Chapter 1

Synopsis of Geometric Algebra

This chapter summarizes and extends some of the basic ideas and results of *Geometric Algebra* developed in a previous book NFCM (New Foundations for Classical Mechanics). To make the summary self-contained, all essential definitions and notations will be explained, and geometric interpretations of algebraic expressions will be reviewed. However, many algebraic identities and theorems from NFCM will be stated without repeating proofs or detailed discussions. And some of the more elementary results and common mathematical notions will be taken for granted. As new theorems are introduced their proofs will be sketched, but readers will be expected to fill-in straightforward algebraic steps as needed. Readers who have undue difficulties with this chapter should refer to NFCM for more details. Those who want a more advanced mathematical treatment should refer to the book GC (Clifford Algebra to Geometric Calculus).

1-1. The Geometric Algebra of a Vector Space

The construction of Geometric Algebras can be approached in many ways. The quickest (but not the deepest) approach presumes familiarity with the conventional concept of a vector space. Geometric algebras can then be defined simply by specifying appropriate rules for multiplying vectors. That is the approach to be taken here.

The terms "linear space" and "vector space" are usually regarded as synonymous, but it will be convenient for us to distinguish between them. We accept the usual definition of a *linear space* as a set of elements which is closed under the operations of addition and scalar multiplication. However, it should be recognized that this definition does not fully characterize the geometrical concept of a vector as and algebraic representation of a "directed line segment." The mathematical characterization of vectors is completed defining a rule for multiplying vectors called the "geometric product." Accordingly, we reserve the term *vector space* for a linear space on which the geometric product is defined. It will be seen that from a vector space many other kinds of linear space are generated.

Let $\mathcal{V}_n$ be an n-dimensional vector space over the real numbers. The *geometric product* of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathcal{V}_n$ is defined by three basic axioms:
The "associative rule"

$$\mathbf{a}(\mathbf{bc}) = (\mathbf{ab})\mathbf{c}, \tag{1.1}$$

The left and right "distributive rules"

$$\begin{aligned}\mathbf{a}(\mathbf{b}+\mathbf{c}) &= \mathbf{ab}+\mathbf{ac}\\ (\mathbf{b}+\mathbf{c})\mathbf{a} &= \mathbf{ba}+\mathbf{ca},\end{aligned} \tag{1.2}$$

and the "contraction rule"

$$\mathbf{a}^2 = |\,\mathbf{a}\,|^2 \tag{1.3}$$

where $|\,\mathbf{a}\,|$ is a positive scalar (= real number) called the *magnitude* or *length* of $\mathbf{a}$, and $|\,\mathbf{a}\,| = 0$ if and only if $a = 0$. Both distributive rules (1.2) are needed, because multiplication is not commutative.

Although the vector space $\mathcal{V}_n$ is closed under vector addition, it is not closed under multiplication, as the contraction rule (1.3) shows. Instead, by multiplication and addition the vectors of $\mathcal{V}_n$ generate a larger linear space $\mathcal{G}_n = \mathcal{G}(\mathcal{V}_n)$ called the *geometric algebra* of $\mathcal{V}_n$. This linear space is, of course, closed under multiplication as well as addition.

The contraction rule (1.3) determines a measure of distance between vectors in $\mathcal{V}_n$ called a *Euclidean geometric algebra*. Thus, the vector space $\mathcal{V}_n$ can be regarded as an *n-dimensional Euclidean space*; when it is desired to emphasize this interpretation later on, we will often write $\mathcal{E}_n$ instead of $\mathcal{V}_n$. Other types of geometric algebra are obtained by modifying the contraction rule to allow the square of some nonzero vectors to be negative or even zero. In general, it is some version of the contraction rule which distinguishes geometric algebra from the other associative algebras.

Inner and Outer Products

The geometric product $\mathbf{ab}$ can be decomposed into symmetric and antisymmetric parts defined by

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(\mathbf{ab}+\mathbf{ba}) \tag{1.4}$$

and

$$\mathbf{a}\wedge\mathbf{b} = \frac{1}{2}(\mathbf{ab}-\mathbf{ba}). \tag{1.5}$$

Thus,

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}. \tag{1.6}$$

By expanding the quantity $(\mathbf{a}+\mathbf{b})^2$, it is easy to prove that the symmetrized product $\mathbf{a}\cdot\mathbf{b}$ is scalar-valued. In fact, $\mathbf{a}\cdot\mathbf{a}$ is precisely the conventional Euclidean *inner product*.

The antisymmetrized product $\mathbf{a}\wedge\mathbf{b}$ is called the *outer product*. It is neither scalar nor vector valued. For any specific vectors $\mathbf{a}$ and $\mathbf{b}$, the quantity $\mathbf{a}\wedge\mathbf{b}$ is a new kind of entity called a *bivector* (or 2-vector). Geometrically, it represents a directed plane segment (Fig. 1.1) in much the same way as a vector represents a directed line segment. We can regard $\mathbf{a}\wedge\mathbf{b}$ as a *directed area*, with a *magnitude* $|\mathbf{a}\wedge\mathbf{b}|$ equal to the usual scalar area of the parallelogram in Fig. 1.1, with the *direction* of the plane in which the parallelogram lies, and with an *orientation* (or sense) which can be assigned to the parallelogram in the plane.

The geometric interpretations of the inner and outer products determine a geometric interpretation for the geometric product. Equation (1.4) implies that

$$\mathbf{ab} = -\mathbf{ba} \quad \text{iff} \quad \mathbf{a}\cdot\mathbf{b} = 0, \tag{1.7a}$$

while (1.5) implies that

$$\mathbf{ab} = \mathbf{ba} \quad \text{iff} \quad \mathbf{a}\wedge\mathbf{b} = 0, \tag{1.7b}$$

where iff means "if and only if." Equation (1.7a) tells us that the geometric product $\mathbf{ab}$ of nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ *anticommutes* iff $\mathbf{a}$ and $\mathbf{b}$ are *orthogonal*, while (1.7b) tells us that the geometric product *commutes* iff the vectors are *collinear*. Equation (1.6) tells us that, in general, $\mathbf{ab}$ is a

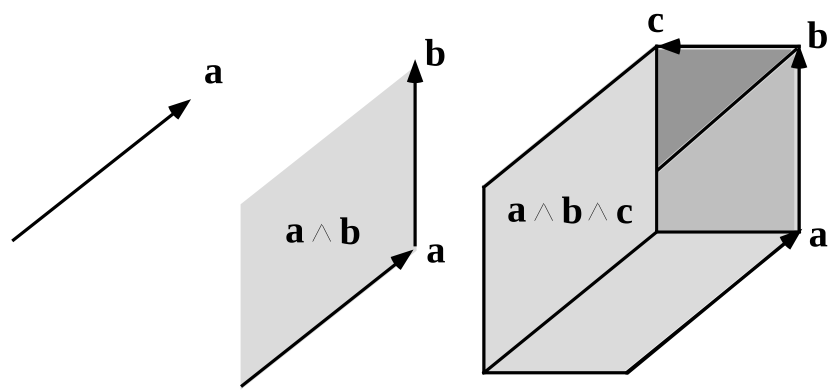

Fig. 1.1. Just as a vector $\mathbf{a}$ represents (or is represented by) a *directed line segment*, a bivector $\mathbf{a}\wedge\mathbf{b}$ represents a *directed plane segment* (the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$, and the trivector (3-vector) $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$ represents a *directed space segment* (the parallelopiped with edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$). An orientation for the bivector (and the parallelogram) is indicated in the diagram by placing the tail of vector $\mathbf{b}$ at the head of vector $\mathbf{a}$. An orientation for the trivector is indicated in a similar way.

mixture of commutative and anticommutative parts. Thus, the degree of commutativity of the product $\mathbf{ab}$ is a measure of the direction of $\mathbf{a}$ relative to $\mathbf{b}$. In fact, the quantity $\mathbf{a}^{-1}\mathbf{b} = \mathbf{ab}/\mathbf{a}^2$ is a *measure of the relative direction and magnitude* of $\mathbf{a}$ and $\mathbf{b}$. It can also be regarded as an operator which transforms $\mathbf{a}$ into $\mathbf{b}$, as expressed by the equation $\mathbf{a}(\mathbf{a}^{-1}\mathbf{b}) = \mathbf{b}$. This property is exploited in the theory of rotations.

Having established a complete geometric interpretation for the product of two vectors, we turn to consider the products of many vectors. The completely antisymmetrized product of k vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ generates a new entity $\mathbf{a}_1\wedge\mathbf{a}_2\wedge\ldots\wedge\mathbf{a}_k$ called a *k-blade*. The integer k is called the *step* (or *grade* of the blade.* A linear combination of blades with the same step is called a *k-vector*. Therefore, every k-blade is a k-vector. The converse that every k-vector is a k-blade holds only in the geometric algebras $\mathcal{G}_n$ with $n \le 3$. Of course, $\mathcal{G}_3$ is the algebra of greatest interest in this book.

From a vector $\mathbf{a}$ and a k-vector $\mathbf{A}_k$, a $(k+1)$-vector $\mathbf{a}\wedge\mathbf{A}_k$ is generated by the *outer product*, which can be defined in terms of the geometric product by

$$\begin{aligned}\mathbf{a}\wedge\mathbf{A}_k &= \frac{1}{2}(\mathbf{a}\mathbf{A}_k + (-1)^k\mathbf{A}_k\mathbf{a})\\ &= (-1)^k\mathbf{A}_k\wedge\mathbf{a}.\end{aligned}\tag{1.8}$$

This reduces to Eq. (1.5) when $k = 1$, where the term "1-vector" is regarded as synonymous with "vector."

The *inner product* of a vector with a k-vector can be defined by

$$\mathbf{a}\cdot\mathbf{A}_k = \frac{1}{2}(\mathbf{a}\mathbf{A}_k - (-1)^k\mathbf{A}_k\mathbf{a}) = (-1)^{k+1}\mathbf{A}_k\cdot\mathbf{a}.\tag{1.9}$$

This generalizes the concept of inner product between vectors. The quantity $\mathbf{a}\cdot\mathbf{A}_k$ is a $(k-1)$-vector. Thus, it is scalar-valued only when $k = 1$, where the term "0-vector" is regarded as synonymous with "scalar."

* The term "grade" is used in NFCM and GC. However, "step" is the term originally used by Grassmann ("steppe" in German), and, unlike "grade," it does not have other common mathematical meanings.

It is very important to note that the outer product (1.8) is a “step up” operation while the inner product (1.9) is a “step down” operation. By combining (1.8) and (1.9) we get

$$\begin{aligned}\mathbf{a}\mathbf{A}_k &= \mathbf{a}\cdot\mathbf{A}_k + \mathbf{a}\wedge\mathbf{A}_k,\\ \mathbf{A}_k\mathbf{a} &= \mathbf{A}_k\cdot\mathbf{a} + \mathbf{A}_k\wedge\mathbf{a}.\end{aligned} \tag{1.10}$$

This can be seen as a decomposition of the geometric product by a vector into step-down and step-up parts.

Identities

To facilitate manipulations with inner and outer products a few algebraic identities are listed here. Let vectors be denoted by bold lower case letters $\mathbf{a}, \mathbf{b}, \ldots$, and let $A, B, \ldots$ indicate quantities of any step, for which a step can be specified by affixing an overbarred suffix. Thus $A_{\bar{k}}$ has step k. The overbar may be omitted when it is clear that the suffix indicates step.

