In: Neural Networks, Vol. **7** No. 1(© Elsevier Science Ltd, 1994), 79–88.

Invariant Body Kinematics:

II. Reaching and Neurogeometry

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Abstract. Invariant methods for formulating and analying the mechanics of the skeleto-muscular system with geometric algebra are further developed and applied to reaching kinematics. This work is set in the context of a neurogeometry research program to develop a coherent mathematical theory of neural sensory-motor control systems.

1. INTRODUCTION

When a primate reaches for an object, the trajectory of the hand is very nearly a straight line with a bell-shaped velocity profile, unless such motion is impeded by external constraints (Morasso, 1981). In contrast, the profiles of joint variables involved in reaching do not display comparable simplicity (Soechting & Lacquanti, 1981). This suggests that kinematics of the hand plays a primary role in the neural planning and control of reaching movements. More specifically, because a straight line trajectory is uniquely determined by its endpoints, it suggests that the hand position end points are *control variables* for reaching movements. This idea has been incorporated in the VITE model of Bullock and Grossberg (1988), which accounts for an impressive range of empirical data, despite its simplicity.

Most studies of reaching motion have constrained the allowed arm movements to two degrees of freedom, in part to avoid the daunting kinematics of unconstrained hand-arm movements. This article aims to show that arm kinematics is not so daunting when expressed in terms of geometric algebra. A completely general and *invariant* formulation of reaching kinematics is presented. This includes a solution of the *inverse kinematics problem* that parametrizes the joint variables in terms of the wrist position endpoints. It should be helpful for generalizing the VITE model to apply to the full kinematic range of arm movements. But, it is equally applicable to the kinematics of any theory of arm-hand movement.

To achieve an invariant formulation of reaching kinematics, we employ the *geometric algebra* expounded and applied to eye-head kinematics in a companion article (Hestenes, 1993). Familiarity with the definitions, techniques, and results in that article is an essential prerequisite to the developments below, so it will be taken for granted henceforth without further comment. As noted there, the formulation in terms of geometric algebra is *invariant* in the sense of being coordinate-free. Everything is expressed in terms of invariants expressing the three-dimensional (3D)-Euclidean structure of physical space and the rigid body constraints imposed by the skeletal system.

The invariant formulation provides a complete and irreducible description of the computational problem that must be solved to achieve kinematic control. Moreover, it suggests computational devices for solving that problem most efficiently. Thus, it provides a well-defined theoretical framework for analyzing how nature has solved the problem.

The invariant treatment of reaching and oculomotor kinematics in this article and its companion generalize without difficulty to kinematic modeling of the entire skeleto-muscular system. The essential mathematical apparatus and special techniques are fully developed in these articles. However, there are generalizations of geometric algebra that may prove to facilitate the treatment of complex kinematics. One of them is described in an appendix.

Although these articles are directed at putting geometric algebra at the service of computational neuroscience, it will be recognized that the mathematical techniques and analysis are of equal value to robotics. In robotics the problem is to design movement control systems, and in neuroscience the problem is to figure out how neural control systems work. Because geometric algebra is so efficient computationally, direct implementations of it in robotic control systems may optimize robotic design.

The concluding section of this article places kinematic analysis in the context of a coherent neuroscience research program aimed at discovering the functional geometry of biological sensory-motor systems. This has been an explicit research program for more than a decade (Pellionisz & Llinas, 1980), and it has been dubbed *neurogeometry* by Andras Pellionisz (Pellionisz & Llinas, 1985), an outspoken advocate for theoretical neurogeometry to systematize, interpret, and explain empirical findings as well as to guide further investigations. The field of neurogeometry is only beginning to take shape. Its immaturity is evident in a lack of synergy among theoretical efforts in the field. There is no lack of empirical fodder for theoretical rumination, however. Indeed, the burgeoning store of data garnered by experimental neuroscience already threatens to become unmanageable for lack of adequate theory. So there is a genuine need to clarify the scientific issues and coordinate efforts in theoretical neuroscience.

2. FORWARD KINEMATICS

To determine the change in arm configuration (or arm position) due to a specified change in joint angles is called a forward kinematics problem in robotics (Spong & Vidyasagar, 1989). The arm configuration can be designated by a set of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as illustrated and explained in Figure 1. We will be concerned only with arm motion relative to the trunk of the body. In other words, we take the trunk as our frame of reference and so regard the position of the shoulder as a fixed point. However, the generalization to arbitrary motions of the trunk does not require any changes in our analysis.

