In: *J. Math. Phys.*, **34** (8) August 1993 pp. 3642–3669.

Lie groups as spin groups

C. Doran
Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, United Kingdom

D. Hestenes
Department of Physics and Astronomy, Arizona State University, Tempe, Arizona 85287

F. Sommen[a)] and N. Van Acker
Department of Mathematical Analysis, University of Gent, Galgaan 2, 9000 Gent, Belgium

Abstract. It is shown that every Lie algebra can be represented as a bivector algebra; hence *every Lie group can be represented as a spin group*. Thus, the computational power of geometric algebra is available to simplify the analysis and applications of Lie groups and Lie algebras. The spin version of the general linear group is thoroughly analyzed, and an invariant method for constructing real spin representations of other classical groups is developed. Moreover, it is demonstrated that *every linear transformation can be represented as a monomial of vectors* in geometric algebra.

1. INTRODUCTION

The *fermion algebra* (generated by fermion creation and annihilation operators) has been widely applied to group theory[1] and many other mathematical problems with no essential relation to fermions. Yet few physicists and mathematicians realize that this mathematical system can be regarded as a *universal geometric algebra* applicable to every mathematical domain with geometric structure. As part of a broad program to make this claim to universality an accomplished fact,[2–5] we show here that this geometric algebra is a viable, if not superior, alternative to matrix algebra for characterizing Lie groups and Lie algebras. As a by-product with even wider ramifications, we show that it is a powerful means for characterizing and manipulating linear transformations in general. We see it as consolidating various insights of many scientists into a coherent mathematical system.

One of the barriers to establishing a *universal geometric algebra*[2] has been a lack of general agreement among mathematicians on the relative status of Grassmann algebra (GA) and Clifford algebra (CA). The disputants can be divided into two camps: call them the "Grassmannians" and the "Cliffordians." *Grassmannian* argue that GA is more fundamental than CA, because it makes no assumptions about a metric on the vector space that generates it. On the contrary, *Cliffordians* argue that CA is more fundamental than GA, because it contains GA as a subalgebra.[6]

[a)] Senior reserch asociate, N.F.W.O., Belgium.

As is usual in scientific disputes, both sides have a valid point to make, but are reluctant (if not unable) to appreciate the viewpoint of the opposition. The issue here is not "Which side is right?" but rather "How should mathematical knowledge be organized?" It is a problem of mathematical design:[2,3] How to design a geometric algebra of maximal scope, coherence, flexibility, and simplicity! We think the solution has been around for a long time, but it has not been widely accepted primarily because the problem it solves has not been recognized.

Our objective in this article is to formulate the universal geometric algebra in a *flexible* way which satisfies the demands of individuals in both the Grassmanian and Cliffordian camps. In the interest of mathematical harmony let us call this construct the *mother algebra*. The mother algebra embraces an enormous range of mathematical structures in both physics and pure mathematics. Here we review the essential formalism and rationale for adopting the mother algebra as a universal foundation for linear algebra as well as for the theory of Lie groups and Lie algebras. This is an elaboration of the approach originally developed in Ref. 4, so for the most part we adopt the same notation, and we refer there for many details.

11. RECONCILING GRASSMANN AND CLIFFORD

There is evidence that Grassmann himself became Cliffordian in his last years. In one of his last publications,[7] ironically dismissed as inconsequential by historians, he took the momentous step of adding, for vectors a and b, his *inner product* $a \cdot b$ to his *outer product* $a \wedge b$ to define a new kind of product ab which he called the *central product*. Thus, he wrote

$$ab = a \cdot b + a \wedge b\,, \tag{2.1}$$

though he employed different notations for the inner and outer products. From the established properties of Grassmann's inner and outer products it can be shown that his *central product* has all the properties of multiplication in Clifford algebra.[5] In a certain sense, therefore, Clifford algebra is inherent in Grassmann's algebra. Moreover, Grassmann published this a year before Clifford.[8] To be sure, Grassmann's intent[7] was only to show that Hamilton's quaternions (a particular Clifford algebra) were inherent in his algebra, but he undoubtedly recognized more general possibilities. Though the addition in Eq. (2.1) is a nontrivial extension of Grassmann's original system, Grassmann plays it down and avoids giving Hamilton credit for inspiring him to do it, perhaps because he was bitter about the lack of recognition for his own work. In Grassmann's defense it can be said that the generalization is straightforward. Clifford was led to the same algebraic structure by asking the same question: How can one combine quaternions and Grassmann's algebra into a single mathematical system? Grassmann expressed his view in these words:[7] "Since extension theory makes only *one* arbitrary assumption, that is that there exist magnitudes that can be numerically derived from more than one unit, and proceeds from this in a completely objective way, all expressions that are numerically derivable from a number of independent units, and in particular the Hamiltonian quaternions, have their definition in extension theory and only find their scientific foundation in it. This was previously not recognized," (translation by L. Kannenberg). No doubt Grassmann would use the same argument to say that Clifford algebra as we know it today is embraced by his extension theory. And Clifford would probably agree, as he referred to his own work[8] as an "application" of Grassmann's algebra. The point of all this is that Grassmann, Hamilton, and Clifford, as well as Lifschitz and many others since have contributed to the development of a single mathematical system which cannot be justifiably associated with the name of a single individual. Today, it is more evident than ever that Clifford's original term *geometric algebra* is the most appropriate name for that system, though the term "Clifford algebra" is more common in the literature.

To reconcile the contemporary views of Grassmann and Clifford algebras, we begin with a standard definition of the *Grassmann algebra* $\Lambda_n = \Lambda(\mathcal{V}^n)$ of an n-dimensional real vector space $\mathcal{V}^n$. This associative algebra is generated from $\mathcal{V}^n$ by Grassmann's *outer product* under the assumption that the product of several vectors vanishes if and only if the vectors are linearly dependent. With the notation in Eq. (2.1) for the outer product, the outer product

$$v_1 \wedge v_2 \wedge \ldots \wedge v_k \tag{2.2}$$

of k linearly independent vectors is called a *k-blade*, and a linear combination of k-blades is called a *k-vector*. The set of all k-vectors is a linear space

$$\Lambda_n^k = \Lambda^k(\mathcal{V}^n)\,, \tag{2.3}$$

with dimension given by the binomial coefficient $\binom{n}{k}$. With the notations $\Lambda_n^1 = \mathcal{V}^n$ and $\Lambda_n^0 = \Re$ for the real scalars, the entire Grassmann algebra can be expressed as a 2^n-dimensional linear space

$$\Lambda_n = \sum_{k=0}^{n} \Lambda_n^k\,. \tag{2.4}$$

This completes our description of Grassmann's "exterior algebra," but ore mathematical structure is needed for applications. Standard practice is to introduce this structure by defining the space of linear forms on Λ_n. However, we think there is a better procedure which is closer to Grassmann's original approach.

We introduce an n-dimensional vector space $\mathcal{V}^{n*}$ dual to $\mathcal{V}^n$ with "duality" defined by the following condition: If $\{w_i\}$ is a basis for $\mathcal{V}^n$, then there is a basis $\{w_i^*\}$ for $\mathcal{V}^{n*}$ defining unique scalar-valued mappings denoted by

$$\mathcal{V}_i^{n*} \cdot \mathcal{V}_j^n = \tfrac{1}{2}\delta_{i,j}\,, \qquad \text{for } i,j = 1,2,\ldots,n\,. \tag{2.5}$$

The dual space generates its own Grassmann algebra

$$\Lambda^*(\mathcal{V}^n) = \Lambda_n^* = \sum_{k=0}^{n} \Lambda_n^{k\,*}\,, \tag{2.6}$$

The inner product (2.5) can be extended to a product between k-vectors, so that each k-vector in $\mathcal{V}^{n*}$ determines a unique k-form on $\mathcal{V}^n$, that is, a linear mapping of Λ_n^k into the scalars. In other words, $\Lambda_n^{k\,*}$ can be regarded as the linear space of all k-forms.

This much is equivalent to the standard theory of linear forms, though Eq. (2.5) is not a standard notation defining one-forms. The notation has been adopted here so Eq. (2.5) can be interpreted as Grassmann's inner product, and Λ_n and Λ_n^* can be imbedded in a single geometric algebra with a single *central product* defined by Eq. (2.1). One way to do that is by identifying Λ_n with Λ_n^* but then Eq. (2.5) defines a nondegenerate metric on $\mathcal{V}^n$, and Grassmannians claim that that is a loss in generality. Cliffordians counter that the loss is illusory, for the interpretation of Eq. (2.5) as a metric tensor is not necessary if it is not wanted; with one variable held fixed, it can equally well be interpreted as a "contraction" defining a linear form. Be that as it may, there really is an advantage to keeping Λ_n and Λ_n^* distinct, in fact maximally distinct, as we see next. We turn Λ_n and Λ_n^* into geometric algebras by defining the inner products

$$w_i \cdot w_j = 0 \quad \text{and} \quad w_i^* \cdot w_j^* = 0\,, \tag{2.7a}$$

so Eq. (2.1) gives

$$w_i \wedge w_j = w_i w_j = -w_j w_i \quad \text{and} \quad w_i^* \wedge w_j^* = w_i^* w_j^* = -w_j^* w_i^*\,. \tag{2.7b}$$

Also we assume that the w_i and the w_i^* are linearly independent vectors spanning a $2n$-dimensional vector space

$$\mathcal{R}^{n,n} = \mathcal{V}^n \otimes \mathcal{V}^{n*}\,, \tag{2.8}$$

with an inner product defined by Eqs. (2.5) and (2.7a). This generates a 2^{2n}-dimensional *geometric algebra* which we denote by

$$\mathcal{R}_{n,n} = \mathcal{G}(\mathcal{R}^{n,n}) = \sum_{k=0}^{n} \mathcal{R}_{n,n}^k\,, \tag{2.9}$$

with k-vector subspaces $\mathcal{R}_{n,n}^k = \mathcal{G}^k(\mathcal{R}^{n,n}) = \mathcal{G}^k(\mathcal{R}_{n,n})$. Anticipating the conclusion that it will prove to be an ideal tool for characterizing linear and multilinear functions on an n-dimensional vector space, let us refer to $\mathcal{R}_{n,n}$ as the *mother algebra*.

III. STRUCTURE OF THE MOTHER ALGEBRA

Before continuing our study of the mother algebra, we review some definitions and results from Ref. 4 which enable algebraic manipulations in any geometric algebra without referring to a basis.

A generic element M of the algebra is called a *multivector*, and it can be decomposed into a sum of its k-vector parts, that is, parts $\langle M \rangle_k$ of *grade* k, thus,

$$M = \langle M \rangle_0 + \langle M \rangle_1 + \langle M \rangle_2 + \cdots\,. \tag{3.1}$$

The geometric product is denoted by MN and the "main antiautomorphism" (or *reversion*) is defined and denoted by

$$(MN)^\dagger = N^\dagger M^\dagger\,, \tag{3.2a}$$

$$\langle M \rangle_1^\dagger = \langle M \rangle_1\,. \tag{3.2b}$$

The geometric product AB of an r-vector $A = \langle A \rangle_r$ with an s-vector $B = \langle B \rangle_s$ has the decomposition

$$AB = \langle AB \rangle_{r+s} + \langle AB \rangle_{r+s-2} + \cdots + \langle AB \rangle_{|r-s|}\,. \tag{3.3}$$

Grassmann's *inner product* $A \cdot B$ and *outer product* $A \wedge B$ can be defined in terms of the geometric product by

$$A \cdot B \;=\; \langle AB \rangle_{|\, r-s\, |} \,, \tag{3.4}$$

$$A \wedge B \;=\; \langle AB \rangle_{r+s} \,, \tag{3.5}$$

For vectors $a \;=\; \langle a \rangle_1$ and $b \;=\; \langle b \rangle_1$, Eq. (3.3) reduces to

$$ab = a \cdot b + a \wedge b \,, \tag{3.6}$$

with

$$a \cdot b = \tfrac{1}{2}(ab + ba) = b \cdot a \tag{3.7}$$

$$a \wedge b = \tfrac{1}{2}(ab - ba) = -b \wedge a \,. \tag{3.8}$$

As Eq. (3.6) is identical with Eq. (2.1), *we can identify the geometric product with Grassmann's central product.* However, the logic is reversed here, and the inner and outer products are derived from the central product, as in Eqs. (3.7) and (3.8) or, more generally, in Eqs. (3.4) and (3.5).