The outer product is associative and antisymmetric. Associativity is expressed by

$$A\wedge(B\wedge C) = (A\wedge B)\wedge C = A\wedge B\wedge C \tag{1.11}$$

The outer product is antisymmetric in the sense that is reverses sign under interchange of any two vectors in a multiple outer product, as expressed by

$$A\wedge\mathbf{a}\wedge B\wedge\mathbf{b}\wedge C = -A\wedge\mathbf{b}\wedge B\wedge\mathbf{a}\wedge C \tag{1.12}$$

The inner product is not strictly associative. It obeys the associative relation

$$A_{\bar{r}}\cdot(B_{\bar{s}}\cdot C_{\bar{t}}) = (A_{\bar{r}}\cdot B_{\bar{s}})\cdot C_{\bar{t}} \qquad \text{for } r+t\le s. \tag{1.13}$$

However, it relates to the outer product by

$$A_{\bar{r}}\cdot(B_{\bar{s}}\cdot C_{\bar{t}}) = (A_{\bar{r}}\wedge B_{\bar{s}})\cdot C_{\bar{t}} \qquad \text{for } r+s\le t\,. \tag{1.14}$$

Our other identity relating inner and outer products is needed to carry out step reduction explicitly:

$$\mathbf{a}\cdot(\mathbf{b}\wedge C) = (\mathbf{a}\cdot\mathbf{b})C + \mathbf{b}\wedge(\mathbf{a}\cdot C). \tag{1.15}$$

In particular, by iterating (1.15) one can derive the reduction formula

$$\begin{aligned}&\mathbf{a}\cdot(\mathbf{b}_1\wedge\mathbf{b}_2\wedge\cdots\wedge\mathbf{b}_r)\\ &\quad = \sum_{k=1}^{r}(-1)^{r+1}(\mathbf{a}\cdot\mathbf{b}_k)\mathbf{b}_1\wedge\cdots\wedge\check{\mathbf{b}}_k\wedge\cdots\wedge\mathbf{b}_r\end{aligned} \tag{1.16}$$

where $\check{\mathbf{b}}_k$ means that the kth vector $\mathbf{b}_k$ is omitted from the product.

Many other useful identities can be generated from the above identities. For example, from (1.14) and (1.16) one can derive

$$\begin{aligned}&(\mathbf{a}_r\wedge\cdots\wedge\mathbf{a}_1)\cdot(\mathbf{b}_1\wedge\cdots\wedge\mathbf{b}_r)\\ &\quad = (\mathbf{a}_r\wedge\cdots\wedge\mathbf{a}_2)\cdot[\mathbf{a}_1\cdot(\mathbf{b}_1\wedge\cdots\wedge\mathbf{b}_r)]\\ &\quad = \sum_{k=1}^{r}(-1)^{k+1}(\mathbf{a}_1\cdot\mathbf{b}_k)(\mathbf{a}_r\wedge\cdots\wedge\mathbf{a}_2)\cdot(\mathbf{b}_1\wedge\cdots\wedge\check{\mathbf{b}}_k\wedge\cdots\wedge\mathbf{b}_r).\end{aligned} \tag{1.17}$$

This is the well-known *Laplace expansion* of a determinant about its first row. That can be established by noting that the matrix elements of any matrix $[a_{ij}]$ can be expressed as inner products between two sets of vectors, thus,

$$a_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j. \tag{1.18}$$

Then the determinant of the matrix can be defined by

$$\det [a_{ij}] = (\mathbf{a}_r \wedge \cdots \wedge \mathbf{a}_1) \cdot (\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_r). \tag{1.19}$$

From this definition all the properties of determinants follow, including the Laplace expansion (1.17).

We will encounter many identities and equations involving the geometric product along with inner and outer products. To reduce the number of parentheses in such expressions, it is sometimes convenient to employ the following *precedence conventions*

$$\begin{aligned} &(A \wedge B)C = A \wedge BC \neq A \wedge (BC) \\ &(A \cdot B)C = A \cdot BC \neq A \cdot (BC) \\ &A \cdot (B \wedge C) = A \cdot B \wedge C \neq (A \cdot B) \wedge C \end{aligned} \tag{1.20}$$

This is to say that when there is ambiguity in the order for performing adjacent products, outer products take precedence over (are performed before) inner products, and both inner and outer products take precedence over the geometric products. For example, the simplest (and consequently the most common and useful) special case of (1.16) is the identity

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b}. \tag{1.21}$$

The precedence conventions allow us to write this as

$$\mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c} = \mathbf{a} \cdot \mathbf{b}\, \mathbf{c} - \mathbf{a} \cdot \mathbf{c}\, \mathbf{b}.$$

The Additive Structure of $\mathcal{G}_n$

The vectors of a set $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r$ are *linearly independent* if and only if the r-blade

$$A_{\overline{r}} = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_r \tag{1.22}$$

is not zero. Every such r-blade determines a unique r-dimensional subspace $\mathcal{V}_r$ of the vector space $\mathcal{V}_n$, consisting of all vectors $\mathbf{a}$ in $\mathcal{V}_n$ which satisfy the equation

$$\mathbf{a} \wedge A_{\overline{r}} = 0. \tag{1.23}$$

The blade $A_{\overline{r}}$ is a *directed volume* for $\mathcal{V}_r$ with magnitude (scalar volume) $|\, A_{\overline{r}}\, |$. These are the fundamental geometric properties of blades.

An ordered set of vectors $\{\mathbf{a}_k |\, k = 1, 2, \ldots n\}$ in $\mathcal{V}_n$ is a *basis* for $\mathcal{V}_n$ if and only if it generates a nonzero n-blade $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_n$. The n-blades are called *pseudoscalars* or $\mathcal{V}_n$ or $\mathcal{G}_n$. They make up a 1-dimensional linear space, so we may choose a unit pseudoscalar I and write

$$\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_n = \alpha I, \tag{1.24}$$

where α is scalar. This can be solved for the scalar volume $|\, \alpha\, |$ expressed as a determinant,

$$\alpha = (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n)I^{-1} = (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n) \cdot I^{-1} \tag{1.25}$$

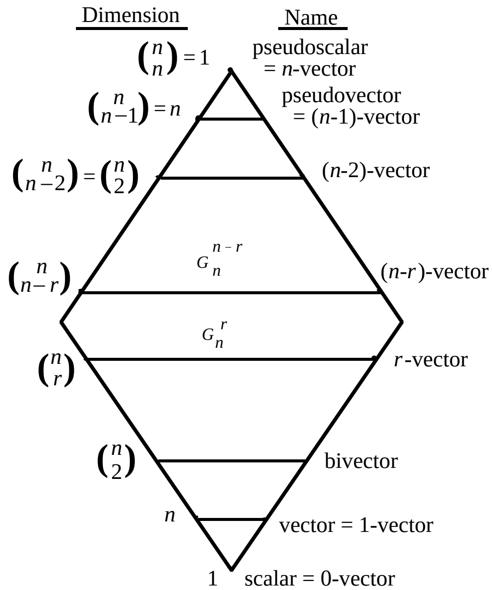

Fig. 1.2. A schematic representation of $\mathcal{G}_n$ showing its duality symmetry. From the bottom up, the rth rung of the ladder represents the space of r-vectors $\binom{r}{n}$. The length of the rung corresponds to the dimension of the space, which increases from the bottom up and the top down to the middle. The 1-dimensional space of scalars is represented by a single point. A duality transformation $\mathcal{G}_n$ simply flips the diagram about its horizontal symmetry axis.

The pseudoscalars $\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n$ and I are said to have the same *orientation* if and only if α is positive. Thus, the choice of I assigns an orientation to $\mathcal{V}_n$ and $\mathcal{V}_n$ is said to be *oriented*. The opposite orientation is obtained by selecting $-I$ as unit pseudoscalar.

From any basis $\{\mathbf{a}_r\}$ for $\mathcal{V}_n$ we can generate a basis for the linear space $\mathcal{G}_n^r$ of r-vectors by forming all independent outer products of r vectors from the set $\{\mathbf{a}_k\}$. The number of ways this can be done is given by the binomial coefficient $\binom{n}{r}$. Therefore, $\mathcal{G}_n^r$ is a linear space of dimension $\binom{n}{r}$. The entire geometric algebra $\mathcal{G}_n = \mathcal{G}(\mathcal{V}_n)$ can be described as a sum of subspaces with different grade, that is,

$$\mathcal{G}_n = \mathcal{G}_n^0 + \mathcal{G}_n^1 + \ldots + \mathcal{G}_n^r + \ldots + \mathcal{G}_n^n = \sum_{r=0}^{n} \mathcal{G}_n^r \tag{1.26}$$

Thus $\mathcal{G}_n$ is a linear space of dimension

$$\dim \mathcal{G}_n = \sum_{r=0}^{n} \dim \mathcal{G}_n^r = \sum_{r=0}^{n} \binom{n}{r} = 2^n. \tag{1.27}$$

The elements of n are called *multivectors* or *quantities*. In accordance with (1.26), any multivector A can be expressed uniquely as a sum of its r-vector parts $A_{\overline{r}}$, that is,

$$A = A_{\overline{0}} + A_{\overline{1}} + \cdots + A_{\overline{r}} + \cdots + A_{\overline{n}} = \sum_{r=0}^{n} A_{\overline{r}}\ . \tag{1.28}$$

The structure of $\mathcal{G}_n$ is represented schematically in Fig. 1.2. Note that the outer product (1.8) with a vector "moves" a k-vector up one rung of the ladder in Fig. 1.2, while the inner product "moves" down a rung. The figure also shows the symmetry of $\mathcal{G}_n$ under a *duality transformation*. The *dual* AI of a multivector A is obtained simply by multiplication with the unit pseudoscalar I. This operation maps scalars into pseudoscalars, vectors into $(n-1)$-vectors and vice-versa. In general, it transforms on r-vector $A_{\overline{r}}$ into an $(n-r)$-vector $A_{\overline{r}}I$. For any arbitrary multivector, the decomposition (1.28) gives the term-by-term equivalence

$$AI = \sum_{r=0}^{n} A_{\overline{r}}\, I = \sum_{r=0}^{n} (AI)_{\overline{n-r}} \tag{1.29}$$

A duality transformation of $\mathcal{G}_n$ interchanges up and down in Fig. 1.2; therefore, it must interchange inner and outer products. That is expressed algebraically by the identity

$$(\mathbf{a}\cdot A_{\overline{r}})I = \mathbf{a}\wedge (A_{\overline{r}}\, I) \tag{1.30}$$

Note that this can be solved to express the inner product in terms of the outer product and two duality transformations,

$$\mathbf{a}\cdot A_{\overline{r}} = [\mathbf{a}\wedge (A_{\overline{r}}\, I)]I^{-1} \tag{1.31}$$

This identity can be interpreted intuitively as follows: one can take a step down the ladder in $\mathcal{G}_n$ by using duality to flip the algebra upside down, taking a step up the ladder and then flipping the algebra back to right side up.

Reversion and Scalar Parts

Computations with geometric algebra are facilitated by the operation of *reversion* which reverses the order of vector factors in any multivector. Reversion can be defined by

$$(\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_r)^\dagger = \mathbf{a}_r\mathbf{a}_{r-1}\cdots\mathbf{a}_1. \tag{1.32}$$

It follows that $\mathbf{a}^\dagger = \mathbf{a}$ and $\alpha^\dagger = \alpha$ for vector $\mathbf{a}$ and scalar α. In general, the *reverse* of the r-vector part of any multivector A is given by

$$(A^\dagger)_{\overline{r}} = (A_{\overline{r}})^\dagger = (-1)^{r(r-1)/2}A_{\overline{r}}. \tag{1.33}$$

Moreover,

$$(AB)^\dagger = B^\dagger A^\dagger, \tag{1.34}$$

$$(A+B)^\dagger = A^\dagger + B^\dagger. \tag{1.35}$$

The operation of selecting the scalar part of a multivector is so important that it deserves the special notation

$$\langle A\rangle \equiv A_{\overline{0}}. \tag{1.36}$$

This generalizes the operation of selecting the real part of a complex number and corresponds to computing the trace of a matrix in matrix algebra. Like the matrix trace, it has the important permutation property

$$\langle ABC\rangle = \langle BCA\rangle. \tag{1.37}$$

A natural *scalar product* is defined for all multivectors in the 2^n-dimensional linear space $\mathcal{G}_n$ by

$$\langle A^\dagger B\rangle = \sum_{r=0}^{n}\langle A_{\overline{r}}^\dagger B_{\overline{r}}\rangle = \langle B^\dagger A\rangle. \tag{1.38}$$

For every multivector A this determines a scalar *magnitude* or *modulus* $|\,A\,|$ given by

$$|\,A\,|^2 = \langle A^\dagger A\rangle = \sum_r |\,A_{\overline{r}}\,|^2 = \sum_r A^\dagger_{\overline{r}} A_{\overline{r}} \tag{1.39}$$

The measure for the "size" of a multivector has the Euclidean property

$$|\,A\,|^2 \geq 0 \tag{1.40}$$

with $|\,A\,| = 0$ if and only if $A = 0$. Accordingly, the scalar product is said to be "Euclidean" or "positive definite." The multivector is said to be *unimodular* if $|\,A\,| = 1$.

An important consequence of (1.39) is the fact that every r-vector has a multiplicative inverse given by

$$A_r^{-1} = \frac{A^\dagger_{\overline{r}}}{|\,A_{\overline{r}}\,|^2}. \tag{1.41}$$

However, some multivectors of mixed step do not have inverses.

