}$ be a solid ball with radius R and center at $\mathbf{r} = 0$. Use Gauss' theorem to help evaluate the volume integral

$$\int_{\mathcal{B}} d^3r\, \mathbf{r}^{2k} = \frac{4\pi R^{2k+3}}{2k+3}$$

for any integer k.

(4.7) For a continuous body with a uniform mass density and volume V, derive the following formulas expressing the center of mass $\mathbf{X}$ and the inertia tensor $\mathcal{I}(\mathbf{a})$ as integrals over the surface of the body:

$$\mathbf{X} = \frac{1}{2V}\oint d^2x\mathbf{n}\,\mathbf{x}^2,$$

$$\mathcal{I}(\mathbf{a}) \equiv \frac{1}{V}\int d^3r\,\mathbf{r}\,\mathbf{r}\wedge\mathbf{a} = \frac{i}{V}\oint d^2\mathbf{r}\,\frac{r^2}{10}(3\mathbf{r}\wedge\mathbf{a} - 2\mathbf{r}\cdot\mathbf{a}),$$

where $\mathbf{v} = \mathbf{x} - \mathbf{X}$.