The defintions of inner and outer products greatly facilitate manipulations without specifying a basis in the algebra, and for this purpose, a system of identities interrelating inner and outer products has been developed in Chap. 1 of Ref. 4. As shown, these indentities sufice for the developing of the entire theory of determinants. To counter the mistaken impression that use of the inner product limits the theory to metric spaces, we point out that it embraces the standard theory of determinants on the Grassmann algebra Λ_n simply by imbedding Λ_n in the mother algebra $\mathcal{R}_{n,n}$. Thus, every determinant of rank r can be represented by

$$A \cdot B^* \;=\; \langle AB^* \rangle_0 \,, \tag{3.9}$$

where $A = \langle A \rangle_r$ is in Λ_n^r and $B^* = \langle B^* \rangle_s$ is in $\Lambda_n^{s\,*}$. The *Laplace expansion* and many other classical theorems of determinant theory are derived in Ref. 4, Chap. 1. For $a = \langle a \rangle_1$ and $B = \langle B \rangle_s$, Eq. (3.3) generalizes Eq. (3.6) to

$$aB = a \cdot B + a \wedge B \,. \tag{3.10}$$

For a *bivector* (or two-vector) $A = \langle A \rangle_2$, Eq. (3.3) yields

$$AB = A \cdot B + A \times B + A \wedge B \,. \tag{3.11}$$

where $A \times B$ is the *commutator product*, defined by

$$A \times B = \tfrac{1}{2}(AB - BA) \,. \tag{3.12}$$

This product is a "derivation" on the algebra, as expressed by

$$A \times (BC) = (A \times B)C + B(A \times C) \,. \tag{3.13}$$

This implies the *Jacobi identity*

$$A \times (B \times C) = (A \times B) \times C + B \times (A \times C) \,. \tag{3.14}$$

For a bivector $A = \langle A \rangle_2$, the commutator product is also "grade-preserving," that is, for any multivector M

$$A \times \langle M \rangle_r \;=\; \langle A \times M \rangle_r \,. \tag{3.15}$$

It follows that the space of bivectors is closed under the commutator product, so it forms a Lie Algebra (called a *bivector algebra*). It was conjectured in Chap. 8 of Ref. 4 that *every Lie algebra is isomorphic to a bivector algebra*. We shall see how to prove that in Sec. IV.

Returning to the study $\mathcal{R}_{n,n}$, we first examine the properties of alternative bases for the generating vector space $\mathcal{R}^{n,n}$, According to Eq. (2.5), the basis $\{w_i, w_i^*\}$ consists entirely of null uectors. Nevertheless, we can construct from these an orthonormal basis $\{e_i, \overline{e}_i\}$ defined by

$$e_i = w_i + w_i^* \,, \tag{3.16a}$$

$$\overline{e}_i = w_i - w_i^* \,. \tag{3.16b}$$

From Eqs. (2.5) and (2.7) it follows that

$$e_i \cdot e_j = \delta_{ij}, \quad e_i \cdot \overline{e}_j = 0, \quad \overline{e}_i \cdot \overline{e}_j = -\delta_{ij} \,. \tag{3.17}$$

The basis $\{e_i\}$ spans a real *Euclidean* vector space $\mathcal{R}^n$ while $\{\overline{e}_i\}$ spans an *anti-Euclidean* space $\overline{\mathcal{R}}^n$, Therefore, as an alternative to Eq. (2.8), $\mathcal{R}^{n,n}$ admits the decomposition where

$$\mathcal{R}^{n,n} = \mathcal{R}^n \otimes \overline{\mathcal{R}}^n \,. \tag{3.18}$$

From the basis $\{e_i, \overline{e}_i\}$ we can construct $(p+q)$-blades

$$E_{p,q} = E_p \overline{E}_q^\dagger = E_p \wedge \overline{E}_q^\dagger \,, \tag{3.19a}$$

where

$$E_p = e_1 e_2 \ldots e_p = E_{p,0} \,, \tag{3.19b}$$

$$\overline{E}_q = \overline{e}_1 \overline{e}_2 \ldots \overline{e}_p = E_{0,p} \,. \tag{3.19c}$$

Each blade determines a projection $\underline{E}_{p,q}$ of $\mathcal{R}^{n,n}$ into a $(p+q)$-dimensional subspace $\mathcal{R}^{p,q}$ defined by

$$\underline{E}_{p,q}(a) = (a \cdot E_{p,q}) E_{p,q}^{-1} = \tfrac{1}{2}\left[a - (-1)^{p+q} E_{p,q} a E_{p,q}^{-1}\right] . \tag{3.20}$$

The vector a resides in $\mathcal{R}^{p,q}$ if and only if

$$a \wedge E_{p,q} = 0 = a E_{p,q} + (-1)^{p+q} E_{p,q} a \,. \tag{3.21}$$

Incidentally, we use an *underbar* to distinguish linear operators from elements of the algebra. This notation has the advantage of allowing us to designate the operator by a multivector which determines it, as in Eq. (3.20), where the operator $\underline{E}_{p,q}$ is determined by the blade $E_{p,q}$. Reference 4 develops many properties and applications of projection operators like Eq. (3.20). For $p + q = n$, the blade $E_{p,q}$ determines a split of $\mathcal{R}^{n,n}$ into orthogonal subspaces with *complementary signature*, as expressed by

$$\mathcal{R}^{n,n} = \mathcal{R}^{p,q} \otimes \overline{\mathcal{R}}^{p,q} \,. \tag{3.22}$$

This generalizes Eq. (3.18), and Eq. (3.20) shows how a similar split is determined by every invertible n-blade in $\mathcal{R}^n_{n,n}$ without referring to any basis vectors. For the case $q = 0$, Eq. (3.20) can be written

$$\underline{E}_n(a) = \tfrac{1}{2}(a + a^*)\,, \tag{3.23}$$

where a^* is defined by

$$a^* = (-1)^{n+1} E_n a E_n^{-1}\,. \tag{3.24}$$

It follows immediately that $e_i^* = e_i$ and $(\overline{e}_i)^* = -\overline{e}_i$. Comparison with Eqs. (3.16a) and (3.16b) shows that w_i^* can indeed be obtained from w_i by applying Eq. (3.24), so the notations are consistent.

The split (2.8) of $\mathcal{R}^{n,n}$ into subspaces of null vectors cannot be obtained in the same way as the split (3.22), because the Grassmann algebra Λ_n does not contain any invertible n vectors. To describe such a split in an invariant way we need a new concept.

Let K be any bivector in $\mathcal{R}^2_{n,n}$ which can be expressed as a sum of n distinct commuting blades K_i with unit square, thus

$$K = \sum_{i=0}^{n} K_i\,, \tag{3.25}$$

where

$$K_i \times K_j = 0 \quad \text{and} \quad K_i^2 = 1\,. \tag{3.26}$$

For given k and $n \geq 2$ the decomposition of K into blades is unique if and only if distinct blades have different magnitudes, as shown in Secs. III or IV of Ref. 4. The bivector K determines an automorphism of $\mathcal{R}^{n,n}$

$$\underline{K} : a \to \bar{a} = \underline{K}a = a \times K = a \cdot K\,. \tag{3.27}$$

This maps each vector a into a vector $\bar{a}$ which we call *the complement of a (with respect to K)*.

It is readily verified that $\underline{K}\bar{a} = \underline{K}^2 a = a$, or, as an operator equation,

$$\underline{K}^2 = \underline{1}\,. \tag{3.28}$$

Thus, $\underline{K}$ is an involution. Furthermore,

$$a \cdot \bar{a} = 0\,, \tag{3.29}$$

and the vectors

$$a_{\pm} = a \pm \bar{a} = a \pm a \cdot K \tag{3.30}$$

are null vectors. In fact, the sets $\{a_+\}$ and $\{a_-\}$ of all such vectors are dual n-dimensional vector spaces, so K determines the desired null space decomposition of the form (2.8) without referring to a vector basis.

From the basis $\{e_i, \overline{e}_i\}$ a suitable K can be constructed by taking

$$K_i = e_i \overline{e}_i = e_i \wedge \overline{e}_i\,. \tag{3.31}$$

Then, indeed,

$$\underline{K}e_i = e_i \times K = e_i \cdot K = \overline{e}_i\,, \tag{3.32a}$$

and, in accordance with Eq. (3.28),

$$\underline{K}\bar{e}_i = e_i \,. \tag{3.32b}$$

Therefore, e_i and $\bar{e}_i$ are *complementary pairs*, as the overbar notation was chosen to indicate. Now it is evident that $\underline{K}$ determines a unique correspondence between the complementary spaces $\mathcal{R}^{p,q}$ and $\bar{\mathcal{R}}^{p,q}$.

From Eqs. (3.16) or (3.30) it is easily seen that the null basis $\{w_i, w_i^*\}$ consists of $\underline{K}$ eigenvectors with

$$\underline{K}w_i = w_i \times K = w_i \,, \tag{3.33a}$$

$$\underline{K}w_i^* = w_i^* \times K = -w_i^* \,. \tag{3.33b}$$

The basis $\{w_i, w_i^*\}$ is called a *Witt basis* in the theory of quadratic forms. The conventional approach to quadratic forms, as elegantly expounded in Ref. 9, for example, laboriously establishes many theorems before arriving after a long detour at the concept of Clifford algebra as the algebra of a quadratic form. Even then the significance of the mother algebra as a covering algebra for quadratic forms of every possible signature and degeneracy is not recognized. We submit that the theory can be greatly simplified and unified by introducing the mother algebra and establishing its properties at the outset. The standard theorems about bilinear and quadratic forms can then be established more simply and directly from these properties. Moreover, form theory is thereby automatically related to the vast range of other applications of the mother algebra and its subalgebras. This will be evident in our treatment of group theory in subsequent sections.

The mother algebra $\mathcal{R}_{n,n}$ is, of course, a subalgebra of the infinite dimensional algebra $\mathcal{R}_{\infty,\infty}$, which might be called the *grandmother algebra* or "Eve". Reference 4 contends that "Eve" should be regarded as *the universal geometric algebra* and adopted as the arena for developing a coordinate-free formulation of manifold theory. Eve has already been employed in physics as the ∞-dimensional algebra of fermion creation and annihilation operators. Indeed, using Eq. (3.7) to rewrite Eq. (2.5), we obtain

$$w_i w_j^* + w_j w_i^* = \delta_{ij} \,. \tag{3.34}$$

This will be recognized as the fundamental equation for fermion operators (See Ref. 10, for example). However, in this general mathematical context the anticommutivity of "fermion operators" expressed by Eq. (2.7b) is not an expression of the Pauli principle as it is in quantum field theory; it is merely an expression of linear independence.