Involutions

An *automorphism* of an algebra $\mathcal{G}_n$ is a one-to-one mapping of $\mathcal{G}_n$ onto $\mathcal{G}_n$ which preserves the algebraic operations of addition and multiplication. An *involution* $\underline{\alpha}$ is an automorphism which equals the identity mapping when applied twice. Thus, for any element A in $\mathcal{G}_n$,

$$\underline{\alpha}^2(A) = \underline{\alpha}\big(\underline{\alpha}(A)\big) = A. \tag{1.42}$$

Geometric algebra has two fundamental involutions. The first is reversion, which has just been discussed with the notation $A^\dagger = \underline{\alpha}(A)$. The second is sometimes called the *main involution* and denoted by $A^* = \underline{\alpha}(A)$. It is defined as follows: Any multivector A can be decomposed into a part $A_+ = A_{\overline{0}} + A_{\overline{2}} + \cdots$ of *even* step and a part $A_- = A_{\overline{1}} + A_{\overline{3}} + \cdots$ of *odd* step; thus

$$A = A_+ + A_-. \tag{1.43}$$

Now the *involute* of A can be defined by

$$A^* = A_+ - A_- \tag{1.44}$$

It follows that, for the geometric product of any multivectors,

$$(AB)^* = A^* B^*. \tag{1.45}$$

1-2. The Algebra of Euclidean 3-Space

Throughout most of this book we will be employing the geometric algebra $\mathcal{G}_3 = \mathcal{G}(\mathcal{E}_3)$, because we use vectors in $\mathcal{E}_3$ to represent places in *physical* (position) *space*. The properties of geometric algebra which are peculiar to the three-dimensional case are summarized in this section. They all derive from special properties of the pseudoscalar and duality in $\mathcal{G}_3$.

The unit pseudoscalar for $\mathcal{G}_3$ is so important that the special symbol i is reserved to denote it. This symbol is particularly apt because i has the algebraic properties of a conventional unit imaginary. Thus, it satisfies the equations

$$i^2 = -1 \tag{2.1}$$

and

$$i\mathbf{a} = \mathbf{a}i \tag{2.2}$$

for any vector $\mathbf{a}$. According to (2.2), the "imaginary number" i commutes with vectors, just like the real numbers (scalars). It follows that i commutes with every multivector in $\mathcal{G}_3$.

The algebraic properties (2.1) and (2.2) allow us to treat i as if it were an imaginary scalar, but i has other properties deriving from the fact that it is the unit pseudoscalar. In particular, i relates scalars and vectors in $\mathcal{G}_3$ to bivectors and pseudoscalars by duality. To make this relation explicit, consider an arbitrary multivector A written in the expanded form

$$A = A_{\overline{0}} + A_{\overline{1}} + A_{\overline{2}} + A_{\overline{3}}. \tag{2.3}$$

For $\mathcal{G}_3$ the general ladder diagram in Fig. 1.2 takes the specific form in Fig. 2.1. As the diagram shows, the bivectors of $\mathcal{G}_3$ are pseudovectors, therefore, every bivector in $\mathcal{G}_3$ can be expressed as the dual of a vector. Thus, the bivector in $\mathcal{G}_3$ can be expressed as the dual of a vector. Thus, the bivector part of A can be written in the form $A_{\overline{2}} = i\mathbf{b}$, where $\mathbf{b}$ is a vector. Similarly, the trivector (=pseudoscalar) part of A can be expressed as the dual of a scalar β by writing $A_{\overline{3}} = i\beta$. Introducing the notations $\alpha = A_{\overline{0}}$ and $\mathbf{a} = A_{\overline{1}}$ for the scalar and vector parts of A, the expansion (2.3) can be written in the equivalent form

$$A = \alpha + \mathbf{a} + i\mathbf{b} + i\beta. \tag{2.4}$$

This shows that A has the formal algebraic structure of a *complex scalar* $\alpha + i\beta$ added to a *complex vector* $\mathbf{a} + i\mathbf{b}$. The algebraic advantages of the "complex expanded form" (2.4) are such that we shall use the form often.

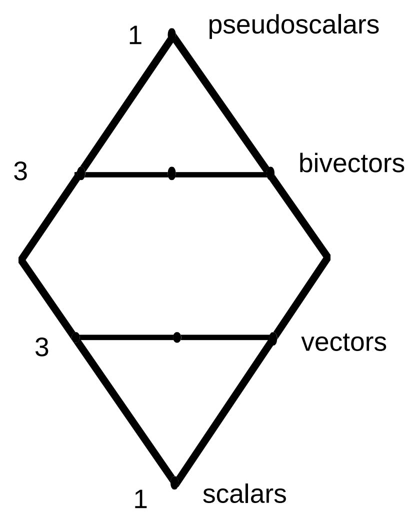

Fig. 2.1. Ladder diagram for G_3. Each dot represents one dimension.

Using the fact that the dual of the bivector $\mathbf{a} \wedge \mathbf{b}$ is a vector, the *vector cross product* $\mathbf{a}\times\mathbf{b}$ can be defined by

$$\mathbf{a}\times\mathbf{b} = -i(\mathbf{a} \wedge \mathbf{b}), \tag{2.5}$$

or, equivalently, by

$$\mathbf{a} \wedge \mathbf{b} = i(\mathbf{a}\times\mathbf{b}). \tag{2.6}$$

Clearly, a change in the sign (orientation) of i in (2.5) will reverse the direction (orientation) of $\mathbf{a}\times\mathbf{b}$. The orientation of i can be chosen so the vector $\mathbf{a}\times\mathbf{b}$ is related to the vectors $\mathbf{a}$ and $\mathbf{b}$ by the standard *right hand rule* (Fig. 2.2). This choice of orientation can be expressed by calling i the *dextral* (or *righthanded*) *pseudoscalar* of $\mathcal{G}_3$.

Now the geometric product of vectors in $\mathcal{E}_3$ can be decomposed in two ways:

$$\mathbf{a}\mathbf{b} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b} = \mathbf{a}\cdot\mathbf{b} + i(\mathbf{a}\times\mathbf{b}) \tag{2.7}$$

The outer product can thus be replaced by the cross product in $\mathcal{G}_3$ provided the unit pseudoscalar is introduced explicitly. This is not always advisable, however, because the outer product is geometrically more fundamental than the cross product.

Another special property of the pseudoscalar i is the duality relation between inner and outer products expressed by the identity (1.30). In $\mathcal{G}_3$ two specific cases of that identity are easily derived by expanding the associative identity $(\mathbf{a}\mathbf{b})i = \mathbf{a}(\mathbf{b}i)$ in terms of inner and outer products and separately equating vector and trivector parts. The result is

$$(\mathbf{a}\wedge\mathbf{b})i = \mathbf{a}\cdot(\mathbf{b}i), \tag{2.8a}$$

$$(\mathbf{a}\cdot\mathbf{b})i = \mathbf{a}\wedge(\mathbf{b}i). \tag{2.8b}$$

Applying (2.8b) to the directed volume $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = \mathbf{a}\wedge(i\mathbf{b}\times\mathbf{c})$, we obtain

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})i \tag{2.9}$$

This tells us that the scalar $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ gives the magnitude and orientation of the pseudoscalar $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$. If $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is positive (negative), then $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$ has the same (opposite) orientation as i; in other words, it is a right-(left-)handed pseudoscalar.

Applying (2.5) and (2.8b) to the double cross product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ and then using the expansion (1.21), we obtain

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = -\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) = -\mathbf{a}\cdot\mathbf{b}\,\mathbf{c} + \mathbf{a}\cdot\mathbf{c}\,\mathbf{b}. \tag{2.10}$$

This reveals a geometrical reason why the cross product is not associative. The double cross product hides the facts that the geometrically different inner and outer products are actually involved.

Reversion and Involution

Reversion of the pseudoscalar i gives

$$i^\dagger = -i \tag{2.11}$$

This can be concluded from (1.32) with $r = 3$ or from $(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^\dagger = \mathbf{c}\wedge\mathbf{b}\wedge\mathbf{a} = -\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$. Also, $(i\mathbf{b})^\dagger = b^\dagger i^\dagger = \mathbf{b}(-i) = -i\mathbf{b}$. Hence, for a multivector A in thestandard form (2.4), the reverse is given explicitly by

$$A^\dagger = \alpha + \mathbf{a} - i\mathbf{b} - i\beta. \tag{2.12}$$

The magnitude of A is now given by the obviously positive definite expression

$$|\,A\,|^2 = \langle AA^\dagger\rangle = \alpha^2 + \mathbf{a}^2 + \mathbf{b}^2 + \beta^2 \geq 0. \tag{2.13}$$

Equation (2.12) expresses reversion in $\mathcal{G}_3$ as a kind of "complex conjugation."

Physicists define *space inversion* as a reversal of all directions in space. This maps every vector $\mathbf{a}$ in $\mathcal{E}_3$ into a vector

$$\mathbf{a}^* = -\mathbf{a}. \tag{2.14}$$

Since $\mathcal{E}_3$ generates $\mathcal{G}_3$, this induces an automorphism of $\mathcal{G}_3$ which, in fact, is precisely the *main involution* defined in the preceeding section. Thus, the induced mapping of a k-blade is given by

$$(\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_k)^* = \mathbf{a}_1^*\wedge\cdots\wedge\mathbf{a}_k^* = (-1)^k\mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_k. \tag{2.15}$$

This sign factor $(-1)^k$ is called the *parity* of the blade. The parity is said to be *even* (*odd*) when k is an even (odd) integer.

The term "space inversion" is not well chosen, because no notion of inverse is involved. Moreover, the term "inversion" is commonly applied to another kind of mapping (discussed later) which does involve inverses. For that reason, the term *space involution* will be adopted in this book.

The composite of reversion and space involution produces a new operation which is important enough in some applications to deserve a name and a notation. The *conjugate* $\widetilde{A}$ can be defined by

$$\widetilde{A} = (A^\dagger)^* = (A^*)^\dagger. \tag{2.16}$$

It follows from (1.34) and (2.15) that the conjugate of a product is given by

$$(AB)^\sim = \widetilde{B}\widetilde{A}. \tag{2.17}$$

From (2.16) and (2.12) we get

$$\widetilde{A} = \alpha - \mathbf{a} - i\mathbf{b} + i\beta. \tag{2.18}$$

This could be adopted as an alternative definition of *conjugation*. It should not be thought that conjugation is less important than reversion and space involution. Actually, we shall have more occasion to employ the operations of conjugation and reversion than space involution.

Polar and Axial Vectors

Physicists commonly define two kinds of vector distinguished by opposite parities under space inversion. *Polar vectors* are defined to have odd parity, as expressed by (2.14). *Axial vectors* are defined to have even parity, and the cross product of polar vectors $\mathbf{a}$ and $\mathbf{b}$ is defined to be axial, so

$$(\mathbf{a}\times\mathbf{b})^* = \mathbf{a}^*\times\mathbf{b}^* = (-\mathbf{a})\times(-\mathbf{b}) = \mathbf{a}\times\mathbf{b}. \tag{2.19}$$

This property is inconsistent with our definition (2.5) of the cross product, for (1.45) and (2.15) imply

$$(\mathbf{a}\times\mathbf{b})^* = -i^*(\mathbf{a}^*\wedge\mathbf{b}^*) = -(-i)(-\mathbf{a})\wedge(-\mathbf{b}) = -\mathbf{a}\times\mathbf{b}. \tag{2.20}$$

Note that our definition recognized the duality implicit in the definition of the cross product, so we get a sign change from the pseudoscalar under space inversion. Thus, every vector is a polar vector for us, whether obtained from a cross product or not.

On the other hand, it should be noted that bivectors have the same parity as axial vectors, for

$$(\mathbf{a}\wedge\mathbf{b})^* = \mathbf{a}^*\wedge\mathbf{b}^* = (-\mathbf{a})\wedge(-\mathbf{b}) = \mathbf{a}\wedge\mathbf{b}. \tag{2.21}$$

This shows that the concept of an axial vector is just a way of representing bivectors by vectors. In geometric algebra the conventional distinction between polar and axial vectors is replaced by the deeper distinction between vectors and bivectors.

