

Multivector Functions

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In a previous paper [1], the fundamentals of differential and integral calculus on Euclidean n -space were expressed in terms of multivector algebra. The theory is used here to derive some powerful theorems which generalize well-known theorems of potential theory and the theory of functions of a complex variable. *Analytic multivector functions* on \mathcal{E}_n are defined and shown to be appropriate generalizations of analytic functions of a complex variable. Some of their basic properties are pointed out. These results have important applications to physics which will be discussed in detail elsewhere.

1. INTEGRAL OF THE GRADIENT OF A FUNCTION

A multivector function f defined on a region \mathcal{R} in \mathcal{E}_n is said to be *differentiable* on \mathcal{R} if its gradient $\nabla f(x)$ exists in some sense at each point x in \mathcal{R} . If f and g are differentiable on \mathcal{R} , then

$$\int_{\mathcal{R}} g \, dv \, \nabla f + (-1)^{n+1} \int_{\mathcal{R}} (g \overline{\nabla}) \, dv f = \int_{\partial \mathcal{R}} g \, da \, f. \quad (1.1)$$

This formula still holds if either f or g is a generalized function [2] (distribution), a fact which often simplifies integration. Here it is used to integrate ∇f .

Let ∇ represent the gradient operating at the point x and ∇' the gradient operating at the point x' . If $r = x - x'$, then

$$\nabla |r| = \frac{r}{|r|} = -\nabla' |r| \quad (1.2)$$

$$\nabla r = n = -\nabla' r. \quad (1.3)$$

It can be readily verified that the equation

$$g(r) \overline{\nabla} = \delta(r) \quad (1.4)$$

admits the particular solution

$$g(r) = \frac{1}{\Omega_n} \frac{r}{|r|^n}, \quad (1.5)$$

where

$$\Omega_n = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{n}{2})} \quad (1.6)$$

is the area of a unit sphere in \mathcal{E}_n . Moreover, g is the gradient of the scalar function

$$G(r) = \frac{|r|^{2-n}}{(2-n)\Omega_n} \quad \text{if} \quad n \neq 2 \quad (1.7a)$$

$$= \frac{1}{2\pi} \ln |r| \quad \text{if} \quad n = 2. \quad (1.7b)$$

Thus

$$g = \nabla G = -\nabla' G. \quad (1.8)$$

The function G is a particular solution of Laplace's equation for a "point source" in \mathcal{E}_n .

$$\nabla^2 G(r) = \delta(r). \quad (1.9)$$

Now substitute the function g just defined into (1.1). By virtue of (1.4), if x' is in \mathcal{R} , then

$$\int_{\mathcal{R}} (g \overline{\nabla}) dv f = \int_{\mathcal{R}} \delta(x - x') |dv| i f(x) = i f(x'), \quad (1.10)$$

where the symbol i denotes the unitary tangent (unit volume element) of \mathcal{R} . Since $i^{-1}i = 1$, (1.1) becomes

$$f(x') = \frac{(-1)^n}{\Omega_n i} \left\{ \int_{\mathcal{R}} \frac{r}{|r|^n} dv \nabla f - \int_{\partial \mathcal{R}} \frac{r}{|r|^n} da f \right\}. \quad (1.11)$$

This shows that a multivector function differentiable in a region \mathcal{R} of \mathcal{E}_n is uniquely determined by its gradient in \mathcal{R} and its value on the boundary of \mathcal{R} .

By using the fact that $ir = (-1)^{n-1}ri$ and defining the normal n to the boundary of the region \mathcal{R} by the equation

$$i^{-1}da = n |da|, \quad (1.12)$$

Eq. (1.11) can be written

$$f(x') = \frac{1}{\Omega_n} \left\{ - \int_{\mathcal{R}} |dv| \frac{r}{|r|^n} \nabla f + \int_{\partial \mathcal{R}} |da| \frac{r}{|r|^n} n f \right\}. \quad (1.13)$$

Equation (1.13) implies that every differentiable multivector function is the gradient of another function. For, by virtue of (1.8),

$$f(x') = \nabla' \varphi(x'), \quad (1.14)$$

where, if $n \neq 2$,

$$\begin{aligned} \varphi(x') &= \frac{1}{(n-2)\Omega_n} \left\{ \int_{\mathcal{R}} \frac{|dv|}{|r|^{n-2}} \nabla f - \int_{\partial \mathcal{R}} \frac{|da|}{|r|^{n-2}} n f + C \right\} \\ &= \frac{1}{(n-2)\Omega_n} \left\{ \int_{\mathcal{R}} \frac{|dv|}{|r|^{n-2}} \nabla^2 \varphi - \int_{\partial \mathcal{R}} \frac{|da|}{|r|^{n-2}} n \nabla \varphi + C \right\} \end{aligned} \quad (1.15)$$

and

$$\nabla' C(x') = 0. \quad (1.16)$$

The following choice of C eliminates the tangential derivative of φ in the second term of the right of (1.15):

$$C(x') = \int_{\partial \mathcal{R}} |da| \left(\nabla \frac{1}{|r|^{n-2}} \right) n \varphi. \quad (1.17)$$