IV. THE GENERAL LINEAR GROUP AS A SPIN GROUP

There are many kinds of *linear functions*, but those mapping vectors to vectors are especially significant, so we reserve the term *linear transformation* to refer to them. Moreover, adopting the perspective of geometric algebra, we associate with every vector space a geometric algebra generated by the geometric product. In other words, along with scalar multiplication and vector addition, we regard the geometric product as a defining property of the vector concept. One advantage of this perspective is that geometric algebra contains all the apparatus needed to characterize and analyze linear transformations. In fact, we shall prove that all linear transformations can be represented as geometric products. Let f

be a linear transformation defined on a given vector space. The characterization of $\underline{f}$ is facilitated by its *outermorphism*,[3,4] a grade-preserving extension of $\underline{f}$ to the entire geometric algebra, which is defined, for vectors $a, b, \ldots$, by

$$\underline{f}(a \wedge b \wedge \cdots) = (\underline{f}a) \wedge (\underline{f}a) \wedge \cdots . \tag{4.1}$$

The outermorphism derives its name from the fact that it preserves the outer product. It describes the essential mathematical structure underlying the concept of determinant. In fact, if P is a pseudoscalar for the vector space on which $\underline{f}$ is defined, the determinant of $\underline{f}$ is defined by the "eigenblade equation"

$$\underline{f}(P) = (\det \underline{f})P . \tag{4.2}$$

We are concerned here with linear transformations on $\mathcal{R}_{n,n}$ and its subspaces, especially orthogonal transformations. An orthogonal transformation $\underline{R}$ is defined by the condition that it leaves the inner product invariant, that is,

$$(\underline{R}a) \cdot (\underline{R}b) = a \cdot b . \tag{4.3}$$

It s called a *rotation* if $\det \underline{R} = 1$, that is, if

$$\underline{R}(E_{n,n}) = E_{n,n} , \tag{4.4}$$

where, as defined by Eq. (3.19a), $E_{n,n} = E_n \overline{E}_n^\dagger$ is the unit pseudoscalar for $\mathcal{R}_{n,n}$. The rotations form a group called the *special orthogonal group* $\mathrm{SO}(n, n)$.

Geometrc algebra makes it possible to express every rotation in the canonical form

$$\underline{R}a = RaR^\dagger , \tag{4.5}$$

where R is an *even* multivector (called a *rotor*) satisfying

$$RR^\dagger = 1 . \tag{4.6}$$

The rotors form a multiplicative group called the *spin group or spin representation* of $\mathrm{SO}(n, n)$, and it is denoted by $\mathrm{Spin}(n, n)$. $\mathrm{Spin}(n, n)$ is said to be a *double covering* of $\mathrm{SO}(n, n)$, since Eq. (4.5) shows that both $\pm R$ correspond to the same $\underline{R}$.

It follows from Eq. (4.6) that the inverse $R^{-1} = R^\dagger$ of the rotation (4.5) is given by

$$\underline{R}^\dagger a = R^\dagger a R . \tag{4.7}$$

This implies that

$$a \cdot (\underline{R}b) = \langle\, aRbR^\dagger \,\rangle_0 = \langle\, bR^\dagger aR \,\rangle_0 = b \cdot (\underline{R}^\dagger a) , \tag{4.8}$$

where the fact that $\langle\, ABC \,\rangle_0 = \langle\, BCA \,\rangle_0$ has been used. In other words, the adjoint of a rotation is equal to its inverse. It is worth remarking that sometimes the spin group is defined by writing R^{-1} instead of $R^\dagger$ in Eq. (4.5). Then it contains additional elements [the K_i in Eq. (3.31)] which are not continuously connected to the identity. We exclude those elements from the group, though it will be seen that they belong to the Lie algebra of the group.

It can be shown that every rotor can be expressed in the exponential form

$$R = \pm e^{\frac{1}{2}B}, \quad \text{with} \quad R^\dagger = \pm e^{-\frac{1}{2}B}, \tag{4.9}$$

where B is a bivector called the *generator* of R or $\underline{R}$, and the minus sign can usually be eliminated by a change in the definition of B. Thus, every bivector determines a unique rotation. The bivector generators of a spin or rotation group form a Lie algebra under the commutator product. This reduces the description of Lie groups to Lie algebras. The Lie algebra of $\mathrm{SO}(n,n)$ and $\mathrm{Spin}(n,n)$ is designated by the lower case notation $\mathrm{so}(n,n)$. It consists of the entire bivector space $\mathcal{R}^2_{n,n}$. Remarkably, *every Lie algebra is a subalgebra* of $\mathrm{so}(n,n)$. Our task will be to prove that and develop a systematic way to find them.

Lie groups are classified according to their invariants. For the *classical groups*[11] the invariants are nondegenerate bilinear or quadratic forms. Geometric algebra supplies us with a simpler alternative system of invariants, namely, the multivectors which determine the bilinear forms. As emphasized in Ref. 4, every bilinear form can be written as $a \cdot (\underline{Q}b)$, where $\underline{Q}$ is a linear operator, and the form is nondegenerate if $\underline{Q}$ is nonsingular (i.e., $\det \underline{Q} \neq 0$). Invariance under a rotation R is expressed by

$$(\underline{R}a) \cdot (\underline{Q}\underline{R}b) = a \cdot (\underline{Q}b)\,. \tag{4.10}$$

Using Eq. (4.8) this can be reformulated as

$$a \cdot (\underline{R}^\dagger \underline{Q}\underline{R}b) = a \cdot (\underline{Q}b)\,. \tag{4.11}$$

Expressed as an operator equation this condition becomes

$$\underline{R}^\dagger \underline{Q}\underline{R} = \underline{Q} = \underline{R}\underline{Q}\underline{R}^\dagger\,, \tag{4.12}$$

or equivalently,

$$\underline{Q}\underline{R} = \underline{R}\underline{Q}\,. \tag{4.13}$$

Thus, the invariance group of the quadratic form consists of those rotations which commute with $\underline{Q}$.

As a simple example, consider the bilinear form $a \cdot b^*$ determined by the involution (3.24) which distinguishes the subspaces $\mathcal{R}^n$ and $\bar{\mathcal{R}}^n$. From Eqs. (3.24) and (4.7), the condition (4.12) in this case reduces to an equivalent multivector equation

$$\underline{R}E_n = RE_nR^\dagger = E_n\,. \tag{4.14}$$

Thus, invariance of the bilinear form $a \cdot b^*$ is equivalent to invariance of the n-blade E_n. Using this fact, we can immediately construct a basis for the Lie algebra from the vector basis $\{e_i, \bar{e}_i\}$ of the * operator. Thus, we obtain the generator basis

$$\begin{aligned} e_{ij} &= e_i e_j, \qquad \text{for} \quad i < j = 1,2,\ldots,n, \\ \bar{e}_{kl} &= \bar{e}_k \bar{e}_l, \qquad \text{for} \quad k < l = 1,2,\ldots,n. \end{aligned} \tag{4.15}$$

Any generator B in the algebra can be written in the form

$$B = \alpha{:}e + \beta{:}\overline{e}\,, \tag{4.16}$$

where

$$\alpha{:}e = \sum_{i<j} \alpha^{ij} e_{ij} \tag{4.17}$$

denotes a linear combination with scalar coefficients α^{ij}. The corresponding group rotor is

$$R = e^{\frac{1}{2}(\alpha:e+\beta:\overline{e})} = e^{\frac{1}{2}\alpha:e} + e^{\frac{1}{2}\beta:\overline{e}}\,. \tag{4.18}$$

This, of course, is the spin representation for the product group $\mathrm{SO}(n) \otimes \mathrm{SO}(n)$. Since it is determined by the invariance of E_n in Eq. (4.10), it is said to be the *stability group* of E_n. No direct reference to a quadratic form is needed to characterize it.

To facilitate the systematic analysis of less obvious cases, we need some general theorems. As proved in Ref. 4, every skew-symmetric bilinear form can be written in the form

$$a \cdot (\underline{Q}b) = a \cdot (b \cdot \underline{Q}) = (a \wedge b) \cdot Q\,, \tag{4.19}$$

where Q is a bivector, and, of course, $\underline{Q}$ is the corresponding linear transformation. We say that the bivector Q is *involutory* if $\underline{Q}$ is nonsingular and

$$\underline{Q}^2 = \pm \underline{1}\,. \tag{4.20}$$

At this point a warning is in order. The operator equation (4.20) applies only to the action of $\underline{Q}$ on vectors and not to the outermorphism acting on multivectors of higher grade, as will be demonstrated below.

By virtue of the fact that $(\underline{R}a \wedge \underline{R}b) \cdot Q = (a \wedge b) \cdot \underline{R}^\dagger Q$, invariance of Eq. (4.19) is equivalent to the *stability condition*

$$\underline{R}^\dagger Q = R^\dagger Q R = Q\,. \tag{4.21}$$

In other words, the invariance group of any skew-symmetric bilinear form is the *stability group* of a bivector.

From Eq. (4.21) it follows that generators of the stability group $G(Q)$ for Q must commute with Q. To ascertain the consequences of this requirement, we study the commutator of Q with an arbitrary two-blade $a \wedge b$. Since $a \wedge b = a \times b$, the Jacobi identity (3.14) implies

$$(a \wedge b) \times Q = (a \times Q) \wedge b + a \wedge (b \times Q) = (\underline{Q}a) \wedge b + a \wedge (\underline{Q}b)\,, \tag{4.22}$$

and then

$$\begin{aligned} [(a \wedge b) \times Q] \times Q &= [(\underline{Q}a) \wedge b + a \wedge (\underline{Q}b)] \times Q \\ &= (\underline{Q}^2 a) \wedge b + 2\underline{Q}(a \wedge b) + a \wedge (\underline{Q}^2 b)\,. \end{aligned} \tag{4.23}$$

Applying the condition (4.20) and extending (4.23) by linearity, we arrive at the *theorem*

$$(B \times Q) \times Q = 2(\underline{Q}B \pm B) \tag{4.24}$$

for any bivector B. Thus if B commutes with Q, then

$$\underline{Q}B = \mp B\,, \tag{4.25}$$

where the signs are opposite to those in Eq. (4.20). In other words, the generators of $G(Q)$ are eigenbivectors of $\underline{Q}$ with eigenvalues ∓ 1.

Now, by employing Eq. (4.22) we verify that for any vectors a and b the condition (4.25) can only be satisfied by the bivectors

$$E(a,b) = a \wedge b \mp (\underline{Q}a) \wedge (\underline{Q}b)\,. \tag{4.26a}$$

and

$$F(a,b) = a \wedge (\underline{Q}b) - (\underline{Q}a) \wedge b\,. \tag{4.26b}$$

This is to say that

$$E(a,b) \times Q = 0 = F(a,b) \times Q\,. \tag{4.27}$$

Thus $E(a,b)$ *and* $F(a,b)$ *are the desired generators of the stability group for* Q. A basis for the Lie algebra is obtained by inserting basis vectors for a and b. The commutation relations for the generators $E(a,b)$ and $F(c,d)$ can be found from Eqs. (4.26a) and (4.26b) by applying the following identity from Ref. 4, which is just a two-fold application of the Jacobi identity:

$$(a \wedge b) \times (c \wedge d) = (b \cdot c)a \wedge d - (b \cdot d)a \wedge c + (a \cdot d)b \wedge c - (a \cdot c)b \wedge d\,. \tag{4.28}$$

This is, in fact, the so-called *structural equation* for the Lie algebra of the orthogonal group. The "structure" is all contained in the inner and outer products; no special Lie structure coefficients need be mentioned. Equations (4.26a) and (4.26b) show how the structure is changed by $\underline{Q}$ to get the subalgebra for the stability group of Q. Evaluation of the commutation relations is simplified by using the eigenvectors of $\underline{Q}$ for a basis, so it is best to defer that task until $\underline{Q}$ is completely specified.

Now as an example application of these results, we identify Q with the *complementation bivector* K in Eq. (3.25), and we note from Eq. (3.28) that $\underline{K}^2 = 1$. We choose an orthonormal basis which factors the component blades K_i into orthogonal factors as in Eq. (3.31). Then, using Eqs. (3.32a) and (3.32b) we obtain immediately a generator basis for the stability group of K, namely,

$$E_{ij} = E(e_i, e_j) = e_i e_j - \overline{e}_i \overline{e}_j \qquad (i < j)\,, \tag{4.29a}$$

$$F_{ij} = F(e_i, e_j) = e_i \overline{e}_j - \overline{e}_i e_j \qquad (i < j)\,, \tag{4.29b}$$

$$K_i = \tfrac{1}{2} F_{ii} = e_i \overline{e}_i\,, \tag{4.29c}$$

for $i, j = 1, 2, \ldots, n$.