One more remark should help avoid confusion in applications of the parity concept. The quantity $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is commonly said to be a pseudoscalar in the physics literature. That is because it changes sign under space inversion if $\mathbf{b}\times\mathbf{c}$ is an axial vector. In geometric algebra it is also true that *pseudoscalars have odd parity*; however, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is a scalar with even parity. The change in sign comes from the unit pseudoscalar in (2.9); thus,

$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^* = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})i^* = -\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}. \tag{2.22}$$

Quaternions

It follows from (1.45) that the product of two multivectors with the same parity always has even parity. Therefore, the linear space of all even multivectors,

$$\mathcal{G}_3^+ = \mathcal{G}_3^0 + \mathcal{G}_3^2 \tag{2.23}$$

is closed under multiplication, whereas the space of odd (parity) multivectors is not. Thus, $\mathcal{G}_3^+$ is a subalgebra of $\mathcal{G}_3$. It is called the *even subalgebra* of $\mathcal{G}_3$ or the *quaternion algebra*. Its elements are called *quaternions*. The name derives from the fact that $\mathcal{G}_3^+$ is a 4-dimensional linear space, with one scalar and three bivector dimensions. Thus, any quaternion Q can be written in the expanded form

$$Q = \alpha + i\mathbf{b} \tag{2.24}$$

Every quaternion has a *conjugate*

$$\widetilde{Q} = Q^\dagger = \alpha - i\mathbf{b}, \tag{2.25}$$

and, if it is nonzero, an inverse

$$Q^{-1} = \frac{1}{Q} = \frac{Q^\dagger}{QQ^\dagger} = \frac{Q^\dagger}{|\,Q\,|^2}. \tag{2.26}$$

In expanded form,

$$Q^{-1} = \frac{\alpha - i\mathbf{b}}{\alpha^2 + \mathbf{b}^2}. \tag{2.27}$$

Most of the literature on the mathematical and physical applications of quaternions fails to recognized how quaternions fit into the more comprehensive mathematical language of geometric algebra. It is only within geometric algebra that quaternions can be exploited to fullest advantage.

As a point of historical interest, it may be noted that all multivectors in $\mathcal{G}_3$ can be regarded formally as *complex quaternions*, also called *biquaternions*. To make the point explicit, note that the expanded form (2.4) is equivalent to

$$A = Q + iP, \tag{2.28}$$

where $Q = \alpha + i\mathbf{b}$ and $P = \beta - i\mathbf{a}$ are "real quaternions." Here the unit imaginary in the biquaternion appears explicitly.

In biquaternion theory the "complex conjugate" is therefore given by

$$A^* = Q - iP, \tag{2.29}$$

and the "quaternion conjugate" is given by

$$\widetilde{A} = \widetilde{Q} + i\widetilde{P}. \tag{2.30}$$

These correspond exactly to our space involution (2.14) and conjugation (2.19). During the last century applications of biquaternions to physics have been repeatedly rediscovered without being developed very far. The literature on biquaternions is limited by a failure to recognize that the unit imaginary in (2.28) should be interpreted as a unit pseudoscalar. The notion of a "complex quaternion" is of little interest in geometric algebra, because it does not give sufficient emphasis to geometric meaning, and the decomposition (2.28) is of interest mainly because it is a separation of even and odd parts.

1-3 Differentiation by Vectors

In this section differentiation by vectors is defined in terms of partial derivatives and is used to establish basic relations needed to carry out differentiation efficiently. This is the quickest way to introduce vector differentiation to readers who are already familiar with partial differentiation. A more fundamental approach is taken in §2–3 where the vector derivative is generalized and defined *ab initio* without employing coordinates.

Let $F = F(\mathbf{x})$ be a multivector-valued function of a vector variable $\mathbf{x}$ in $\mathcal{E}_n$. The variable $\mathbf{x}$ can be parameterized by rectangular coordinates $x_1, x_2, \ldots, x_n$ by writing

$$\mathbf{x} = \boldsymbol{\sigma}_1 x_1 + \boldsymbol{\sigma}_2 x_2 + \boldsymbol{\sigma}_2 x_2 + \cdots + \boldsymbol{\sigma}_n x_n = \sum \boldsymbol{\sigma}_k x_k, \tag{3.1}$$

where $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_n\}$ is a fixed orthonormal basis. The function $F = F(\mathbf{x})$ is then expressed as a function of the coordinates by

$$F(\mathbf{x}) = F(\boldsymbol{\sigma}_1 x_1 + \boldsymbol{\sigma}_2 x_2 + \cdots + \boldsymbol{\sigma}_n x_n) = F(x_1, x_2, \ldots, x_n). \tag{3.2}$$

It is convenient to express the derivative with respect to the kth coordinate as an operator $\boldsymbol{\partial}_{x_k}$ abbreviated to $\boldsymbol{\partial}_k$ and defined by

$$\boldsymbol{\partial}_k F = \boldsymbol{\partial}_{x_k} F = \frac{\partial F}{\partial x_k}. \tag{3.3}$$

Then the *vector derivative* can be introduced as a differential operator $\boldsymbol{\nabla}$ (called "del") defined by

$$\boldsymbol{\nabla} = \sum \boldsymbol{\sigma}_k \boldsymbol{\partial}_k. \tag{3.4}$$

Operating on the function $F = F(\mathbf{x})$, this gives

$$\boldsymbol{\nabla} F = \boldsymbol{\sigma}_1 \boldsymbol{\partial}_1 F + \boldsymbol{\sigma}_2 \boldsymbol{\partial}_2 F + \cdots + \boldsymbol{\sigma}_n \boldsymbol{\partial}_n F. \tag{3.5}$$

The operation of $\boldsymbol{\nabla}$ on the function $F = F(\mathbf{x})$ thus produces a new function $\boldsymbol{\nabla} F$ called "the *gradient* of F," which means that it is "the *derivative* of F" with respect to a vector variable. This concept of the vector derivative along with its notation and nomenclature is unconventional in that it employs the geometric product in an essential way. However, as its implications are developed it will be seen to unify the various conventional notions of differentiation.

The variable $\mathbf{x}$ is suppressed in the notation $\boldsymbol{\nabla} F$. The dependence can be made explicit by writing $\boldsymbol{\nabla} = \boldsymbol{\nabla}_{\mathbf{x}}$ and $\boldsymbol{\nabla}_{\mathbf{x}} F = \boldsymbol{\nabla}_{\mathbf{x}} F(\mathbf{x})$. When dealing with functions of two or more vector variables, the subscript on $\boldsymbol{\nabla}_{\mathbf{x}}$ indicates which variable is being differentiated.

The partial derivative obeys the three general rules for scalar differentiation: the distributive, product, and chain rules. For functions $F = F(\mathbf{x})$ and $G = G(\mathbf{x})$ the *distributive rule* is

$$\boldsymbol{\partial}_k (F + G) = \boldsymbol{\partial}_k F + \boldsymbol{\partial}_k G, \tag{3.6}$$

and the *product rule* is

$$\boldsymbol{\partial}_k (FG) = (\boldsymbol{\partial}_k F) G + F \boldsymbol{\partial}_k G. \tag{3.7}$$

If $G = \alpha$ is a scalar constant, the produce rule reduces to

$$\boldsymbol{\partial}_k (\alpha F) = \alpha \boldsymbol{\partial}_k F. \tag{3.8}$$

Note that (3.6) along with (3.8) are the requirements making the derivative a linear operator in G_n. Finally, for a function of the form $F = F(\lambda(\mathbf{x}))$ where $\lambda = \lambda(\mathbf{x})$ is scalar-valued, the *chain rule* is

$$\boldsymbol{\partial}_k F = (\boldsymbol{\partial}_k \lambda)\frac{\boldsymbol{\partial} F}{\boldsymbol{\partial}\lambda} = (\boldsymbol{\partial}_k F)\left[\frac{\boldsymbol{\partial} F(\lambda)}{\boldsymbol{\partial}\lambda}\right]_{\lambda=\lambda(\mathbf{x})} \tag{3.9}$$

It should be noted that the symbol λ is used in two distinct but related senses here; in the expression $\boldsymbol{\partial} F(\lambda)/\boldsymbol{\partial}\lambda$ it denotes an independent scalar variable, while in $\lambda = \lambda(\mathbf{x})$ it denotes a scalar-valued function. This kind of ambiguity is frequently introduced by physicists to reduce the proliferation of symbols, but it is not so often employed by mathematicians. The reader should be able to resolve the ambiguity from the context.

The general rules for vector differentiation can be obtained from those of scalar differentiation by using the definition (3.4). Multiplying (3.6) by $\boldsymbol{\sigma}_k$ and summing, we obtain the *distributive rule*

$$\boldsymbol{\nabla}(F+G) = \boldsymbol{\nabla}F + \boldsymbol{\nabla}G. \tag{3.10}$$

Similarly, from (3.7) we obtain the *product rule*

$$\boldsymbol{\nabla}(FG) = (\boldsymbol{\nabla}F)G + \sum \boldsymbol{\sigma}_k F \boldsymbol{\partial}_k G \tag{3.11}$$

In general $\boldsymbol{\sigma}_k$ will not commute with F. In $\mathcal{G}_3$, since only scalars and pseudoscalars commute with all vectors, the product rule can be put in the form

$$\boldsymbol{\nabla}(FG) = (\boldsymbol{\nabla}F)G + F(\boldsymbol{\nabla}G) \tag{3.12}$$

if and only if $F = F_{\overline{0}} + F_{\overline{3}}$. The general product rule (3.11) can be put in a coordinate-free form by introducing various notations. For example, we can write

$$\boldsymbol{\nabla}(FG) = \grave{\boldsymbol{\nabla}}\grave{F}G + \grave{\boldsymbol{\nabla}}F\grave{G}, \tag{3.13}$$

where the accent indicates which function is being differentiated while the other is held fixed. Another version of the product rule is

$$G\boldsymbol{\nabla}F = G\grave{\boldsymbol{\nabla}}\grave{F} + \grave{G}\grave{\boldsymbol{\nabla}}F. \tag{3.14}$$

In contrast to scalar calculus, the two versions of the product rule (3.13) and (3.14) are needed because $\boldsymbol{\nabla}$ need not commute with factors which are not differentiated.

In accordance with the *usual convention* for differential operators, the operator $\boldsymbol{\nabla}$ differentiates quantities immediately to its right in a product. So in the expression $\boldsymbol{\nabla}(FG)$, it is understood that both $F = F(\mathbf{x})$ and $G = G(\mathbf{x})$ are differentiated. But in the expression $G\boldsymbol{\nabla}F$ only F is differentiated. The parenthesis or the accents in $(\boldsymbol{\nabla}F)G = \grave{\boldsymbol{\nabla}}\grave{F}G$ indicates that F but not G is differentiated. Accents can also be used to indicate differentiation to the left as in (3.14). Of course, the *left derivative* $\grave{F}\grave{\boldsymbol{\nabla}}$ is not generally equal to the *right derivative* $\boldsymbol{\nabla}F$, but note that

$$\grave{F}\grave{\boldsymbol{\nabla}} = (\boldsymbol{\nabla}F^\dagger)^\dagger. \tag{3.15}$$

Besides this, the accent notation helps formulate a useful *generalization of the product rule*:

$$\begin{aligned}\boldsymbol{\nabla}H(\mathbf{x},\mathbf{x}) &= \grave{\boldsymbol{\nabla}}\underline{H}(\grave{\mathbf{x}},\mathbf{x}) + \grave{\boldsymbol{\nabla}}\underline{H}(\mathbf{x},\grave{\mathbf{x}}) \\ &= \left[\boldsymbol{\nabla}_{\mathbf{y}}H(\mathbf{y},\mathbf{x}) + \boldsymbol{\nabla}_{\mathbf{y}}H(\mathbf{x},\mathbf{y})\right]_{\mathbf{y}=\mathbf{x}},\end{aligned} \tag{3.16}$$

where $H(\mathbf{x},\mathbf{x})$ is a function of a single variable formed from a function of two variables $H(\mathbf{x},\mathbf{y})$, and the accents indicate which variable is differentiated.