So (1.15) becomes

$$\begin{aligned} \varphi(x') = \frac{1}{(n-2)\Omega_n} \left\{ \int_{\mathcal{R}} \frac{|dv|}{|r|^{n-2}} \nabla^2 \varphi \right. \\ \left. + \int_{\partial\mathcal{R}} |da| \left[\left(n \cdot \nabla \frac{1}{|r|^{n-2}} \right) \varphi - \frac{1}{|r|^{n-2}} n \cdot \nabla \varphi \right] \right\}, \end{aligned} \quad (1.18)$$

a familiar formula from potential theory, although, of course, φ is a multivector field here.

2. RELATION TO COMPLEX VARIABLE THEORY

A vector function on \mathcal{E}_2 is equivalent to a complex function of a complex variable. Let f be such a function. Taking $g = 1$ in \mathcal{R} , Eq. (1.1) can be written

$$\int_{\mathcal{R}} dv \nabla f = \int_{\partial\mathcal{R}} dx f. \quad (2.1)$$

So, if

$$\nabla f = 0 \quad (2.2)$$

in \mathcal{R} , then

$$\int_{\partial\mathcal{R}} dx f(x) = 0. \quad (2.3)$$

A vector function satisfying (2.2) and (2.3) is equivalent to an analytic function of a complex variable. Equation (2.2) corresponds to the Cauchy-Riemann equations, and (2.3) corresponds to Cauchy's theorem. If f is analytic except at poles x_k in \mathcal{R} , then

$$\nabla f(x) = 2\pi \sum_k R_k \delta(x - x_k). \quad (2.4)$$

The "residue theorem" is, obtained by substituting this into (2.1):

$$\int_{\partial\mathcal{R}} dx f = 2\pi i \sum_k R_k, \quad (2.5)$$

where, of course, R_k is the residue of f at x_k . For the case $n = 2$, Equation (1.11) can be written

$$f(x') = \frac{1}{2\pi} \int_{\mathcal{R}} |dv| \frac{1}{x - x'} \nabla f(x) - \frac{1}{2\pi i} \int_{\partial\mathcal{R}} \frac{1}{x - x'} dx f(x). \quad (2.6)$$

For the special case $\nabla f = 0$, (2.4) reduces to Cauchy's integral formula.

It should be clear by now that all of complex variable theory can be readily formulated in the language of multivector calculus.

3. ANALYTIC MULTIVECTOR FUNCTIONS

When formulated in terms of multivector calculus as in the last section, the notion of analytic function in complex variable theory admits to an obvious generalization. Moreover, many theorems and even proofs are essentially unaltered in the process.

A multivector function f is here called *analytic* in a region \mathcal{R} of \mathcal{E}_n if

$$\nabla f(x) = 0 \tag{3.1}$$

for x in \mathcal{R} . By the fundamental theorem, this implies the generalization of Cauchy's theorem:

$$\int_{\partial\mathcal{R}} da f = 0 \tag{3.2}$$

or, since $i^{-1}da = |da|n$ where n is the outward normal,

$$\int_{\partial\mathcal{R}} |da| n f = 0. \tag{3.3}$$

If f is analytic in \mathcal{R} except at poles x_k , then

$$\nabla f = \Omega_n \sum_k R_k \delta(x - x_k). \tag{3.4}$$

This becomes the *residue theorem* when written in integral form:

$$\int_{\partial\mathcal{R}} |da| n f = \Omega_n \sum_k R_k. \tag{3.5}$$

Equation (1.13) yields the generalization of Cauchy's integral formula:

$$f(x') = \frac{1}{\Omega_n} \int_{\partial\mathcal{R}} |da| \frac{r}{|r|^n} n f. \tag{3.6}$$

This formula reveals the fundamental property of analytic functions: If f is analytic in \mathcal{R} , its value at every point in \mathcal{R} is *uniquely determined* by its values on $\partial\mathcal{R}$. From (3.6) it follows that $f(x')$ can be expanded in a power series. But this is not the crucial property of analytic functions; many nonanalytic functions can also be expanded in a power series. Nor is the existence of the complex derivative d/dz a crucial property of analytic functions in \mathcal{E}_2 . The existence of d/dz is an expression of the fact that the derivative with respect to a vector is independent of its direction; it follows directly from $\nabla f = 0$ and the limitation to two dimensions. This feature is peculiar to two dimensions and cannot be generalized. But the essential properties of analytic functions do not depend on the existence of d/dz ; they are contained in Cauchy's integral formula, which generalizes quite nicely to (3.6). Therefore the use of d/dz should be eschewed.