The stability group of K can now be identified as the *general linear group* $\mathrm{GL}(n, \mathcal{R})$. To establish that, we first prove that it leaves the null vector spaces $\mathcal{V}^n$ and $\mathcal{V}^{n*}$ invariant. These spaces have pseudoscalars

$$W_n = w_1 w_2 \cdots w_n \tag{4.30a}$$

and

$$W_n^* = w_1^* w_2^* \cdots w_n^* \,. \tag{4.30b}$$

From the eigenvalue equations (3.33a) and (3.33b) we find immediately that

$$\underline{K}(W_n) = W_n \tag{4.31a}$$

and

$$\underline{K}(W_n^*) = (-1)^n W_n^* \,, \tag{4.31b}$$

which proves that $\mathcal{V}^n$ and $\mathcal{V}^{n*}$ are invariant. These relations will be preserved only by rotations which commute with $\underline{K}$. It follows that W_n must be an eigenblade for every member of the stability group. More about that below.

Since each group element $\underline{R}$ leaves $\mathcal{V}^n$ invariant, we can write

$$\underline{R} w_j = \sum_{k=1}^{n} w_k \rho_{kj} \,. \tag{4.32}$$

Then using Eq. (2.4) we can solve for the matrix elements

$$\rho_{ij} = 2 w_i^* \cdot (\underline{R} w_j) = 2 \langle\, w_i^* R w_j R^\dagger \,\rangle_0 \,. \tag{4.33}$$

This shows us how to compute the matrix elements from the spin representation R of the group. The number of independent elements is n^2, which is precisely the number of linearly independent generators in Eqs. (4.29a)–(4.29c). This completes our proof.

The identification of the bivectors (4.29a)–(4.29c) with the Lie algebra gl$(n, \mathcal{R})$ has important consequences. First, it *proves the conjecture in Ref. 4 that every Lie algebra is isomorphic to some bivector algebra*, for it is well-known that every Lie group is isomorphic to a subgroup of the general linear group. Indeed, all Lie algebras have a real matrix representation via the "adjoint representation," and we have shown how that can be realized in a bivector algebra in general. However, this is not usually a helpful way of constructing the algebras. Explicit construction of bivector versions of the classical Lie algebras is undertaken in Sec. VI.

Another consequence of Eqs. (4.29a)–(4.29c) is that every positive, nonsingular linear transformation can be represented by a spinor of the form

$$R = e^{\frac{1}{2}(\alpha:E+\beta:F+\mu:K)}. \tag{4.34}$$

The composition of linear transformations is then described as the product of such spinors. It is well established that the computation of composite rotations with such "spin representations" is decidedly more efficient than standard matrix methods. So we may expect the same for general linear transformations. Therefore Eq. (4.34) deserves intensive study, and from our knowledge of matrix theory, we can expect a rich body of results to follow.

Some comments on the interpretation of Eq. (4.34) and alternative forms for a spinor are in order. It is known already in the case of rotations that the exponential form for spinors is not optimal for most computational purposes, but it is, of course, appropriate for a Lie algebra analysis. Comparing the E_{ij} in Eq. (4.29a) with Eqs. (4.15) through (4.18), we see that they generate rotations of $\mathcal{R}^n$ and $\bar{\mathcal{R}}^n$ in tandem, and, by virtue of Eq. (3.30), this can be interpreted as the orthogonal group SO(n) on $\mathcal{V}^n$.

The rotations can be described on $\mathcal{V}^n$ without reference to $\mathcal{R}^n$. For any member $\underline{R}$ of GL$(n, \mathcal{R})$, the outermorphism of W_n satisfies

$$\underline{R}W_n = W_n \det{}_0 \underline{R}\,, \tag{4.35}$$

where the subscript on $\det{}_0$ is to distinguish it from the determinant on the whole of $\mathcal{R}^{n,n}$. From Eqs. (4.30a), (4.30b), and (2.5) it is easily ascertained that

$$W_n^* \cdot W_n^\dagger = 2^{-n}. \tag{4.36}$$

Therefore,

$$\det{}_0 \underline{R}^{-1} = 2^n W_n^* \cdot (\underline{R}W_n^\dagger)\,. \tag{4.37}$$

Similarly, since $\underline{R}^{-1} = \underline{R}^\dagger$

$$\det{}_0 \underline{R}^{-1} = 2^n W_n^* \cdot (\underline{R}^\dagger W_n^\dagger) =^n W_n^\dagger \cdot (\underline{R}W_n^*)\,, \tag{4.38}$$

so

$$\underline{R}W_n^* = W_n^* \det{}_0 \underline{R}^{-1}\,. \tag{4.39}$$

Since every $\underline{R}$ is a rotation on the whole of $\mathcal{R}^{n,n}$, we have

$$\underline{R}(W_n \wedge W_n^*) = (\underline{R}W_n) \wedge (\underline{R}W_n^*) = W_n \wedge W_n^*\,, \tag{4.40}$$

whence we obtain the “classical result”

$$(\det{}_0 \underline{R})(\det{}_0 \underline{R}^{-1}) = 1\,. \tag{4.41}$$

Next consider the spinor

$$D = e^{(1/2)\mu:K} = e^{(1/2)\mu_1 K_1} e^{(1/2)\mu_2 K_2} \cdots e^{(1/2)\mu_n K_n}. \tag{4.42}$$

According to Eq. (3.33a) the w_i are eigenvectors of K, whence

$$\underline{D}w_i = Dw_iD^\dagger = \lambda_i w_i\,, \tag{4.43}$$

with eigenvalues

$$\lambda_i = e^{\mu_i}\,. \tag{4.44}$$

Therefore, $\underline{D}$ is a symmetric linear transformation which is *“diagonalized”* by the eigenvectors w_i. According to the “diagonalization theorem,” therefore, any positive definite symmetric linear transformation $\underline{S}$ can be represented as a spinor S of the form

$$S = R_1 D R_1^\dagger\,, \tag{4.45}$$

where

$$R_1 = e^{(1/2)\alpha_1:E} \tag{4.46}$$

represents the rotation which diagonalizes S. The *polar decomposition* theorem asserts that Eq. (4.34) is equivalent to

$$R = R_2 S\,, \tag{4.47}$$

where R_2 "is" another rotation. Since $R_3 = R_2 R_1$ "is" also a rotation, we have the result

$$R = R_3 D R_1 = e^{(1/2)\alpha_3 : E} e^{(1/2)\mu : K} e^{(1/2)\alpha_1 : E}. \tag{4.48}$$

As a check, note that this also has n^2 parameters.

Inserting Eq. (4.47) into Eq. (4.37), we easily obtain the classical result

$$\det{}_0 \underline{R} = \det{}_0 \underline{S} = \det{}_0 \underline{D} = \lambda_1 \lambda_2 \cdots \lambda_n\,. \tag{4.49}$$

Though we have interpreted the above spinors as representing linear transformations on the invariant space $\mathcal{V}^n$, they can also be interpreted as linear transformations on $\mathcal{R}^n$ by employing the projection operator $\underline{E}_n = \underline{E}_{n,0}$ defined by Eq. (3.20) to write

$$\underline{R}_E = \underline{E}_n \underline{R} \tag{4.50}$$

for the corresponding operator on $\mathcal{R}^n$. Of course, the projection operator introduces complications which are avoided by working on $\mathcal{V}^n$.

The above remarks serve to illustrate the powerful potential of the *spinor version* of linear algebra. Its generalization to include arbitrary linear transformations will be obtained in the next section.

Returning to group theory, we note that K commutes with all other elements of the Lie algebra $\mathrm{gl}(n, \mathcal{R})$, so it generates a one-dimensional invariant subgroup of $\mathrm{GL}(n, \mathcal{R})$. We can remove it from the group by replacing the K_i in Eq. (4.29c) by

$$H_i = K_i - K_{i+1} \quad (j = 1, 2, \ldots, n-1)\,. \tag{4.51}$$

Along with E_{ij} and F_{ij} in Eq. (4.29a)–(4.29c), these bivectors generate the *special linear group* $\mathrm{SL}(n, \mathcal{R})$, the subgroup of $\mathrm{GL}(n, \mathcal{R})$ for which the determinant (4.49) is unity.

Finally, we compose the complementation operator $\underline{K}$ with the *-operator (3.24) to produce an operator $\underline{K}_*$ defined by

$$\underline{K}_* a = \underline{K} a^* = (a^*) \cdot K\,. \tag{4.52}$$

It follows that

$$\underline{K}_* e_j = \overline{e}_j, \quad \underline{K}_* \overline{e}_j = -e_j,\,. \tag{4.53}$$

Hence

$$\underline{K}_*^2 = -1\,. \tag{4.54}$$

Also, in analogy to Eq. (4.25), $\underline{K}_*$ defines a new Lie algebra with generators defined by the outermorphism condition

$$\underline{K}_*(B) = -B\,. \tag{4.55}$$

From this we construct a generator basis for the invariance group of $\underline{K}_*$

$$E_{ij} = e_i e_j - \overline{e}_i \overline{e}_j\,, \tag{4.56}$$

$$F_{ij} = e_i\overline{e}_j + \overline{e}_i e_j\,, \tag{4.57}$$

$(i,j = 1,2,\ldots,n, i<j)$. These are generators of the *complex orthogonal group* $\mathrm{SO}(n,\mathcal{C})$. The *complex structure* is defined on $\mathcal{R}^{n,n}$ by the operator $\underline{K}_*$, which plays the role of $\sqrt{-1}$.

For odd n, K is the only kind of involutory bivector. However, there are others when n is even, and their invariants determine other groups which are discussed in Sec. VI.

The general linear group $\mathrm{GL}(p,q)$ can be obtained and analyzed in essentially the same way as the Euclidean case, with E_n replaced by $E_(p,q)$, and the corresponding null space pseudoscalar W_n replaced by $W_{p,q} = w_1\cdots w_p w^*_{p+1}\cdots w^*_n$.

V. ENDOMORPHISMS OF $\mathcal{R}^n$

Now we develop an alternative argument leading to the conclusion that the mother algebra $\mathcal{R}_{n,n}$ is the appropriate arena for the theory of linear transformations and Lie groups. We show how it arises naturally as the *endomorphism algebra* $\mathcal{E}nd(\mathcal{R}_n)$, the algebra of linear maps of the Euclidean geometric algebra $\mathcal{R}_n$ onto itself. This algebra is, of course, isomorphic to the algebra of real $2^n\times 2^n$ matrices, that is,

$$\mathcal{E}nd(\mathcal{R}_n) \simeq \Re(2^n)\,. \tag{5.1}$$

For an arbitrary multivector A in $\mathcal{R}_n$, left and right multiplication by basis vectors e_i determine endomorphisms of $\mathcal{R}_n$ defined by

$$\underline{e}_i : A \to \underline{e}_i(A) = \underline{e}_i A\,, \tag{5.2a}$$

$$\underline{\overline{e}}_i : A \to \underline{\overline{e}}_i(A) = \overline{A}e_i\,, \tag{5.2b}$$

where, for the moment, the overbar indicates the main involution of $\mathcal{R}_n$ defined by

$$\overline{(AB)} = \overline{A}\,\overline{B} \quad \text{and} \quad \overline{e}_i = -e_i\,. \tag{5.3}$$

We shall see below that this is consistent with our overbar notation in $\mathcal{R}_{n,n}$. The operators $\underline{e}_i$ are clearly linearly independent, and they satisfy the operator relations

$$\underline{e}_i\underline{e}_j + \underline{e}_j\underline{e}_i = 2\delta_{ij}\,, \tag{5.4a}$$

$$\underline{\overline{e}}_i\underline{\overline{e}}_j + \underline{\overline{e}}_j\underline{\overline{e}}_i = -2\delta_{ij}\,, \tag{5.4b}$$

$$\underline{e}_i\underline{\overline{e}}_j + \underline{\overline{e}}_j\underline{e}_i = 0\,. \tag{5.4c}$$

By virtue of Eq. (3.7), these relations are isomorphic to the defining relations (3.17) for the vector basis $\{e_i, \overline{e}_i\}$ in $\mathcal{R}_{n,n}$. This establishes the algebra isomorphism

$$\mathcal{R}_{n,n} \simeq \mathcal{E}nd(\mathcal{R}_n)\,. \tag{5.5}$$

The above defining equations for this isomorphism were first formulated in Ref. 12. The role of the main involution in determining the negative signature in Eq. (5.4b) is especially noteworthy. Its significance can be explained as follows.