To complete the list of general rules for vector differentiation, from (3.9) we get the *chain rule*

$$\boldsymbol{\nabla} F = (\boldsymbol{\nabla}\lambda)\frac{\boldsymbol{\partial} F}{\boldsymbol{\partial}\lambda} \tag{3.17}$$

for differentiating $F(\lambda(\mathbf{x}))$. For a function of the form $F = F(\mathbf{y}(\mathbf{x}))$ where $\mathbf{y} = \mathbf{y}(\mathbf{x})$ is a vector-valued function, the chain rule can be derived by decomposing $\mathbf{y}$ into rectangular coordinates y_k and applying (3.17). Thus, differentiating

$$F(\mathbf{y}(\mathbf{x})) = F(\boldsymbol{\sigma}_1 y_1(\mathbf{x}) + \boldsymbol{\sigma}_2 y_2(\mathbf{x}) + \cdots + \boldsymbol{\sigma}_n y_n(\mathbf{x}))$$

we get

$$\boldsymbol{\nabla}_{\mathbf{x}} F = \sum_k (\boldsymbol{\nabla}_{\mathbf{x}} y_k(\mathbf{x}))\frac{\boldsymbol{\partial} F}{\boldsymbol{\partial} y_k} = (\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{y}(\mathbf{x})\cdot\boldsymbol{\nabla}_{\mathbf{y}}\,)\ F(\mathbf{y}).$$

Hence the *chain rule* for differentiating $F(\mathbf{y}(\mathbf{x}))$ can be put in the form

$$\boldsymbol{\nabla}_{\mathbf{x}} F = \grave{\boldsymbol{\nabla}}\grave{\mathbf{y}}\cdot\boldsymbol{\nabla}_{\mathbf{y}} F. \tag{3.18}$$

In a later chapter we will see the deep meaning of this formula, and it will appear somewhat simpler.

Differential Identities

Let $\mathbf{v} = \mathbf{v}(\mathbf{x})$ be vector-valued function. Since $\boldsymbol{\nabla}$ is a vector operator, we can use it in the algebraic formula (2.7) to write

$$\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{\nabla}\cdot\mathbf{v} + \boldsymbol{\nabla}\wedge\mathbf{v} = \boldsymbol{\nabla}\cdot\mathbf{v} + i(\boldsymbol{\nabla}\times\mathbf{v}). \tag{3.19}$$

In (3.19), the last equality holds only in $\mathcal{G}_3$ because it involves the vector cross product, but the first equality holds in $\mathcal{G}_n$ insofar as they involve inner and outer products, but a restriction to $\mathcal{G}_3$ is tacitly understood where the vector cross product appears.

This decomposes the derivative of $\mathbf{v}$ into two parts. The scalar part $\boldsymbol{\nabla}\cdot\mathbf{v}$ is called the divergence of $\mathbf{v}$. The bivector part $\boldsymbol{\nabla}\wedge\mathbf{v}$ is called the *curl* of $\mathbf{v}$, and its dual $\boldsymbol{\nabla}\times\mathbf{v}$ is also called the *curl*. The divergence and curl of a vector as well as the gradient of a scalar are given separate definitions in standard texts on vector calculus. Geometric calculus unites all three in a single vector derivative.

From every algebraic identity involving vectors we can derive a "differential identity" by inserting $\boldsymbol{\nabla}$ as one of the vectors and applying the product rule to account for differentiation of vector functions in the identity. For example, from the algebraic identity

$$-\,\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}\,\mathbf{c} - \mathbf{a}\cdot\mathbf{c}\,\mathbf{b}$$

we can derive several important differential identities. Thus, let $\mathbf{u} = \mathbf{u}(\mathbf{x})$ and $\mathbf{v} = \mathbf{v}(\mathbf{x})$ be vector-valued functions. Then

$$\boldsymbol{\nabla}\cdot(\mathbf{u}\wedge\mathbf{v}) = \boldsymbol{\nabla}\cdot\mathbf{u}\,\mathbf{v} - \boldsymbol{\nabla}\cdot\mathbf{v}\,\mathbf{u}.$$

But, $\boldsymbol{\nabla}\cdot\mathbf{u}\,\mathbf{v} = (\boldsymbol{\nabla}\cdot\mathbf{u})\mathbf{v} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{v}$. Hence, we have the identity

$$\begin{aligned} -\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{v}) &= \boldsymbol{\nabla}\cdot(\mathbf{u}\wedge\mathbf{v}) \\ &= \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{v} - \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{u} + \mathbf{v}\boldsymbol{\nabla}\cdot\mathbf{u} - \mathbf{u}\boldsymbol{\nabla}\cdot\mathbf{v}. \end{aligned} \tag{3.20}$$

Alternatively,

$$\mathbf{u}\cdot(\boldsymbol{\nabla}\wedge\mathbf{v}) = \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{v} - \boldsymbol{\nabla}\mathbf{u}\cdot\mathbf{v} \tag{3.21}$$

But, $\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{v}) = \dot{\boldsymbol{\nabla}}\dot{\mathbf{u}}\cdot\mathbf{v} + \dot{\boldsymbol{\nabla}}\mathbf{u}\cdot\dot{\mathbf{v}}$. Therefore, by adding to (3.21) a copy of (3.21) with $\mathbf{u}$ and $\mathbf{v}$ interchanged, we get the identity

$$\begin{aligned}\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{v}) &= \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{u} - \mathbf{u}\cdot(\boldsymbol{\nabla}\wedge\mathbf{v}) - \mathbf{v}\cdot(\boldsymbol{\nabla}\wedge\mathbf{u})\\ &= \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{u} + \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{v}) + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{u}).\end{aligned}$$

By the way, the differential operator $\mathbf{u}\cdot\boldsymbol{\nabla}$ is called the *directional derivative*.

Differentiating twice with the vector derivative, we get a differential operator $\boldsymbol{\nabla}^2$ called the *Laplacian*. Since $\boldsymbol{\nabla}$ is a vector operator, we can write $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} + \boldsymbol{\nabla}\wedge\boldsymbol{\nabla}$. However, operating on any function $F = F(\mathbf{x})$, and using the commutativity $\boldsymbol{\partial}_j\boldsymbol{\partial}_k = \boldsymbol{\partial}_k\boldsymbol{\partial}_j$ of partial derivatives, we see that

$$\begin{aligned}\boldsymbol{\nabla}\wedge\boldsymbol{\nabla}F &= (\textstyle\sum_j \boldsymbol{\sigma}_j\boldsymbol{\partial}_j)\wedge(\sum_k \boldsymbol{\sigma}_k\boldsymbol{\partial}_k)F = \sum_{j,k}\boldsymbol{\sigma}_j\wedge\boldsymbol{\sigma}_k\boldsymbol{\partial}_j\boldsymbol{\partial}_k F\\ &= \boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_2\boldsymbol{\partial}_1\boldsymbol{\partial}_2 F + \boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_1\boldsymbol{\partial}_2\boldsymbol{\partial}_1 F + \cdots\\ &= \boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_2\boldsymbol{\partial}_1\boldsymbol{\partial}_2 F - \boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_2\boldsymbol{\partial}_1\boldsymbol{\partial}_2 F + \cdots = 0\end{aligned}$$

Thus, we have $\boldsymbol{\nabla}\wedge\boldsymbol{\nabla}F = 0$, or, expressed as an operator identity

$$\boldsymbol{\nabla}\wedge\boldsymbol{\nabla} = i\boldsymbol{\nabla}\times\boldsymbol{\nabla} = 0 \tag{3.22}$$

As a corollary, the Laplacian

$$\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} \tag{3.23}$$

is a "scalar differential operator" which does not alter the step of any function on which it operates.

Basic Derivatives

As in the conventional scalar differential calculus, to carry out computations in geometric calculus we need to know the derivatives of a few elementary functions. The most important functions are algebraic functions of a vector variable, so let us see how to differentiate them.

By taking the partial derivative of (3.1) we get

$$\boldsymbol{\partial}_k\mathbf{x} = \boldsymbol{\sigma}_k \tag{3.24}$$

Then, for a vector variable $\mathbf{x}$ in $\mathcal{E}_n$, the definition (3.4) for $\boldsymbol{\nabla}$ gives us

$$\boldsymbol{\nabla}\mathbf{x} = \sum\boldsymbol{\sigma}_k\boldsymbol{\partial}_k\mathbf{x} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2\boldsymbol{\sigma}_2 + \cdots + \boldsymbol{\sigma}_n\boldsymbol{\sigma}_n.$$

Hence

$$\boldsymbol{\nabla}\mathbf{x} = n. \tag{3.25}$$

Separating scalar and bivector parts, we get

$$\boldsymbol{\nabla}\cdot\mathbf{x} = n, \tag{3.26a}$$

$$\boldsymbol{\nabla}\wedge\mathbf{x} = 0. \tag{3.26b}$$

Similarly, if $\mathbf{a}$ is a constant vector, then

$$\boldsymbol{\nabla}\mathbf{x}\cdot\mathbf{a} = \sum_k\boldsymbol{\sigma}_k(\boldsymbol{\partial}_k\mathbf{x})\cdot\mathbf{a} = \sum\boldsymbol{\sigma}_k\boldsymbol{\sigma}_k\cdot\mathbf{a} = \mathbf{a}$$

and

$$\mathbf{a}\cdot\boldsymbol{\nabla}\mathbf{x} = \sum\mathbf{a}\cdot\boldsymbol{\sigma}_k(\boldsymbol{\partial}_k\mathbf{x}) = \mathbf{a}.$$

Therefore,

$$\boldsymbol{\nabla}\mathbf{x}\cdot\mathbf{a} = \mathbf{a}\cdot\boldsymbol{\nabla}\mathbf{x} = \mathbf{a}, \tag{3.27}$$

where $\mathbf{a}$ is a constant vector.

The two basic formulas (3.25) and (3.27) tell us all we need to know about differentiating the variable $\mathbf{x}$. They enable us to differentiate any algebraic function of $\mathbf{x}$ directly without further appeal to partial derivatives. We need only apply them in conjunction with algebraic manipulations and the general rules for differentiation. By way of example, let us derive a few more formulas which occur so frequently they are worth memorizing. Using algebraic manipulation we evaluate

$$\boldsymbol{\nabla}(\mathbf{x}\wedge\mathbf{a}) = \boldsymbol{\nabla}(\mathbf{x}\,\mathbf{a} - \mathbf{x}\cdot\mathbf{a}) = n\mathbf{a} - \mathbf{a}$$

Therefore, since $\boldsymbol{\nabla}\wedge(\mathbf{x}\wedge\mathbf{a}) = (\boldsymbol{\nabla}\wedge\mathbf{x})\wedge\mathbf{a} = 0$, we have

$$\boldsymbol{\nabla}(\mathbf{x}\wedge\mathbf{a}) = \boldsymbol{\nabla}\cdot(\mathbf{x}\wedge\mathbf{a}) = (n-1)\mathbf{a}, \tag{3.28a}$$

or alternatively in $\mathcal{G}_3$, by (2.10),

$$\boldsymbol{\nabla}\times(\mathbf{x}\times\mathbf{a}) = -2\mathbf{a}. \tag{3.28b}$$

Applying the product rule we evaluate

$$\boldsymbol{\nabla}\mathbf{x}^2 = \grave{\boldsymbol{\nabla}}(\grave{\mathbf{x}}\cdot\mathbf{x} + \mathbf{x}\cdot\grave{\mathbf{x}}) = \mathbf{x} + \mathbf{x}.$$

Therefore,

$$\boldsymbol{\nabla}\mathbf{x}^2 = 2\mathbf{x} \tag{3.29}$$

On the other hand, application of the chain rule to the left side of $|\,\mathbf{x}\,|^2 = \mathbf{x}^2$ yields

$$\boldsymbol{\nabla}|\,\mathbf{x}\,|^2 = 2|\,\mathbf{x}\,|\boldsymbol{\nabla}|\,\mathbf{x}\,| = 2\mathbf{x}$$

Therefore,

$$\boldsymbol{\nabla}|\,\mathbf{x}\,| = \hat{\mathbf{x}} = \frac{\mathbf{x}}{|\,\mathbf{x}\,|}. \tag{3.30}$$

This enables us to evaluate the derivative of $|\,\mathbf{x}\,|^n$ by further application of the chain rule. Thus, for integer n we get

$$\boldsymbol{\nabla}|\,\mathbf{x}\,|^n = n|\,\mathbf{x}\,|^{n-1}\hat{\mathbf{x}} \tag{3.31}$$

This applies when $n < 0$ except at $|\,\mathbf{x}\,| = 0$.

All these basic vector derivatives and their generalizations in the following exercises will be taken for granted throughout the rest of this book.