From (3.6) many theorems follow which are straightforward generalizations of theorems in complex variable theory: Liouville's theorem, the mean value theorem, the maximum

modulus principle, etc. The proofs are so similar to well-known proofs based on Cauchy's integral formula that they need not be given here.

Other properties of analytic functions can be derived with the help of the fundamental theorem of calculus. For example, if f is analytic in a region \mathcal{R} of \mathcal{E}_n , and if \mathcal{V} is a smooth oriented surface in \mathcal{R} with tangent v , then

$$\int_{\mathcal{V}} dv \nabla_n f = - \int_{\partial\mathcal{V}} dv f, \quad (3.7)$$

where $\nabla_n f = \nabla_{iv} f$ is the *normal derivative* of f on \mathcal{V} . Using the fact that

$$\nabla = \nabla_v + \nabla_{iv},$$

the theorem follows immediately from the identity

$$\begin{aligned} \int_{\mathcal{V}} dv \nabla f &= \int_{\mathcal{V}} dv \nabla_v f + \int_{\mathcal{V}} dv \nabla_n f \\ &= \int_{\partial\mathcal{V}} dv f + \int_{\mathcal{V}} dv \nabla_n f. \end{aligned}$$

An obvious corollary of (3.7): the integral of the normal derivative of an analytic function over any closed surface vanishes.

The notion of analytic function can be profitably generalized beyond what has been discussed so far. Let \mathcal{V} be an oriented surface with tangent v . A function f is said to be *analytic* on \mathcal{V} if $\nabla_v f = O$ on \mathcal{V} . With this definition the theory of analytic functions can be developed on curved surfaces in much the same way as it is developed on flat surfaces in this paper.

The theory of analytic functions of several complex variables can be incorporated into the theory of multivector functions in the following way. Consider \mathcal{E}_{2n} as a ‘‘Cartesian product’’ of n planes, and let i_k be the tangent 2-vector to the k th plane. Note that i_k is a ‘‘unit imaginary’’ for the k th plane:

$$i_k^2 = -1. \quad (3.8)$$

The tangents commute with one another:

$$i_j i_k = i_k i_j. \quad (3.9)$$

The pseudoscalar i of \mathcal{E}_{2n} can be factored into the product

$$i = i_1 i_2 \cdots i_n. \quad (3.10)$$

The coordinate x of a point in \mathcal{E}_{2n} can be expressed as the sum

$$x = x_1 + x_2 + \cdots + x_n, \quad (3.11)$$

where, for $k = 1, 2, \dots, n$,

$$x_k = (i_k)^{-1} i_k \cdot x = -i_k i_k \cdot x. \quad (3.12)$$

A function $f(x)$ on \mathcal{E}_{2n} can be regarded as a function $f(x_1, x_2, \dots, x_n)$ of the n variables defined by (3.1). If f is analytic in a region \mathcal{R}_k of the k th plane, then

$$\nabla_{i_k} f = 0 \quad (3.13)$$

when x_k is in \mathcal{R}_k . Let $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \dots \otimes \mathcal{R}_n$ be the Cartesian product of the regions \mathcal{R}_k . If $x' = x'_1 + x'_2 + \dots + x'_n$ is in \mathcal{R} , then by n applications of (2.6),

$$\begin{aligned} f(x') &= \frac{i}{(2\pi)^n} \oint_{\partial\mathcal{R}_1} \frac{1}{x_1 - x'_1} dx_1 \\ &\quad \cdot \oint_{\partial\mathcal{R}_2} \frac{1}{x_2 - x'_2} dx_2 \cdots \oint_{\partial\mathcal{R}_n} \frac{1}{x_n - x'_n} dx_n f(x). \end{aligned} \quad (3.14)$$

This is equivalent to Cauchy's integral formula for an analytic function of n complex variables. The order of integrations in (3.14) is immaterial because of (3.9). Equation (3.14) should be compared with (3.6), which for the present case can be written

$$f(x') = \frac{(n-1)!}{2\pi^n} \int_{\partial\mathcal{R}} |da| \frac{(x-x')}{|x-x'|^{2n}} n f(x). \quad (3.15)$$

This formula applies because, by (3.1),

$$\nabla_{i_1} f(x') + \nabla_{i_2} f(x') + \dots + \nabla_{i_n} f(x') = \nabla f(x') = 0. \quad (3.16)$$

REFERENCES

- [1] D. Hestenes. *J. Math. Anal. and Appl.* **24** (1968), 313–325.
- [2] I. M. Gel'fand and G. E. Shilov. "Generalized Functions," Vol. 1. Academic Press, New York, 1964.