The composite operators

$$\underline{e}_i\underline{\overline{e}}_i : A \to \underline{e}_i\underline{\overline{e}}_i(A) = e_i\overline{A}e_i \tag{5.6}$$

generate $\mathcal{A}ut(\mathcal{R}_n)$, a subgroup of $\mathcal{E}nd(\mathcal{R}_n)$ which preserves the geometric product. It is also a subgroup of the group of nonsingular outermorphism on $\mathcal{R}_n$ which are generated by the general linear group on $\mathcal{R}^n$, In fact, it is the *outermorphism of the orthogonal group*, because it preserves the inner product. Thus, action of the operator (5.6) on vectors defines the fundamental linear transformation

$$\underline{f_i} : a \to \underline{f_i}a = -e_iae_i\,. \tag{5.7}$$

As is well-known, this transformation is a *reflection in a hyperplane* with normal e_i, and the entire orthogonal group $\mathrm{O}(n)$ is generated by products of such reflections with e_i ranging over all unit vectors in $\mathcal{R}^n$. The bivector outermorphism of Eq. (5.7) is

$$\underline{f_i}(a\wedge b) = e_i(a\wedge b)e_i\,. \tag{5.8}$$

Note that the sign difference between (5.7) and (5.8) is just what is required by the involution in Eq. (5.6). Thus, the involution is essential to generating $\mathcal{A}ut(\mathcal{R}_n)$.

As observed in Chap. 3 of Ref. 4, all symmetric transformations are generated by linear combinations of the operators in Eqs. (5.7), while orthogonal transformations are generated by their products. A big advantage of representing these linear transformations in $\mathcal{R}_{n,n}$ is that both symmetric and orthogonal transformations are generated by products, as we saw in the previous section.

Our next task is to prove that the mother algebra contains a $2n$-dimensional subspace $\mathcal{S}_n$ on which all the endomorphisms $\mathcal{R}_n$ are *faithfully* represented by left multiplication with elements of $\mathcal{R}_{n,n}$. The space $\mathcal{S}_n$ is called a *spinor space* and its elements are called *spinors*. It is a *minimal left ideal* of $\mathcal{R}_{n,n}$, and its construction is easily described after establishing the relevant algebraic relations. The same construction is employed in Ref. 13, and it is implicit in many other works on physics and mathematics.

From the null vectors

$$w_i = \tfrac{1}{2}(e_i+\overline{e}_i), \qquad w_i^* = \tfrac{1}{2}(e_i-\overline{e}_i)\,, \tag{5.9}$$

we construct a family of commuting idempotents I_i, which can be expressed in the several different ways

$$I_i = w_i^*w_i = \tfrac{1}{2}(1+e_i\overline{e}_i) = \tfrac{1}{2}(1+K_i) = e_iw_i = w_i^*e_i = w_i^*\overline{e}_i = -\overline{e}_iw_i\,, \tag{5.10}$$

for $i = 1,2,\ldots,n$. The idempotence and commutative properties are expressed by

$$I_i^2 = I_i, \qquad I_i\times I_j = 0\,. \tag{5.11}$$

The relations (5.10) will be more useful as expressions for the effect of left and right multiplication on I_i

$$e_iI_i = \overline{e}_iI_i = w_iI_i = w_i = w_iw_i^*w_i\,, \tag{5.12}$$

$$I_i e_i = -I_i \overline{e}_i = I_i w_i^* = w_i^* = w_i^* w_i w_i^* \,. \tag{5.13}$$

From the I_i we construct the *"mother idempotent"*

$$I = I_1 I_2 \cdots I_n = W_n^* W_n^\dagger = W_n^{*\dagger} W_n \,, \tag{5.14}$$

where it will be recalled from Eq. (4.30a) that W_n is the pseudoscalar for the null space $\mathcal{V}^n$. The impotence property

$$I^2 = I \tag{5.15}$$

follows from Eq. (5.10).

From Eq. (5.12) it follows that

$$e_i I = \overline{e}_i I = w_i I \,, \tag{5.16}$$

and further that

$$E_n I = \overline{E}_n I = W_n I = W_n \,. \tag{5.17}$$

This establishes an equivalency of the vector spaces $\mathcal{R}^n$, $\overline{\mathcal{R}}^n$, and $\mathcal{V}^n$. Indeed, I completely characterizes the relations among them. Thus, it specifies the involutory bivector

$$K = 2^n \langle\, I \,\rangle_2 = K_1 + K_2 + \cdots + K_n \tag{5.18}$$

as well as its relation to the pseudoscalar

$$E_{n,n} = 2^n \langle\, I \,\rangle_{2n} = K_1 K_2 \cdots K_n = E_n^\dagger \overline{E}_n = W_n^* \wedge W_n^\dagger \,. \tag{5.19}$$

Comparison with Eq. (5.17) yields

$$E_{n,n} I = I \,. \tag{5.20}$$

The spinor space $\mathcal{S}_n$ is generated by left multiplication of I by the entire mother algebra, as expressed by

$$\mathcal{S}_n = \mathcal{R}_{n,n} I \,. \tag{5.21}$$

The multiplicative equivalence of e_i and $\overline{e}_i$ implies that $\mathcal{S}_n$ has the dimension of $\mathcal{R}_n$ namely, 2^n, though the operators on it have the dimension of the algebra $\mathcal{R}_{n,n}$, namely $2^n \times 2^n$. With the above preparation, it is easy to establish the interpretation of operators on $\mathcal{S}_n$ as endomorphisms of $\mathcal{R}_n$.

First, using Eqs. (5.20) and (5.16), we see that

$$E_{n,n} e_i I = -e_i I = E_{n,n} \overline{e}_i I = -\overline{e}_i I \,. \tag{5.22}$$

Therefore, multiplication of $\mathcal{S}_n$ by $E_{n,n}$ corresponds to the main involution in $\mathcal{R}_n$, as expressed by

$$E_{n,n} \mathcal{S}_n \Longleftrightarrow \overline{\mathcal{R}}_n \,. \tag{5.23}$$

In view of the operaor relations (5.4a)–(5.4c) the definitions (5.2a) and (5.2b) give the correspondences

$$e_i \mathcal{S}_n \Longleftrightarrow e_i \mathcal{R}_n \,, \tag{5.24}$$

$$\overline{e}_i \mathcal{S}_n \Longleftrightarrow \overline{\mathcal{R}}_n e_i \,, \tag{5.25}$$

and the latter combines with Eq. (5.23) to give

$$\bar{e}_i E_{n,n}\mathcal{S}_n \Longleftrightarrow \mathcal{R}_n e_i \,. \tag{5.26}$$

Lastly, it is easily established that reversion in $\mathcal{R}_{n,n}$ corresponds to reversion in $\mathcal{R}_n$. The interpretation of spinor space operators as $\mathcal{R}_n$ endomorphisms is now completely established. The rest is calculation.

Of special interest is the endomorphism correspondence for $\mathrm{GL}(n,\mathcal{R})$. We consider the orthogonal group first. For *any unit* vectors $u, v, \ldots,$ in $\mathcal{R}_n$, Eqs. (5.24) and (5.25) give the correspondences

$$u\bar{u}\mathcal{S}_n \Longleftrightarrow u\overline{\mathcal{R}}_n u \,, \tag{5.27}$$

$$v\bar{v}u\bar{u}\mathcal{S}_n \Longleftrightarrow vu\mathcal{R}_n uv \,. \tag{5.28}$$

As noted earlier, the first of these is (the outermorphism of) a reflection in $\mathcal{R}^n$, while the second, a double reflection, is a rotation in the $u\wedge v$ plane. The generalization is immediate. For k-vectors $u_1, u_2, \ldots,$ in $\mathcal{R}^n$ let us write

$$U_{(k)} = u_k u_{k-1}\cdots u_2 u_1 \qquad \text{and} \qquad \overline{U}_{(k)} = \bar{u}_k \bar{u}_{k-1}\cdots \bar{u}_2 \bar{u}_1 \,, \tag{5.29}$$

whence

$$\begin{aligned} U_{(k)}\overline{U}_{(k)} &= u_k \cdots u_1 \bar{u}_k \cdots \bar{u}_1 = (-1)^{k-1} u_k \cdots u_2 \bar{u}_k \cdots \bar{u}_2 u_1 \bar{u}_1 \cdots \\ &= \epsilon_k u_k \bar{u}_k u_{k-1}\bar{u}_{k-1}\cdots u_2\bar{u}_2 u_1 \bar{u}_1 \,, \end{aligned} \tag{5.30}$$

where $\epsilon_k = (-1)^{\frac{1}{2}k(k-1)}$. For odd k, therefore,

$$\epsilon_k U_{(k)}\overline{U}_{(k)}\mathcal{S}_n \Longleftrightarrow U_{(k)}\overline{\mathcal{R}}_n U^{\dagger}_{(k)} \,, \tag{5.31}$$

and for even k,

$$\epsilon_k U_{(k)}\overline{U}_{(k)}\mathcal{S}_n \Longleftrightarrow U_{(k)}\mathcal{R}_n U^{\dagger}_{(k)} \,, \tag{5.32}$$

Equations (5.31) and (5.32) describe the complete *orthogonal group* $\mathrm{O}(n)$ as an automorphism group of $\mathcal{R}_n$. The multiplicative group of unit vectors in $\mathcal{R}^n$, exemplified by $U_{(k)}$ in Eq. (5.29), is called the Pin group $\mathcal{R}_n$ and denoted by $\mathrm{Pin}(n)$. Clearly $\mathrm{Pin}(n)$ is a double covering of $\mathrm{O}(n)$. The subgroup for even n is $\mathrm{Spin}(n)$, the double covering of $\mathrm{SO}(n)$. Adopting the notation of Eq. (4.19), it can be shown that, for even I, the $U_{(k)}$ in Eq. (5.29) which are continuously connected with the identity can be written in the exponential form

$$U_{(k)} = e^{(1/2)\alpha:e} \,. \tag{5.33}$$

Therefore

$$\epsilon_k U_{(k)}\overline{U}_{(k)} = e^{(1/2)\alpha:e} e^{-(1/2)\alpha:\bar{e}} = e^{(1/2)\alpha:(e-\bar{e})} \,. \tag{5.34}$$

With $E = e - \bar{e}$, we see that this is exactly the rotor representation for elements of $\mathrm{O}(n)$ given by Eq. (4.56). Moreover, it is absolutely clear that the two-bladed structure of the bivector generators $E_{ij} = e_i e_j - \bar{e}_i\bar{e}_j$ represents concurrent left and right multiplications of $\mathcal{R}_n$ as in Eq. (5.32).