1-3 Exercises

(3.1) Let $\mathbf{r} = \mathbf{r}(\mathbf{x}) = \mathbf{x} - \mathbf{x}'$ and $r = |\,\mathbf{r}\,| = |\,\mathbf{x} - \mathbf{x}'\,|$, where $\mathbf{x}'$ is a vector independent of $\mathbf{x}$. Establish the basic derivatives

$$\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{a}) = \mathbf{a}\cdot\boldsymbol{\nabla}\mathbf{r} = \mathbf{a},$$

$$\boldsymbol{\nabla}r = \hat{\mathbf{r}},$$

$$\boldsymbol{\nabla}\mathbf{r} = n.$$

Note that $\mathbf{r}$ can be regarded as a new variable differing from $\mathbf{x}$ only by a shift of origin from $\mathbf{0}$ to $\mathbf{x}'$. Prove that

$$\boldsymbol{\nabla}_{\mathbf{r}} = \boldsymbol{\nabla}_{\mathbf{x}}.$$

Thus, the vector derivative is invariant under a shift of origin.

Use the above results to evaluate the following derivatives:

$$\boldsymbol{\nabla}\hat{\mathbf{r}} = \frac{2}{r} = \boldsymbol{\nabla}^2 r$$

$$\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{r}) = \boldsymbol{\nabla}\cdot(\mathbf{r}\wedge\mathbf{a}) = 2\mathbf{a}$$

$$\mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{r}) = 0$$

$$\boldsymbol{\nabla} r^k = k r^{k-2}\mathbf{r} \quad \text{(for any scalar } n\text{)}$$

$$\boldsymbol{\nabla}\frac{\mathbf{r}}{r^k} = \frac{2-k}{r^k}$$

$$\boldsymbol{\nabla}\log r = \frac{\mathbf{r}}{r^2} = \mathbf{r}^{-1}$$

Of course, some of these derivatives are not defined at $\mathbf{r} = 0$ for some values of k.

(3.2) For scalar field $\phi = \phi(\mathbf{x})$ and vector fields $\mathbf{u} = \mathbf{u}(\mathbf{x})$ and $\mathbf{v} = \mathbf{v}(\mathbf{x})$, derive the following identities as instances of the product rule:

(a) $\boldsymbol{\nabla}\cdot(\phi\mathbf{v}) = \mathbf{v}\cdot\boldsymbol{\nabla}\phi + \phi\boldsymbol{\nabla}\cdot\mathbf{v}$

(b) $\boldsymbol{\nabla}\wedge(\mathbf{u}\wedge\mathbf{v}) = \mathbf{v}\wedge\boldsymbol{\nabla}\wedge\mathbf{u} - \mathbf{u}\wedge\boldsymbol{\nabla}\wedge\mathbf{v}$

(c) $\boldsymbol{\nabla}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\boldsymbol{\nabla}\times\mathbf{u}) - \mathbf{u}\cdot(\boldsymbol{\nabla}\times\mathbf{v})$

(d) $\boldsymbol{\nabla}\cdot(\mathbf{u}\wedge\mathbf{v}) = \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{u}) = \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{v} - \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{u} + \mathbf{v}\boldsymbol{\nabla}\cdot\mathbf{u} - \mathbf{u}\boldsymbol{\nabla}\cdot\mathbf{v}$

(e) $\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{u} - \mathbf{u}\cdot(\boldsymbol{\nabla}\wedge\mathbf{v}) - \mathbf{v}\cdot(\boldsymbol{\nabla}\wedge\mathbf{u})$

$$= \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{u} + \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{v}) + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{u})$$

(f) $\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{\nabla}(\tfrac{1}{2}\mathbf{v}^2) + \mathbf{v}\cdot(\boldsymbol{\nabla}\wedge\mathbf{v})$

(3.3) (a) In NFCM the *directional derivative* of the function $F = F(\mathbf{x})$ is defined by

$$\mathbf{a}\cdot\boldsymbol{\nabla}F = \left.\frac{dF(\mathbf{x}+\mathbf{a}\tau)}{d\tau}\right|_{\tau=0} = \lim_{\tau\to 0}\frac{F(\mathbf{x}+\tau\mathbf{a}) - F(\mathbf{x})}{\tau}$$

Prove that this is equivalent to the operator $\mathbf{a}\cdot\boldsymbol{\nabla}$ obtained algebraically from the vector derivative $\boldsymbol{\nabla}$ through the identity

$$2\mathbf{a}\cdot\boldsymbol{\nabla}F = \mathbf{a}\boldsymbol{\nabla}F + \grave{\boldsymbol{\nabla}}\mathbf{a}\grave{F},$$

where the accent indicating that only F is differentiated is unnecessary if $\mathbf{a}$ is constant.

(b) The directional derivative of $F = F(\mathbf{x})$ produces a new function of two vector variables

$$F' = F'(\mathbf{a}, \mathbf{x}) = \mathbf{a}\cdot\boldsymbol{\nabla} F(\mathbf{x})$$

called the *differential* of F. Prove that it is a linear function of the first variable $\mathbf{a}$. Prove that

$$\boldsymbol{\nabla} F = \boldsymbol{\nabla}_{\mathbf{a}}(\mathbf{a}\cdot\boldsymbol{\nabla} F) = \boldsymbol{\nabla}_{\mathbf{a}} F'(\mathbf{a}, \mathbf{x}),$$

and so establish the operator equation

$$\boldsymbol{\nabla}_{\mathbf{x}} = \boldsymbol{\nabla}_{\mathbf{a}}\mathbf{a}\cdot\boldsymbol{\nabla}_{\mathbf{x}}.$$

This tells us how to get the vector derivative of F from its differential F' if we know how to differentiate a linear function. It can also be used to establish the properties of the vector derivative without using coordinates and partial derivatives.

(c) The function $F = F(\mathbf{x})$ is said to be *constant* in a region if its differential vanishes at every point $\mathbf{x}$ in the region, that is, if $\mathbf{a}\cdot\boldsymbol{\nabla} F(\mathbf{x}) = 0$ for all $\mathbf{a}$. Prove that

$$\boldsymbol{\nabla} F = 0 \text{ if } F \text{ is constant},$$

and prove that the converse is false by using the results of Exercise (3.1) to construct a counterexample.

(d) The *second differential* of F is defined by

$$F''(\mathbf{a}, \mathbf{b}) = \mathbf{b}\cdot\dot{\boldsymbol{\nabla}}\mathbf{a}\cdot\dot{\boldsymbol{\nabla}} F(\dot{\mathbf{x}})$$

Explain why it is a linear symmetric function of the variables $\mathbf{a}$ and $\mathbf{b}$, and show that

$$\boldsymbol{\nabla}^2 F = \boldsymbol{\nabla}_{\mathbf{b}}\boldsymbol{\nabla}_{\mathbf{a}} F''(\mathbf{a}, \mathbf{b}).$$

Use this to prove the identity (3.22).

(3.4) *Taylor's Formula*. Justify the following version of a "Taylor expansion"

$$\begin{aligned} F(\mathbf{x}+\mathbf{a}) &= F(\mathbf{x}) + \mathbf{a}\cdot\boldsymbol{\nabla} F(\mathbf{x}) + \frac{(\mathbf{a}\cdot\dot{\boldsymbol{\nabla}})^2}{2!} F(\dot{\mathbf{x}}) + \cdots, \\ &= \sum_{k=0}^{\infty} \frac{(\mathbf{a}\cdot\dot{\boldsymbol{\nabla}})^k}{k!} F(\dot{\mathbf{x}}) \equiv e^{\mathbf{a}\cdot\boldsymbol{\nabla}} F(\mathbf{x}) \end{aligned}$$

by reducing it to a standard expansion in terms of a scalar variable.

Note that this leads to a natural definition of the exponential operator $e^{\mathbf{a}\cdot\nabla}$ and its interpretation as a *displacement operator*, shifting the argument of F from $\mathbf{x}$ to $\mathbf{x}+\mathbf{a}$. The operator $\mathbf{a}\cdot\boldsymbol{\nabla}$ is called the *generator* of the displacement.

1-4 Linear Transformations

A function $F = F(A)$ is said to be *linear* if preserves the operations of addition and scalar multiplication, as expressed by the equations

$$F(A+B) = F(A) + F(B) \tag{4.1a}$$

$$F(\alpha A) = \alpha F(A), \tag{4.1b}$$

where α is a scalar. We will refer to a linear function which maps vectors into vectors as a *linear transformation* or a *tensor*. The term "tensor" is preferred especially when the function represents some property of a physical system. Thus, we will be concerned with the "stress tensor" and the "strain tensor" in Chapter 4. In this mathematical section the terms "linear transformation" and "tensor" will be used interchangeably with each other and some times with the term "linear operator."

Outermorphisms

Every linear transformation (or tensor) f on the vector space $\mathcal{E}_3$ induces a unique linear transformation $\underline{f}$ on the entire geometric algebra $\mathcal{G}_3$ called the *outermorphisms* of $\underline{f}$. Supposed that f maps vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ into vectors $\mathbf{a}', \mathbf{b}', \mathbf{c}'$. The outermorphisms of a bivector $\mathbf{a} \wedge \mathbf{b}$ is defined by

$$\underline{f}(\mathbf{a} \wedge \mathbf{b}) = f(\mathbf{a}) \wedge f(\mathbf{b}) = \mathbf{a}' \wedge \mathbf{b}' \tag{4.2}$$

Similarly, the outermorphisms of a trivector $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is defined by

$$\underline{f}(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = f(\mathbf{a}) \wedge f(\mathbf{b}) \wedge f(\mathbf{c}) = \mathbf{a}' \wedge \mathbf{b}' \wedge \mathbf{c}' \tag{4.3}$$

The trivectors $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ and $\mathbf{a}' \wedge \mathbf{b}' \wedge \mathbf{c}'$ can differ only by a scale factor which depends solely on f. That scale factor is called the *determinant* of f and denoted by det f. Thus the determinant of a linear transformation can be defined by

$$\det f = \frac{\underline{f}(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})}{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}} = \frac{\mathbf{a}' \wedge \mathbf{b}' \wedge \mathbf{c}'}{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}}, \tag{4.4}$$

provided $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \neq 0$. We can interpret $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ as a directed volume, so (4.4) tells us that $\det f$ can be interpreted as a dilation of volumes induced by f as well as a change in orientation if $\det f$ is negative. We say that f is *singular* if $\det f = 0$; otherwise f or $\underline{f}$ is said to be *nonsingular*.

Note that (4.4) is independent of the chosen trivector, so the determinant can be defined by

$$\det f = i^{-1}\, \underline{f}(i) = -i\, \underline{f}(i), \tag{4.5}$$

where, as always, i is the unit pseudoscalar. Comparison of (4.2) with (4.3) shows that if f is nonsingular then $\mathbf{a}' \wedge \mathbf{b}'$ cannot vanish unless $\mathbf{a} \wedge \mathbf{b}$ vanishes.

Further insight into the geometrical significance of outermorphisms comes from considering the equation

$$\mathbf{a} \wedge (\mathbf{x} - \mathbf{b}) = 0,$$

where $\mathbf{x}$ is a vector variable. This is the equation for a straight line with direction $\mathbf{a}$ passing through a point $\mathbf{b}$. Under an outermorphisms it is transformed to

$$\underline{f}[\,\mathbf{a} \wedge (\mathbf{x} - \mathbf{b})] = \mathbf{a}' \wedge (\mathbf{x}' - \mathbf{b}') = 0.$$

Thus, every nonsingular linear transformation maps straight lines into straight lines.

The above definition of the outermorphism for k-vectors has a unique extension to all multivectors $A, B, \ldots$ of the algebra $\mathcal{G}_3$. In general, the outermorphisms $\underline{f}$ of a linear transformation f is defined by the following properties: It is *linear*,

$$\underline{f}(\alpha A + \beta B) = \alpha \underline{f}(A) + \beta \underline{f}(B); \tag{4.6}$$

It is identical with f when acting on vectors,

$$\underline{f}(\mathbf{a}) \equiv f(\mathbf{a}); \tag{4.7}$$

It preserves outer products

$$\underline{f}(A \wedge B) = \underline{f}(A) \wedge \underline{f}(B); \tag{4.8}$$

Finally, it leaves scalars invariant,

$$\underline{f}(\alpha) = \alpha \tag{4.9}$$

It follows that $\underline{f}$ is grade-preserving,

$$\underline{f}(A_{\bar{k}}) = [\underline{f}(A)]_{\bar{k}} \,. \tag{4.10}$$

Operator Products and Notation

The composite $h(\mathbf{a}) = g(f(\mathbf{a}))$ of linear functions g and f is often called the *product* of g and f. It is easily proved that the outermorphisms of a product is equal to the product of outermorphisms. This can be expressed as an *operator product*

$$\underline{h} = \underline{g}\,\underline{f} \tag{4.11}$$

provided we follow the common practice of dropping parentheses and writing $f(A) = fA$ for linear operators. To avoid confusion between the operator product (4.11) and the geometric product of multivectors, we need some notation to distinguish operators from multivectors. The underbar notation serves that purpose nicely. Accordingly, we adopt the underbar notation for tensors as well as their induced outermorphisms. The ambiguity in this notation will usually be resolved by the context. But if necessary, the vectorial part can be distinguished from the complete outermorphisms by writing $f\mathbf{a}$ or $f = f\mathbf{a}$. For example, when the operator equation (4.11) is applied to vectors it merely represents the composition of tensors,

$$\underline{h}\mathbf{a} = \underline{g}\,\underline{f}\mathbf{a} = \underline{g}(\underline{f}\mathbf{a}).$$

However, when it is applied to the unit pseudoscalar i it yields

$$\underline{h}i = \underline{g}((\det \underline{f})i) = (\det \underline{f})\,\underline{g}i = (\det \underline{f})(\det \underline{g})i.$$

Thus, we obtain the theorem

$$\det(\underline{g}\,\underline{f}) = (\det \underline{g})(\det \underline{f}). \tag{4.12}$$

As a final remark about the notation for operator products, when working with repeated applications of the same operator, it is convenient to introduced the "power notation"

$$\underline{f}^2 = \underline{f}\,\underline{f} \quad \text{and} \quad \underline{f}^k = \underline{f}\,\underline{f}^{k-1}, \tag{4.13}$$

where k is a positive integer. From (4.12) we immediately obtain the theorem

$$\det(\underline{f}^k) = (\det \underline{f})^k. \tag{4.14}$$

Adjoint Tensors

To every linear transformation f there corresponds another linear transformation $\bar{f}$ defined by the condition that

$$\mathbf{a}\cdot(\underline{f}\mathbf{b}) = (\bar{f}\mathbf{a})\cdot\mathbf{b} \tag{4.15}$$

holds for all vectors $\mathbf{a}, \mathbf{b}$. Differentiation gives

$$\bar{f}\mathbf{a} = \boldsymbol{\nabla}_{\mathbf{b}}(\underline{f}\mathbf{b})\cdot\mathbf{a} = \dot{\boldsymbol{\nabla}}\dot{\mathbf{f}}\cdot\mathbf{a}. \tag{4.16}$$

where $\mathbf{f} = f(\mathbf{b})$. The tensor $\bar{f}$ is called the *adjoint* or *transpose* of f. Being linear, it has a unique outermorphisms also denoted by $\bar{f}$. It follows that for any multivectors A, B

$$\langle A\underline{f}B\rangle = \langle B\bar{f}A\rangle. \tag{4.17}$$

This could be taken as the definition of $\bar{f}$. It reduces to (4.15) when A and B are vectors. Taking $A = i^{-1}$ and $B = i$ in (4.17) yields the theorem

$$\det \underline{f} = i^{-1}\underline{f}i = -\langle i\underline{f}i\rangle = \det \bar{f}. \tag{4.18}$$

When f is nonsingular it has a unique inverse f^{-1} which can be computed from f by using the equation

$$f^{-1}\mathbf{a} = \bar{f}(\mathbf{a}i)/\underline{f}i = \frac{\bar{f}(\mathbf{a}i)}{i\det \underline{f}} \tag{4.19}$$

Of course, the inverse of an outermorphism is equal to the outermorphism of an inverse, so we can write

$$\underline{f}^{-1}\underline{f}A = A \tag{4.20}$$

for any multivector A. As a consequence, by inserting $g = f^{-1}$ into (4.12) we get the theorem

$$\det \underline{f}^{-1} = (\det \underline{f})^{-1}. \tag{4.21}$$

Bases and Matrices

A *basis* $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ for $\mathcal{E}_3$ is said to be *orthogonal* if

$$\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_3\cdot\boldsymbol{\sigma}_1 = 0.$$

It is said to be *normalized* if

$$\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_2^2 = \boldsymbol{\sigma}_3^2 = 1.$$

It is said to be *orthonormal* if both conditions hold. The two conditions can be expressed together by writing

$$\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j = \delta_{ij}, \tag{4.22}$$

where $i, j = 1, 2, 3$ and $[\delta_{ij}]$ is the identity matrix. The orthonormal basis $\{\boldsymbol{\sigma}_k\}$ is said to be *dextral* (or *right-handed*) if it is related to the dextral unit pseudoscalar by

$$\boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i \tag{4.23}$$

We will refer to a given dextral orthonormal basis $\{\boldsymbol{\sigma}_k\}$ as a *standard basis*.

A tensor $\underline{f}$ transforms each basis vector $\boldsymbol{\sigma}_k$ into a vector $\mathbf{f}_k$ which can be expanded in the standard basis; thus

$$\mathbf{f}_k = \underline{f}\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_j f_{jk}, \tag{4.24}$$

where the summation is understood to be over the repeated indices. Each scalar coefficient

$$f_{jk} = \langle \boldsymbol{\sigma}_j \underline{f} \boldsymbol{\sigma}_k \rangle = \boldsymbol{\sigma}_j \cdot \mathbf{f}_k \tag{4.25}$$

is called a *matrix element* of the tensor $\underline{f}$, and $[\,\underline{f}\,] = [f_{jk}]$ denotes *the matrix of* f *in the standard basis*.

The matrix $[f]$ is called a *matrix representation* of the operator f, because operator addition and multiplication are thereby represented as matrix addition and multiplication, as expressed by

$$\begin{aligned} [\,\alpha \underline{f} + \beta \underline{g}\,] &= \alpha[\,\underline{f}\,] + \beta[\,\underline{g}\,], \\ [\,\underline{g}\underline{f}\,] &= [\,\underline{f}\,][\,\underline{g}\,]. \end{aligned} \tag{4.26}$$

Note that the matrix $[\underline{f}]$ does not represents the entire outermorphism, but only the action of $\underline{f}$ on vectors. In fact, the complete properties of outermorphisms are not easily represented by matrices, because matrix algebra does not directly characterize the outer product. However, the determinant of $\underline{f}$ is equal to the determinant of its matrix in any basis; thus,

$$\begin{aligned} \det \underline{f} = i\mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \mathbf{f}_3 &= (\boldsymbol{\sigma}_3 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_1) \cdot (\mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \mathbf{f}_3) \\ &= \det [\boldsymbol{\sigma}_j \cdot \mathbf{f}_k] = \det [\,\underline{f}\,]. \end{aligned} \tag{4.27}$$

This determinant can be evaluated in terms of the matrix elements by the Laplace expansion (1.17).

A tensor $\underline{f}$ is completely determined by specifying a standard basis $\{\boldsymbol{\sigma}_k\}$ and the transformed vectors $\{\mathbf{f}_k = \underline{f}\boldsymbol{\sigma}_k\}$ or, equivalently, the transformation matrix $[f_{jk}] = [\boldsymbol{\sigma}_j \cdot \mathbf{f}_k]$. This has the disadvantage of introducing vectors with no relation to the properties of $\underline{f}$. For this reason, we will take the alternative approach of representing $\underline{f}$ in terms of multivectors and geometric products. One way to do that is with dyadics. A *dyad* $\underline{D}$ is a tensor determined by two vectors $\mathbf{u}, \mathbf{v}$, as expressed by

$$\underline{D} \;=\; \mathbf{u}\mathbf{v}\cdot. \tag{4.28}$$

Here the incomplete dot product $\mathbf{v}\cdot$ is regarded as an operator which, in accordance our "precedence convention," is to be completed before multiplication by $\mathbf{u}$; thus,

$$\underline{D}\mathbf{a} = \mathbf{v}\mathbf{u}\cdot\mathbf{a}. \tag{4.29}$$

A *dyadic* is a tensor expressed as a sum of dyads. Any tensor can be represented in dyadic form, but other representations are more suitable for some tensors or some purposes. The dyadic form is especially well suited for symmetric tensors.

Symmetric Tensors

A tensor $\underline{S}$ is said to be *symmetric* if

$$\mathbf{a}\cdot(\underline{S}\mathbf{b}) = \mathbf{b}\cdot(\underline{S}\mathbf{a}) \tag{4.30}$$

for all vectors $\mathbf{a}, \mathbf{b}$. Comparison with (4.15) shows that this is equivalent to the condition

$$\overline{S} = \underline{S}. \tag{4.31}$$

The *canonical form* for a symmetric tensor is determined by the

Spectral Theorem:

For every symmetric tensor $\underline{S}$ *there exists an orthonormal basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *and scalars* $\lambda_1, \lambda_2, \lambda_3$ *so that* $\underline{S}$ *can be represented in the dyadic form*

$$\underline{S} = \sum \lambda_k \mathbf{e}_k \mathbf{e}_k \cdot . \tag{4.32}$$

The λ_k are *eigenvalues* and the $\mathbf{e}_k$ are *eigenvectors* of $\underline{S}$, for they satisfy the equation

$$\underline{S}\mathbf{e}_k = \lambda_k \mathbf{e}_k. \tag{4.33}$$

With the eigenvectors $\mathbf{e}_k$ as basis, the matrix representation has the diagonal form

$$[\,\underline{S}\,] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{4.34}$$

The representation (4.32) is called the *spectral decomposition* of $\underline{S}$.

The set of eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ is called the spectrum of $\underline{S}$. If all the eigenvalues are distinct the spectrum is said to be *nondegenerate*; otherwise it is *degenerate*. The unit eigenvectors of a symmetric tensor are unique (except for sign) if and only if its spectrum is nondegenerate. If only two of the eigenvalues are distinct with $\lambda_2 = \lambda_3$, say, then (4.32) implies that all vectors in the $\mathbf{e}_2 \wedge \mathbf{e}_3$-plane are eigenvectors of $\underline{S}$ with eigenvalue λ_2. If the spectrum is completely degenerate $(\lambda_1 = \lambda_2 = \lambda_3)$, then $\underline{S}\mathbf{a} = \lambda \mathbf{a}$ for every vector $\mathbf{a}$, that is, $\underline{S}$ reduces to a multiple of the identity operator.

A symmetric tensor is said to be *positive definite* if

$$\mathbf{a} \cdot (\underline{S}\mathbf{a}) > 0 \tag{4.35}$$

for every nonzero vector $\mathbf{a}$. This is equivalent to the condition that all the eigenvalues of $\underline{S}$ be positive. The terminology helps us formulate the

Square Root Theorem:

Every positive definite symmetric tensor $\underline{S}$ *has a unique positive definite and symmetric square root* $\underline{S}^{\frac{1}{2}}$ *with the canonical form*

$$\underline{S}^{\frac{1}{2}} = \sum_k (\lambda_k)^{\frac{1}{2}} \mathbf{e}_k \mathbf{e}_k \cdot . \tag{4.36}$$

When (4.32) *is the canonical form for* $\underline{S}$. The terminology "square root" is justified by the operator equation

$$(\underline{S}^{\frac{1}{2}})^2 = \underline{S} \tag{4.37}$$

The Square Root Theorem is needed for the Polar Decomposition Theorem given in the next section.