Next we determine the "correspondence rule" for the diagonal operators

$$D_i = e^{(1/2)\mu_i K_i} = e^{(1/2)\mu_i e_i \overline{e}_i} . \tag{5.35}$$

It will be convenient to drop the subscripts and write

$$D = e^{(1/2)\mu e\overline{e}} = \alpha + \beta e\overline{e} = uv . \tag{5.36}$$

This is the well-known "spin representation" for a Lorentz transformation or *boost* in the hyperbolic plane of the bivector $e\overline{e} = e \wedge \overline{e}$. It has been thoroughly studied in Ref. 3, where it is used to represent dilatations in the conformal group, and the decomposition into a product of unit vectors u and v on the right side of Eq. (5.36) is explained. The parameters α and β are related by

$$DD^\dagger = u^2 v^2 = \alpha^2 - \beta^2 = 1 . \tag{5.37}$$

For comparison, we consider first the action of D on the null vector $w = \frac{1}{2}(e + \overline{e})$. From Eqs. (5.9) and/or (5.10) we note

$$e\overline{e}w = -w = -we\overline{e} \quad \text{or} \quad w \times (e\overline{e}) = w , \tag{5.38}$$

whence,

$$\underline{D}w = DwD^\dagger = D^2 w = (\alpha - \beta)^2 w . \tag{5.39}$$

The projection of this into $\mathcal{R}^n$ defined by Eq. (4.50) gives

$$\underline{D}_E(e) = \underline{E}_n \underline{D}(w) = (\alpha - \beta)^2 e , \tag{5.40a}$$

and for any vector $e_\perp$ orthogonal to e,

$$\underline{D}_E(e_\perp) = e_\perp , \tag{5.40b}$$

thus $\underline{D}_E$ describes a stretch along e by the positive factor $(\alpha - \beta)^2$.

In contrast, from Eq. (5.36) we get the endomorphism correspondence

$$D\mathcal{S}_n = uv\mathcal{S}_n \Longleftrightarrow \underline{D}\mathcal{R}_n = \alpha\mathcal{R}_n + \beta e\overline{\mathcal{R}}_n e . \tag{5.41}$$

For a vector a in $\mathcal{R}^n$, by virtue of Eq. (5.38) and the identity $ae = -ea + 2a \cdot e$, Eq. (5.41) gives

$$\underline{D}a = (\alpha + \beta)a - 2\beta a \cdot ee . \tag{5.42}$$

This is the composite of a dilatation of $\mathcal{R}^n$ by $(\alpha + \beta)$ with a stretch along e by the factor $(\alpha - \beta)(\alpha + \beta)^{-1} = (\alpha - \beta)^2$, in agreement with Eq. (5.40a).

These results generalize trivially to give the correspondence theorem for an arbitrary "diagonal" transformation represented by Eq. (4.42). The correspondence for any other symmetric transformation follows from Eq. (4.45) by composition with a rotation. This suffices to establish the correspondences for GL$(n, \mathcal{R})$, though much more can, and no doubt will, be said about the subject. Incidentally, it should be evident from the foregoing that to each symmetric linear transformation $\underline{S}$ on $\mathcal{R}^n$ there corresponds a decomposition of the involutory bivector K into commuting blades which represent the eigenvectors of $\underline{S}$.

There is one more basic type of transformation to consider. Combining Eqs. (5.24) and (5.25), we obtain

$$w_i\mathcal{S}_n = \tfrac{1}{2}(e_i + \overline{e}_i)\mathcal{S}_n \Longleftrightarrow \tfrac{1}{2}(e_i\mathcal{R}_n + \overline{\mathcal{R}}_n e_i) = e_i \wedge \mathcal{R}_n\,, \tag{5.43}$$

$$w_i^*\mathcal{S}_n = \tfrac{1}{2}(e_i - \overline{e}_i)\mathcal{S}_n \Longleftrightarrow \tfrac{1}{2}(e_i\mathcal{R}_n - \overline{\mathcal{R}}_n e_i) = e_i \cdot \mathcal{R}_n\,. \tag{5.44}$$

Thus, the "fermion creation operator" w_i can be represented in the real Euclidean $\mathcal{R}_n$ by an outer product which raises the grade of every multivector by one unit. Similarly, the "annihilation operator" w_i^* can be represented in $\mathcal{R}_n$ by the grade lowering inner product. This correspondence has been exploited in Ref. 14 to reformulate Grassmann/Berezin calculus in $\mathcal{R}_n$, leading to simplifications in the theory of pseudoclassical mechanics.

Composing Eqs. (5.43) and (5.44), we obtain the grade-preserving outermorphisms

$$I_i\mathcal{S}_n = w_i w_i^*\mathcal{S}_n \Longleftrightarrow \underline{P}_i^{\perp}\mathcal{R}_n = \tfrac{1}{2}(\mathcal{R}_n + e_i\overline{\mathcal{R}}_n e_i) = e_i \cdot (e_i \wedge \mathcal{R}_n)\,, \tag{5.45a}$$

$$I_i^{\dagger}\mathcal{S}_n = w_i^* w_i\mathcal{S}_n \Longleftrightarrow \underline{P}_i\mathcal{R}_n = \tfrac{1}{2}(\mathcal{R}_n - e_i\overline{\mathcal{R}}_n e_i) = e_i \wedge (e_i \cdot \mathcal{R}_n)\,, \tag{5.45b}$$

Operating on vectors in $\mathcal{R}^n$, they become

$$\underline{P}_i^{\perp}(a) = \tfrac{1}{2}(a + e_i a e_i) = (a \wedge e_i) \cdot e_i\,, \tag{5.46a}$$

$$\underline{P}_i(a) = \tfrac{1}{2}(a - e_i a e_i) = (a \cdot e_i)e_i\,. \tag{5.46b}$$

The first of these is the projection onto the orthogonal complement of e_i, called a *rejection* in Ref. 4. Thus, I_i represents a projection operator which annihilates the e_i direction in $\mathcal{R}^n$. Similarly, Eq. (5.46b) is a projection onto the e_i subspace in $\mathcal{R}^n$, which is represented by $I_i^{\dagger}$ in $\mathcal{R}_{n,n}$.

Having shown how projections as well as orthogonal and symmetric transformations on $\mathcal{R}^n$ can be represented in $\mathcal{R}_{n,n}$ as even monomials (that is, products of an even number of vectors), we can draw a major conclusion: *Every linear transformation in $\mathcal{R}^n$ can be represented in $\mathcal{R}_{n,n}$ as an even multivector which commutes with the complementation bivector K.* This reduces the composition of linear transformations to geometric products among idempotents and rotors in Spin(n,n). The commutativity with K simplifies many manipulations, as is implicit in the reordering of vectors in Eq. (5.30).

VI. CLASSIFICATION OF THE CLASSICAL GROUPS

The classical groups are traditionally distinguished by the various quadratic or bilinear forms they leave invariant.[15] In Sec. IV we saw that the quadratic form which distinguishes GL$(n,\mathcal{R})$ is determined by an involutory bivector K and this provides an alternative specification of GL$(n,\mathcal{R})$ as the stability group of K. Here we show that many *classical groups can be similarly classified as stability groups of various involutory bivectors*. This approach appears to be simpler and more systematic than the traditional approach, because it fully exploits the power of geometric algebra. However, for reference purposes we show how the two approaches are related.

Our approach is to systematically search for involutory bivectors and invariant relations among them. As all the groups are subgroups of an orthogonal group $\mathrm{O}(p,q)$, the inner product $a \cdot b$ is always available as an invariant form, and the pseudoscalar $E_{p,q}$ is necessarily invariant. Taking this for granted, we search for involutory bivectors in $\mathcal{R}_{p,q}$. As defined in Sec. IV, each involutory bivector Q determines a skew-symmetric linear transformation $\underline{Q}$ satisfying one of the two conditions $\underline{Q}^2 = \pm 1$, and it can exist only for vector spaces of even dimension. For odd n the only possibilities in $\mathcal{R}_{n,n}$ is the complementation bivector K, but for even $n = 2m$ new possibilities arise which we now explore.

A. Subgroups of 0(2m)

In the geometric algebra $\mathcal{R}_{2m} = \mathcal{G}(\mathcal{R}^{2m})$, from an orthonormal basis $\{e_i, \widetilde{e}_i\}$ on $\mathcal{R}_{2m}$, we construct the involutory bivector

$$J = \sum_{i=1}^{m} e_i \widetilde{e}_i \, . \tag{6.1}$$

This determines the skew-symmetric transformation

$$\underline{J}a = a \cdot J = \widetilde{a} \, , \tag{6.2}$$

with the involutory property

$$\underline{J}^2 = (\widetilde{a})^{\widetilde{}} = -a \, . \tag{6.3}$$

Thus, J induces a *complex structure* on $\mathcal{R}^{2m}$. From our theorem (4.25) about Q, it follows that the stability group of J is the invariance group of this complex structure. This is the *unitary group* $\mathrm{U}(m)$. It has the same dimension as $\mathrm{GL}(m, \mathcal{R})$, and its structure differs only in replacing $\underline{K}^2 = \underline{1}$ by $\underline{J}^2 = \underline{1}$.

Like $\mathrm{gl}(m, \mathcal{R})$, a generator basis for the Lie algebra $\mathrm{u}(m)$ can be written down at once from Eq. (4.26a) and (4.26b), namely,

$$E_{ij} = e_i e_j + \widetilde{e}_i \widetilde{e}_j \qquad (i < j) \tag{6.4a}$$

$$F_{ij} = e_i \widetilde{e}_j - \widetilde{e}_i e_j \qquad (i < j) \tag{6.4b}$$

$$J_i = e_i \widetilde{e}_i \, , \tag{6.4c}$$

for $i, j = 1, 2, \ldots, m$.

In analogy with the restriction of $\mathrm{gl}(m, \mathcal{R})$ to $\mathrm{sl}(m, \mathcal{R})$, $\mathrm{u}(m)$ contains J, which commutes with all other elements and so generates an invariant $\mathrm{U}(1)$ subgroup. We can remove J from the $\mathrm{u}(m)$ to produce generators for the *special unitary group* $\mathrm{SU}(m)$ by replacing the J_i in Eq. (6.4c) by

$$H_i = J_i - J_{i+1} \qquad (i = 1, 2, \ldots, n-1) \, . \tag{6.5}$$

In passing we note the interesting relation

$$e^{(\pi/2)J} = J_1 J_2 \cdots J_m = E_{2m} \, . \tag{6.6}$$

From the invariant bilinear forms $a \cdot b$ and $a \cdot \underline{J}b$ we can construct a *Hermitian symmetric bilinear form* by introducing a "unit imaginary" $\mathbf{i}$ (which could be a bivector) and writing

$$\epsilon(a, b) = a \cdot b + \mathbf{i}(a \cdot \underline{J}b) \, . \tag{6.7}$$

With the definition $\mathbf{i}^\dagger = -\mathbf{i}$, this has the symmetry property

$$\epsilon(a,b) = \epsilon^\dagger(b,a) = b \cdot a - \mathbf{i}(b \cdot \underline{J}a)\,. \tag{6.8}$$

This introduction of $\mathbf{i}$ is clearly an artifice for expressing the fact that U(n) leaves two distinct bilinear forms invariant, one of which is skew-symmetric. The properties assigned to $\mathbf{i}$ have no essential relation to the underlying group structure, though standard choice $\mathbf{i}^2 = -1$ reflects the involutory relation (6.3). Moreover, the use of Hermitian forms hides the essential role of J and so makes it difficult to relate to other groups. In particular, the stability group of J is the *symplectic group* Sp($m, \mathcal{R}$). Since U(m) leaves $a \cdot b$ invariant as well as J, we have the group relation

$$\mathrm{U}(m) = \mathrm{O}(2m) \cap \mathrm{SP}(m, \mathcal{R})\,. \tag{6.9}$$

We have introduced J as a *splitting operator* which splits $\mathcal{R}^{2m}$ into equivalent orthogonal subspaces $\mathcal{R}^m$ and $\widetilde{\mathcal{R}}^m$ with bases $\{e_i\}$ and $\{\widetilde{e}_i\}$. Alternatively, it is often convenient to regard J as a *doubling operator* which generates $\mathcal{R}^{2m} = \mathcal{R}^m \oplus \overline{\mathcal{R}}^m$ from $\mathcal{R}^m$.

B. Subgroups of 0(2m,2m)

To import the complex structure in $\mathcal{R}_{2m}$ into the general linear group, we simply double the dimension to $\mathcal{R}_{2m,2m}$ using a complementing bivector K which commutes with J. This can be done by introducing a basis $\{e_i, \widetilde{e}_i, f_i, \widetilde{f}_i\}$ satisfying

$$e_i \cdot e_j = \widetilde{e}_i \cdot \widetilde{e}_j = \delta_{ij} = -f_i \cdot f_j = -\widetilde{f}_i \cdot \widetilde{f}_j \tag{6.10}$$

for $i, j = 1, 2, \ldots, m$. Then

$$J = \sum_{i=1}^{m} (e_i \widetilde{e}_i + f_i \widetilde{f}_i)\,, \tag{6.11}$$

$$K = \sum_{i=1}^{m} (e_i f_i - \widetilde{e}_i \widetilde{f}_i)\,. \tag{6.12}$$

We verify that

$$J \times K = \langle\, JK \,\rangle_2 = \sum_{i=1}^{m} (f_i \widetilde{e}_i + e_i \widetilde{f}_i - f_i \widetilde{e}_i + e_i \widetilde{f}_i) = 0\,. \tag{6.13}$$

The invariance group of both J and K is the *complex general linear group* GL($m, \mathcal{C}$) . It is the subgroup of GL($2m, \mathcal{R}$) which leaves J invariant or, equivalently, the subgroup of U($2m$) which leaves K invariant. In other words

$$\mathrm{GL}(m, \mathcal{C}) = \mathrm{U}(2m) \cap \mathrm{GL}(2m, \mathcal{R})\,. \tag{6.14}$$

We can derive a basis for gl$(m,\mathcal{C})$ by applying Eq. (4.26a) and (4.26b) with $K^2 = \underline{1}$ to double the basis for u(m) in Eqs. (6.4a)–(6.4c). Thus, we obtain

$$\begin{aligned} E_{ij} &= e_i e_j + \widetilde{e}_i\widetilde{e}_j - f_i f_j + \widetilde{f}_i\widetilde{f}_j\,, \\ F_{ij} &= e_i\widetilde{e}_j - \widetilde{e}_i e_j + f_i\widetilde{f}_j - \widetilde{f}_i f_j\,, \\ G_{ij} &= e_i f_j - f_i e_j - \widetilde{e}_i\widetilde{f}_j + \widetilde{f}_i\widetilde{e}_j\,, \\ H_{ij} &= e_i\widetilde{f}_j + f_i\widetilde{e}_j + \widetilde{f}_i e_j + \widetilde{e}_i f_j \quad (i<j=1,2,\ldots,m)\,, \\ J_i &= e_i\widetilde{e}_i + f_i\widetilde{f}_i\,, \\ K_i &= e_i f_i - \widetilde{e}_i\widetilde{f}_i \quad (i=1,2,\ldots,m)\,, \end{aligned} \tag{6.15}$$

for a total of $4\times\frac{1}{2}m(m-1)+2m = 2m^2$ generators. Both J and K can be eliminated as before by replacing J_i, K_i by

$$H_i = J_i - J_{i+1}, \qquad G_i = K_i - K_{i+1} \quad (i=1,2,\ldots,n-1)\,. \tag{6.16}$$

The result is the Lie algebra for SL$(m,\mathcal{C})$. This completes our discussion of GL$(m,\mathcal{C})$.

We can describe the extension from $\mathcal{R}_{2m}$ to $\mathcal{R}_{2m,2m}$ in a different way. If we replace K in Eq. (6.12) by

$$K_1 = \sum_{i=1}^{m}(e_i f_i + \widetilde{e}_i\widetilde{f}_i)\,, \tag{6.17}$$

then Eq. (6.13) is replaced by

$$J\times K_1 = 2K_2\,, \tag{6.18}$$

where

$$K_2 = \sum_{i=1}^{m}(f_i\widetilde{e}_i + e_i\widetilde{f}_i)\,. \tag{6.19}$$

Thus we have another stability group for J which leaves K_1 and hence K_2 invariant instead of K. This is the full *symplectic group* Sp$(m,\mathcal{R})$. It is a subgroup of GL$(m,\mathcal{R})$, since the latter is defined by K_1 invariance. In contrast to U(m), however, it is not a subgroup of O$(2m)$; see Eq. (6.9). A basis for sp$(m,\mathcal{R})$ can be written down from the gl$(m,\mathcal{R})$ basis (6.15) simply by noting the effect of switching the sign in replacing K by K_1 and replacing K_i by

$$K_{1i} = e_i f_i + \widetilde{e}_i\widetilde{f}_i \quad \text{and} \quad K_{2i} = e_i\widetilde{f}_i + f_i\widetilde{e}_i\,. \tag{6.20}$$

Thus sp$(m,\mathcal{R})$ contains m more generators than gl$(m,\mathcal{C})$.

By combining the *-operator (3.24) with K_1, as we did in Sec. IV, we define the operators

$$\underline{K}_{1*}a = \underline{K}_1 a^* = a^*\cdot K_1\,, \tag{6.21a}$$

and

$$\underline{K}_{2*}a = \underline{K}_2 a^* = a^*\cdot K_2\,. \tag{6.21b}$$

Then we have an algebra of three operators satisfying

$$\underline{K}_{1*}^2 = \underline{K}_{2*}^2 = \underline{J}^2 = -\underline{1}\,, \tag{6.22a}$$

and

$$[\underline{J}, \underline{K}_{1*}] = -2\underline{K}_{2*}\,. \tag{6.22b}$$

As established in Sec. IV, the invariance group of $\underline{K}_{1*}$ is SO($2m$). The subgroup which also leaves $\underline{J}$ in variant is denoted by Sk($m, \mathcal{Q}$) or So*($2m$). The $\mathcal{Q}$ refers to the quaternionic structure specified by the invariant relations (6.22a) and (6.22b). With the artifice of introducing quaternionic units **i**, **j**, **k** (which could be bivectors in $\mathcal{R}_3$), Sk($m, \mathcal{Q}$) can be regarded as the invariance group of the *quaternion skew-Hermitian form*

$$\epsilon(a,b) = a \cdot \underline{J}b + (a \cdot b)\mathbf{i} + (a \cdot \underline{K}_{1*}b)\mathbf{j} + (a \cdot \underline{K}_{2*}b)\mathbf{k}\,. \tag{6.23}$$

This defines a mapping from $\mathcal{R}^{2m} \times \mathcal{R}^{2m}$ to the quaternions. Of course, the four coefficients are separately group invariants. With quaternion conjugation denoted by a dagger, the skewHermitian property is expressed by

$$\epsilon(a,b) = -\epsilon^{\dagger}(b,a)\,. \tag{6.24}$$

A basis for the Lie algebra of Sk($m, \mathcal{Q}$) is easily constructed by doubling the u(n) basis (6.4a)–(6.4c) with the symmetry of the so($n, \mathcal{C}$) basis (4.56) and (4.57).

C. Subgroups of 0(4m)

On doubling $\mathcal{R}^{2m}$ to $\mathcal{R}^{4m}$ we gain the possibility of two invariant involutory bivectors. Expressing this doubling by writing an orthonormal vector basis in the form $\{e_i, g_i, \widetilde{e}_i, \widetilde{g}_i\}$, one involutory bivector is defined by

$$J = \sum_{i=1}^{m} (e_i\widetilde{e}_i + g_i\widetilde{g}_i)\,. \tag{6.25}$$

Another is defined by

$$J' = \sum_{i=1}^{m} (e_i\widetilde{g}_i + g_i\widetilde{e}_i)\,. \tag{6.26}$$

Their commutator is

$$J \times J' = \langle\, JJ'\,\rangle_2 = \sum_{i=1}^{m} (\widetilde{g}_i\widetilde{e}_i + e_i g_i - e_i g_i - \widetilde{g}_i\widetilde{e}_i) = 0\,. \tag{6.27}$$

Since J and J' commute, the group which leaves them both invariant is U(m) $\otimes$ U(m).

Alternatively, if we write $J = J_1$ and replace J' by

$$J_2 = \sum_{i=1}^{m} (e_i\widetilde{g}_i - g_i\widetilde{e}_i)\,, \tag{6.28}$$

we get the quaternionic structure

$$J_1 \times J_2 = 2J_3\,, \tag{6.29}$$

where

$$J_3 = \sum_{i=1}^{m}(e_i g_i + \widetilde{g}_i \widetilde{e}_i)\,. \tag{6.30}$$

The invariance group of this structure is $\mathrm{U}(m,\mathcal{Q})$, [also denoted by $\mathrm{HU}(m)$, $\mathrm{USp}(m)$, or $\mathrm{Sp}(m)$ in the literature]. It can be described as the invariance group of the Hermitian symmetric quaternion-valued bilinear form

$$\epsilon(a,b) = a\cdot b + \mathbf{i}(a\wedge b)\cdot J_1 + \mathbf{j}(a\wedge b)\cdot J_2 + \mathbf{k}(a\wedge b)\cdot J_3\,. \tag{6.31}$$

The Hermitian symmetry is, of course, expressed by $\epsilon(a,b) = \epsilon^\dagger(b,a)$. A basis for the Lie algebra $\mathrm{u}(m,\mathcal{Q})$ can be constructed by double application of Eqs. (4.26a) and (4.26b), with the result

$$\begin{aligned}
E_{ij} &= e_i e_j + \widetilde{e}_i \widetilde{e}_j + g_i g_j + \widetilde{g}_i \widetilde{g}_j\,,\\
F_{ij} &= e_i \widetilde{e}_j - \widetilde{e}_i e_j + g_i \widetilde{g}_j + \widetilde{g}_i g_j\,,\\
G_{ij} &= e_i g_j - g_i e_j - \widetilde{g}_i \widetilde{e}_j + \widetilde{e}_i \widetilde{g}_j\,,\\
H_{ij} &= e_i \widetilde{g}_j - \widetilde{g}_i e_j - \widetilde{e}_i g_j + g_i \widetilde{e}_j \quad (i<j=1,2,\ldots,m)\,,\\
F_i &= e_i \widetilde{e}_i - g_i \widetilde{g}_i \quad G_i = e_i g_i - \widetilde{g}_i \widetilde{e}_i \quad H_i = e_i \widetilde{g}_i - \widetilde{e}_i g_i \quad (i=1,2,\ldots,m)\,,
\end{aligned} \tag{6.32}$$

for a total of $4\times\frac{1}{2}m(m-1)+3m = 2m^2+m$ generators.

D. Subgroups of 0(4m,4m)

The doubling of $\mathcal{R}_{4m}$ to $\mathcal{R}_{4m,4m}$ produces new groups analogous to those in $\mathcal{R}_{2m,2m}$, but, of course, with quaternionic instead of complex structure. The analysis is similar to the previous case, so our discussion will be limited to describing the group invariants.

The general linear group with quaternionic structure $\mathrm{GL}(m,\mathcal{Q})$ is the invariance group of bivectors K, J_1, J_2, J_3 satisfying

$$K\times J_i = 0, \qquad J_1\times J_2 = 2J_3\,, \tag{6.33}$$

and

$$\underline{K}^2 = \underline{1}, \qquad \underline{J}^2 = -\underline{1}\,. \tag{6.34}$$

This bivector algebra has an m-dimensional representation in $\mathcal{R}^2_{4m,4m}$. The special linear group $\mathrm{SL}(m,\mathcal{Q}) \simeq \mathrm{Su}^*(2m)$ is obtained by eliminating the Abelian subgroup generated by K from $\mathrm{GL}(m,\mathcal{Q})$.