Invariants of a Linear Transformation

By vector differentiation of a tensor and its outermorphisms we obtain a complete set of invariants which characterize the tensor. Thus, writing $\mathbf{f} = \underline{f}\mathbf{x}$, we obtain

$$\boldsymbol{\nabla}\mathbf{f} = \boldsymbol{\nabla}\cdot\mathbf{f} + \boldsymbol{\nabla}\wedge\mathbf{f}. \tag{4.38}$$

To see the significance of the curl, consider

$$\mathbf{a}\cdot(\boldsymbol{\nabla}\wedge\mathbf{f}) = \mathbf{a}\cdot\boldsymbol{\nabla}\mathbf{f} - \boldsymbol{\nabla}\mathbf{f}\cdot\mathbf{a}$$

Since $\underline{f}$ is linear and $\mathbf{a}\cdot\boldsymbol{\nabla}$ is a scalar operator

$$\mathbf{a}\cdot\boldsymbol{\nabla}\mathbf{f} = \mathbf{a}\cdot\boldsymbol{\nabla}\underline{f}\mathbf{x} = \underline{f}(\mathbf{a}\cdot\boldsymbol{\nabla}\mathbf{x}) = \underline{f}\mathbf{a}. \tag{4.39}$$

Along with (4.16), this gives us

$$\mathbf{a}\cdot(\boldsymbol{\nabla}\wedge\mathbf{f}) = \underline{f}\mathbf{a} - \bar{f}\mathbf{a}. \tag{4.40}$$

The right side vanishes when $\underline{f}$ is symmetric. Therefore, *a linear transformation is symmetric if and only if its curl is zero.* On the other hand, for a *skew symmetric tensor* (defined by the condition $\underline{f}\mathbf{a} = -\bar{f}\mathbf{a}$), (4.40) gives the canonical form

$$\underline{f}\mathbf{a} = \mathbf{a}\cdot\boldsymbol{\Omega} = \boldsymbol{\omega}\times\mathbf{a} \tag{4.41a}$$

where the bivector $\boldsymbol{\Omega}$ and its dual vector $\boldsymbol{\omega}$ are given by

$$\boldsymbol{\Omega} = i\boldsymbol{\omega} = \tfrac{1}{2}\boldsymbol{\nabla}\wedge\mathbf{f}. \tag{4.41b}$$

Taking the divergence of (4.41a) we get

$$\boldsymbol{\nabla}\cdot\mathbf{f} = \boldsymbol{\nabla}\cdot(\mathbf{x}\cdot\boldsymbol{\Omega}) = (\boldsymbol{\nabla}\wedge\mathbf{x})\cdot\boldsymbol{\Omega} = 0$$

Thus, *the divergence of a skewsymmetric tensor vanishes and the tensor is completely determined by its curl.*

For an arbitrary tensor $\underline{f}$, its divergence is related to its matrix representation by

$$\boldsymbol{\nabla}\cdot\mathbf{f} = \sum \boldsymbol{\sigma}_k\cdot\underline{f}(\boldsymbol{\partial}_k\mathbf{x}) = \boldsymbol{\sigma}_k\cdot\mathbf{f}_k = \sum f_{kk}$$

The sum of diagonal matrix elements f_{kk} is called the *trace* of the matrix $[f_{jk}]$, so let us define the *trace* of the operator $\underline{f}$ by

$$\mathrm{Tr}\,\underline{f} = \boldsymbol{\nabla}\cdot\mathbf{f} = \mathrm{Tr}[\,\underline{f}\,] = \sum f_{kk} \tag{4.42}$$

The trace is an invariant of $\underline{f}$ in the sense that it is independent of any particular matrix representation for $\underline{f}$ as (4.42) shows.

Another invariant is obtained by differentiating the outermorphism

$$\underline{f}(\mathbf{x}_1\wedge\mathbf{x}_2) = (\underline{f}\mathbf{x}_1)\wedge(\underline{f}\mathbf{x}_2).$$

For the calculation, it will be convenient to simplify our notation with the abbreviations $\mathbf{f}_k \equiv \underline{f}\mathbf{x}_k$ and $\boldsymbol{\nabla}_k$ for the derivative with respect to $\mathbf{x}_k$. Differentiating with respect to the first variable, we obtain

$$\boldsymbol{\nabla}_1\cdot(\mathbf{f}_1\wedge\mathbf{f}_2) = (\boldsymbol{\nabla}_1\cdot\mathbf{f}_1)\mathbf{f}_2 - \mathbf{f}_2\cdot\boldsymbol{\nabla}_1\mathbf{f}_1 = (\boldsymbol{\nabla}\cdot\mathbf{f})\mathbf{f}_2 - \mathbf{f}_2^2,$$

where (4.39) was used to get

$$\mathbf{f}_2 \cdot \boldsymbol{\nabla}_1 f \mathbf{x}_1 = \underline{f}(\mathbf{f}_2) = \underline{f}(\underline{f}\mathbf{x}_2) = \underline{f}^2 \mathbf{x}_2 \equiv \mathbf{f}_2^2 \tag{4.43}$$

Now differentiating with respect to $\mathbf{x}_2$ we get the invariant

$$\boldsymbol{\nabla}_2 \cdot [\boldsymbol{\nabla}_1 \cdot (\mathbf{f}_1 \wedge \mathbf{f}_2)] = (\boldsymbol{\nabla} \cdot \mathbf{f}) \boldsymbol{\nabla}_2 \mathbf{f}_2 - \boldsymbol{\nabla}_2 \cdot \mathbf{f}_2^2$$

or,

$$\begin{aligned}(\boldsymbol{\nabla}_{\mathbf{b}} \wedge \boldsymbol{\nabla}_{\mathbf{a}}) \cdot \underline{f}(\mathbf{a} \wedge \mathbf{b}) &= (\boldsymbol{\nabla}_2 \wedge \boldsymbol{\nabla}_1) \cdot (\mathbf{f}_1 \wedge \mathbf{f}_2) \\ &= (\boldsymbol{\nabla} \cdot \mathbf{f})^2 - \boldsymbol{\nabla} \cdot \mathbf{f}^2 = (\mathrm{Tr}\ \underline{f})^2 - \mathrm{Tr}\ \underline{f}^2 \end{aligned} \tag{4.44}$$

We get a third invariant from

$$\begin{aligned}(\boldsymbol{\nabla}_{\mathbf{c}} \wedge \boldsymbol{\nabla}_{\mathbf{b}} \wedge \boldsymbol{\nabla}_{\mathbf{a}}) \underline{f}(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) &= \sum_i \sum_j \sum_k \boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j \wedge \boldsymbol{\sigma}_k \underline{f}(\boldsymbol{\sigma}_k \wedge \boldsymbol{\sigma}_j \wedge \boldsymbol{\sigma}_i) \\ &= 6\, \boldsymbol{\sigma}_3 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_1 \underline{f}(\boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_2 \wedge \boldsymbol{\sigma}_3) = 6 \det \underline{f}. \end{aligned} \tag{4.45}$$

Summarizing, we have the following list of principal invariants for any tensor $\underline{f}$.

$$\begin{aligned} \alpha_1(\underline{f}) &\equiv \boldsymbol{\nabla} \cdot \mathbf{f} = \mathrm{Tr}\ \underline{f}, \\ \alpha_2(\underline{f}) &\equiv \tfrac{1}{2}(\boldsymbol{\nabla}_{\mathbf{b}} \wedge \boldsymbol{\nabla}_{\mathbf{a}}) \cdot \underline{f}(\mathbf{a} \wedge \mathbf{b}) = \tfrac{1}{2}[(\mathrm{Tr}\ \underline{f})^2 - \mathrm{Tr}\ .\ \underline{f}^2], \\ \alpha_3(\underline{f}) &\equiv \tfrac{1}{6}(\boldsymbol{\nabla}_{\mathbf{c}} \wedge \boldsymbol{\nabla}_{\mathbf{b}} \wedge \boldsymbol{\nabla}_{\mathbf{a}}) \cdot \underline{f}(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) = \det\ \underline{f}. \end{aligned} \tag{4.46}$$

For a symmetric tensor $\underline{S}$ the principal invariants can be expressed in terms of its eigenvalues $\{\lambda_k\}$ by a simple calculation from the canonical form (4.32) or its matrix (4.34). The result is

$$\begin{aligned} \alpha_1(\underline{S}) &= \lambda_1 + \lambda_2 + \lambda_3 \\ \alpha_2(\underline{S}) &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ \alpha_3(\underline{S}) &= \lambda_1\lambda_2\lambda_3 \end{aligned} \tag{4.47}$$

The principal invariants appear in one of the most celebrated theorems of linear algebra, to which we now turn.

Cayley-Hamilton Theorem

Repeated application of a tensor $\underline{f}$ to a vector $\mathbf{a}$ generates a series of vectors $\mathbf{a}, \underline{f}\mathbf{a}, \underline{f}^2\mathbf{a}, \underline{f}^3\mathbf{a}, \ldots$. At the most three of these vectors can be linearly independent in $\mathcal{E}_3$. Therefore, it must be possible to express all higher powers of the tensor as linear combinations of the three lowest powers. The precise relation among tensor powers is given by the

Cayley-Hamilton Theorem: *Every tensor $\underline{f}$ on $\mathcal{E}_3$ satisfies the operator equation*

$$\underline{f}^3 - \alpha_1 \underline{f}^2 + \alpha_2 \underline{f} - \alpha_3 \underline{f}^0 = 0, \tag{4.48}$$

where $\underline{f}^0 = 1$ is the identity operator and the $\alpha_k = \alpha_k(\underline{f})$ are the principal invariants given by (4.45). It should be understood that this operator equation holds for the tensor operator on $\mathcal{E}_3$ but not for its outermorphisms.

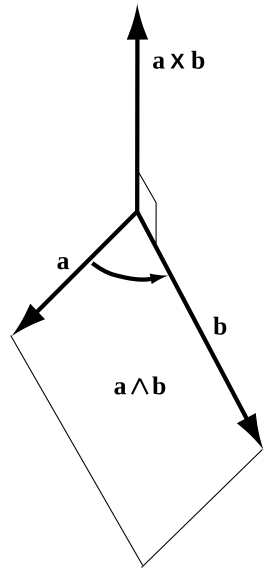

Fig. 2.2. The right-hand rule for the vector cross product.

If $\mathbf{e}$ is an eigenvector of $\underline{f}$ with eigenvalue λ then $\underline{f}^k\mathbf{e} = \lambda^k\mathbf{e}$, so operation on $\mathbf{e}$ with the Cayley-Hamilton equation (4.48) produces a cubic equation for the eigenvalue:

$$\lambda^3 - \alpha_1\lambda^2 + \alpha_2\lambda - \alpha_3 = 0. \tag{4.49}$$

This is called the *characteristic* (or *secular*) *equation* for $\underline{f}$. Being cubic, it has at least one real root. Therefore every tensor on $\mathcal{E}_3$ has at least one eigenvector.

We turn now to a proof of the Cayley-Hamilton equation, obtaining it from a differential identity applied to a linear transformation. We employ the abbreviated notation and results from the above discussion of principal invariants along with several algebraic tricks which the reader is invited to identify.

$$\begin{aligned}
6\mathbf{a}\alpha_3 &= \mathbf{a}\cdot(\boldsymbol{\nabla}_3\wedge\boldsymbol{\nabla}_2\wedge\boldsymbol{\nabla}_1)\mathbf{f}_1\wedge\mathbf{f}_2\wedge\mathbf{f}_3\\
&= \boldsymbol{\nabla}_2\wedge\boldsymbol{\nabla}_1\mathbf{f}_1\wedge\mathbf{f}_2\wedge\mathbf{f} - \boldsymbol{\nabla}_3\wedge\boldsymbol{\nabla}_1\mathbf{f}_1\wedge\mathbf{f}\wedge\mathbf{f}_3 + \boldsymbol{\nabla}_3\wedge\boldsymbol{\nabla}_2\mathbf{f}\wedge\mathbf{f}_2\wedge\mathbf{f}_3\\
&= 3\boldsymbol{\nabla}_2\wedge\boldsymbol{\nabla}_1\mathbf{f}_1\wedge\mathbf{f}_2\wedge\mathbf{f} = 3(\boldsymbol{\nabla}_2\wedge\boldsymbol{\nabla}_1)\cdot(\mathbf{f}_1\wedge\mathbf{f}_2\wedge\mathbf{f})\\
&= 3\{(\boldsymbol{\nabla}_2\wedge\boldsymbol{\nabla}_1)\cdot(\mathbf{f}_1\wedge\mathbf{f}_2)\mathbf{f} - (\boldsymbol{\nabla}_2\wedge\boldsymbol{\nabla}_1)\cdot(\mathbf{f}_1\wedge\mathbf{f})\mathbf{f}_2 + (\boldsymbol{\nabla}_2\wedge\boldsymbol{\nabla}_1)\cdot(\mathbf{f}_2\wedge\mathbf{f})\mathbf{f}_1\}\\
&= 6\{\alpha_2\mathbf{f} - (\boldsymbol{\nabla}_2\wedge\boldsymbol{\nabla}_1)\cdot(\mathbf{f}_1\wedge\mathbf{f})\mathbf{f}_2\}\\
&= 6\{\alpha_2\mathbf{f} - [(\boldsymbol{\nabla}\cdot\mathbf{f})\mathbf{f}\cdot\boldsymbol{\nabla}_2 - \mathbf{f}\cdot\boldsymbol{\nabla}_1\mathbf{f}_1\cdot\boldsymbol{\nabla}_2]\mathbf{f}_2\}\\
&= 6\{\alpha_2\mathbf{f} - \alpha_1\mathbf{f}^2 + \mathbf{f}^3\}.
\end{aligned}$$