We now turn to the other new group structure obtained from doubling $\mathcal{R}_{4m}$ It differs from $\mathrm{GL}(m,\mathcal{Q})$ in the way that the quaternionic structure on the complementary spaces $\mathcal{R}_{4m}$ and $\overline{\mathcal{R}}_{4m}$ are linked by group elements. The complex skew norm in $2m$-dimensional complex space is represented in $\mathcal{R}_{4m}$ by

$$\epsilon(a,b) = (a\wedge b)\cdot J_1 + \mathbf{i}(a\wedge b)\cdot J_2\,, \tag{6.35}$$

with

$$J_1\times J_2 = 2J_3\,. \tag{6.36}$$

TABLE 1. The eight types of bilinear form and their groups.

Type	Form	Base space	Group
$\mathcal{R}$-symmetric	$\epsilon(a,b)=\epsilon(b,a)=a\cdot b$	$\mathcal{R}^{n}$	SO(n)
$\mathcal{R}$-skew	$\epsilon(a,b)=-\epsilon(b,a)=a\cdot \underline{J}b$	$\mathcal{R}^{2n,2n}$	Sp$(n,\mathcal{R})$
$\mathcal{C}$-symmetric	$\epsilon(a,b)=-\epsilon(b,a)=a\cdot b+\mathbf{i}(a\cdot \underline{K}_{*})b$	$\mathcal{R}^{n,n}$	SO$(n,\mathcal{C})$
$\mathcal{C}$-skew	$\epsilon(a,b)=-\epsilon(b,a)=a\cdot \underline{J}_1 b+\mathbf{i}(a\cdot \underline{J}_2 b)$	$\mathcal{R}^{4n,4n}$	Sp$(n,\mathcal{C})$
$\mathcal{C}$-Hermitian symmetric	$\epsilon(a,b)=\epsilon^{\dagger}(b,a)=a\cdot b+\mathbf{i}(a\cdot \underline{J}b)$	$\mathcal{R}^{2n}$	U(n)
$\mathcal{C}$-Hermitian skew	$\epsilon(a,b)=-\epsilon(b,a)=a\cdot \underline{J}b+\mathbf{i}(a\cdot b)$	$\mathcal{R}^{2n}$	U(n)
$\mathcal{Q}$-Hermitian symmetric	$\epsilon(a,b)=\epsilon^{\dagger}(b,a)=a\cdot b+(a\cdot \underline{J}_1 b)\mathbf{i}+(a\cdot \underline{J}_2 b)\mathbf{j}$ $+(a\cdot \underline{J}_3 b)\mathbf{k}$	$\mathcal{R}^{4n}$	U$(n,\mathcal{Q})$
$\mathcal{Q}$-Hermitian skew	$\epsilon(a,b)=-\epsilon^{\dagger}(b,a)=a\cdot \underline{J}b+(a\cdot b)\mathbf{i}+(a\cdot \underline{K}_{1*} b)\mathbf{j}$ $+(a\cdot \underline{K}_{2*} b)\mathbf{k}$	$\mathcal{R}^{4n,4n}$	Sk$(n,\mathcal{Q})$

To obtain its complete invariance group Sp$(m,\mathcal{C})$, we must double to $\mathcal{R}_{4m,4m}$ and find it as a subgroup of GL$(2m,\mathcal{C})$, just like we did to get the real symplectic group Sp$(m,\mathcal{R})$ for a single involutory bivector.

The quaternionic structure (6.36) can be preserved while the group structure generated by J_1, J_2, J_3 is broken by composing the *-operator (3.24) with J_1 and J_2 to get

$$\underline{J}_{1*}a=\underline{J}_1 a^{*}\,,\quad \underline{J}_{2*}a=\underline{J}_2 a^{*}\,. \tag{6.37}$$

From Eqs. (6.35) and (6.36) we get the operator relations

$$\underline{J}_{1*}^2=\underline{J}_{2*}^2=\underline{1}\,,\quad \underline{J}_3^2=-\underline{1}\,, \tag{6.38a}$$

$$[\underline{J}_{1*},\underline{J}_{2*}]=[\underline{J}_1,\underline{J}_2]=2\underline{J}_3\,. \tag{6.38b}$$

Sp$(m,\mathcal{C})$ is the invariance group of these relations. Note the similarity of these relations with Eqs. (6.21a) and (6.21b), where the * operator was used to generate quaternionic structure from the $\underline{K}_i$.

This completes our characterization of the major "classical groups." It includes all real forms of Cartan's series of complexified semisimple groups: $A_{n-1}\simeq$ SL$(n,\mathcal{C})$, $B_n\simeq$ SO$(2n+1,\mathcal{C})$, $C_n\simeq Sp(n,\mathcal{C})$, $D_n\simeq$ SO$(2n,\mathcal{C})$. The exceptional semisimple groups are also subgroups of Spin(n,n), but their invariants are not just pseudoscalars and/or bivectors.[16] However, that topic deserves a separate article. The classical classification of groups according to bilinear forms is given in Table 1 to summarize the results of this section. These groups are all subgroups of the general linear groups, which are, in turn, are subgroups of O(n,n), as summarized in Table II.

Vll. PROJECTIVE SPLITS AND OTHER FACTORS

The classical groups which we have discussed so far are all subgroups of GL$(n,\mathcal{R})$ which, in turn, is a subgroup of the *mother group* O(n,n). But there is more to the mother group than that. In fact, the mother algebra $\mathcal{R}_{n,n}$ embraces all of projective geometry and its group structure. The essential ideas and techniques to explicate this projective structure

TABLE II. The general linear groups as subgroups of $\mathrm{O}(n,n)$.

Group	Invariants
$\mathrm{O}(n,n)$	$a\cdot b$
$\mathrm{GL}(n,\mathcal{R})\, subset\, \mathrm{O}(n,n)$	K
$\mathrm{GL}(n,\mathcal{C})\, subset\, GL(2m,\mathcal{R})$	K,J
$\mathrm{GL}(n,\mathcal{Q})\, subset\, GL(4m,\mathcal{R})$	K,J_1,J_2,J_3

are laid out in Ref. 3, so we remark here only on how the present perspective generalizes and, perhaps, *perfects* the approach there.

Reference 3 explains the geometric meaning of two kinds of multiplicative splits (or factorizations) of geometric algebra. The first kind, a split with respect to a vector, was called a "projective split" there. For the Euclidean and anti-Euclidean algebras $\mathcal{R}_n$ and $\overline{\mathcal{R}}_n$ the projective split can be described as the decomposition

$$\mathcal{R}_{n+1} = \overline{\mathcal{R}}_n \bar{\otimes} \mathcal{R}_1, \qquad \overline{\mathcal{R}}_{n+1} = \mathcal{R}_n \bar{\otimes} \overline{\mathcal{R}}_1\,, \tag{7.1}$$

where $\bar{\otimes}$ means that vectors in the generating vector spaces of the factored algebras mutually anticommute. As explained in Ref. 3, the group structure associated with these factorizations is the "metric affine group." This group extends the orthogonal group to include translations. To get the full affine group, we need to generalize the orthogonal group to the general linear group, and we now know that the way to do this is to extend $\mathcal{R}_n$ to

$$\mathcal{R}_{n,n} = \mathcal{R}_n \bar{\otimes} \overline{\mathcal{R}}_n\,. \tag{7.2}$$

This entails a generalization of Eq. (7.1) to

$$\mathcal{R}_{n+1,n+1} = \overline{\mathcal{R}}_{n,n} \bar{\otimes} \mathcal{R}_{1,1}\,, \tag{7.3}$$

where the elements of $\mathcal{R}_{1,1}$ now commute with the elements of $\mathcal{R}_{n,n}$. The split (7.3) was called a "conformal split" in Ref. 3, because its invariance group is the conformal group on $\mathcal{R}^{n,n}$, but now we see it as a generalized projective split. In recognition of this synthesis, we propose, henceforth, to refer to Eq. (7.3) as a *projective split*, discarding the term "conformal split." Since the affine group is a subgroup of the conformal group which preserves the split

$$\overline{\mathcal{R}}_{1,1} = \mathcal{R}_1 \bar{\otimes} \overline{\mathcal{R}}_1\,, \tag{7.4}$$

it would be appropriate to call Eq. (7.1) an *affine split*.

A projective split is determined by a single two-blade K_1 with positive signature, say $K^2 = 1$. The "factor algebra" $\mathcal{R}_{1,1}$ is generated by all vectors which anticommute with K_1. To describe the projective split of $\mathcal{R}_{n,n}$ in more detail, we adopt an orthonormal basis $\{e_i, \bar{e}_i\}$ with "complementary" blades $K_i = e_i\bar{e}_i$. To split $\mathcal{R}_{n,n}$ with respect to K_1, we define a new basis

$$e_2' = e_2 K_1, \quad e_3' = e_3 K_1, \quad \ldots, \quad e_n' = e_n K_1\,. \tag{7.5}$$

Since these basis elements anticommute and have unit square, we can regard them as vectors generating a Euclidean algebra $\mathcal{R}_{n-1}$ which commutes with $\mathcal{R}_{1,1}$. The complementary vectors $\overline{e}'_i = e_i K_i$ generate the corresponding anti-Euclidean algebra $\overline{\mathcal{R}}_{n-1}$ Thus we obtain an explicit projective split of $\mathcal{R}_{n,n}$

This process can, of course, be repeated to express $\mathcal{R}_{n,n}$ as an n-fold product of commuting $\mathcal{R}_{1,1}$ algebras. Also, similar splits can be made with respect to two-blades with negative signature. We cannot analyze, here, the rich group structure associated with the various splits. Our aim is only to call attention to the possibilities.

ACKNOWLEDGMENTS

D. H. and F. S. were partially supported by a NATO Grant. C.D. is supported by a SERC studentship.

[1] H. Georgi, *Lie Algebras in Particle Physics* (Benjamin/Cummings, Reading, 1982).

[2] D. Hestenes, "A Unified Language for Mathematics and Physics," in *Clifford Algebras and their Applications in Mathematical Physics*, edited by J. S. R. Chisholm and A. K. Common (Reidel, Dordrecht/Boston, 1986), pp. 1–23.

[3] D. Hestenes, *Acta Appl. Math.* **23**, 65 (1991).

[4] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus, A Unified Language for Mathematics and Physics* (Reidel, Dordrecht/Boston, 1984).

[5] D. Hestenes and R. Ziegler, *Acta Appl. Math.* **23**, 25 (1991).

[6] D. Hestenes "Mathematical Viruses," in *Proceddings of the Second International Conference on Clifford Algebras and their Applications in Mathematical Physics*, edited by A. Micali (Kluwer, Dordrecht/Boston, 1991), pp. 3–16.

[7] H. Grassmann, *Math. Commun.* **12**, 375 (1877).

[8] W. Clifford, *Am. J. Math.* **1**, 350 (1878); *Mathematical Papers* (1882), pp. 266–276.

[9] A. Crumeyrolle *Orthogonal and Symplectic Clifford Algebras*, (Kluwer, Dordrecht/Boston, 1991).

[10] E. Caianello, *Combinatorics and Renormalization in Quantum Field Theory* (Benjamin, Reading, MA, 1973).

[11] H. Weyl, *The Classical Groups* (Princeton University, Princeton, 1939).

[12] F. Sommen and N. Van Acker, *Found. Phys.* (in press, 1993).

[13] R. Delanghe, F. Sommen, and V. Soucek, *Clifford Algebra and Spinor-Valued Functions* (Kluwer, Dordrecht/Boston, 1992).

[14] A. Lasenby, C. Doran, and S. Gull, *J. Math. Phys.* **34**, 3683 (1993).

[15] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York, 1974).

[16] R. Ablamowicz, P. Lounesto, and J. Maks, *Found. Phys.* **21**, 735 (1991).