Changes in the *shoulder*, *elbow*, and *wrist* joints are all rotations, so they can be characterized by attitude spinors A, B, and C, respectively. Accordingly, a change in elbow can be described by the equation

$$\mathbf{a} = A\mathbf{a}_0A^{-1}\,. \tag{1}$$

From a fixed elbow position, bending the elbow produces a rotation of the wrist position described by

$$\mathbf{b}' = B\mathbf{b}_0B^{-1}\,. \tag{2}$$

with $B = B(t)$ satisfying $B(0) = 1$. Simultaneous rotation at the shoulder produces the composite rotation

$$\mathbf{b} = AB\mathbf{b}_0B^{-1}A^{-1}\,. \tag{3}$$

The trajectory $\mathbf{r} = \mathbf{r}(t)$ of the wrist is therefore given by

$$\mathbf{r} = \mathbf{a} + \mathbf{b} = A(\mathbf{a}_0 + B\mathbf{b}_0 B^{-1})A^{-1} . \tag{4}$$

This solves the forward kinematics problem for wrist motion, as it expresses the wrist position as a function of the joint attitude spinors A and B.

For a fixed wrist position, a wrist rotation takes an arbitrary point c_0 in the hand to a new relative position

$$c' = C\mathbf{c}_0 C^{-1} , \tag{5}$$

If $\mathbf{a} = \mathbf{a}(t)$ is the trajectory of the elbow with initial position $\mathbf{a}_0 = \mathbf{a}(0)$, then $A = A(t)$ can be taken to satisfy the initial condition $A(0) = 1.$ with $C(0) = 1.$ We do not consider here the independent motions of fingers and thumb, as in grasping. The hand is taken to be in some rigid configuration that can be rotated about the wrist. When the wrist rotation (5) is combined with elbow and shoulder rotations,
the result is

$$\mathbf{c} = W\mathbf{C}_0 W^{-1} , \tag{6a}$$

where the spinor

$$W = ABC \tag{6b}$$

completely describes the attitude of the hand with respect to the trunk. With respect to the trunk, the position $\mathbf{f}$ of any point in the hand (say, a fingertip) is accordingly described in complete generality by

$$\mathbf{f} = \mathbf{a} + \mathbf{b} + \mathbf{c}$$
$$= A[\,\mathbf{a}_0 + B(\mathbf{b}_0 + C\mathbf{c}_0 C^{-1})B^{-1}\,]A^{-1} . \tag{7}$$

This completes the general solution of the forward kinematics problem for arm motion.

The general *reaching equation* (7) describes the finite displacement from an arbitrary initial arm configuration $\{\mathbf{a}_0, \mathbf{b}_0, \mathbf{c}_0\}$ to an arbitrary final configuration $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ as a function of the changes in joint attitude described by spinors $\{A, B, C\}$. A great advantage of this formulation is that it incorporates the rigid body constraints of the skeletal system in a completely coordinate-free fashion. To be sure, the arm has an intrinsic coordinate system determined by the lengths and attachments of muscles, and ultimately any arm motion must be expressed in terms of these *muscle coordinates*.

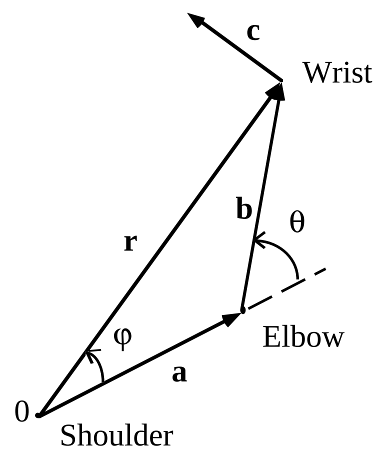

FIGURE 1. Arm configuration or posture is described by a set of relative position vectors **a**,**b**,**c** relating the joints. The position of a joint is defined to be its center of rotation. The vector **a** designates the position of the elbow relative to the shoulder; **b** designates the position of the wrist relative to the elbow; **c** designates the position of any point in the hand relative to the wrist.

Moreover, geometric algebra can be a great help in modeling the action of muscle coordinates by explicit parametrizations of the joint spinors A, B, C. However, muscle coordinates are certainly not appropriate variables for the planning and control of movements, so the coordinate-free formulation simplifies the task of analyzing alternative parametrizations by avoiding the complexities of muscle coordinates. Limitations on the range of motion imposed by muscle and joint structure can easily be expressed as restrictions on the allowed values for the spinors A, B, C without reference to muscle coordinates, but that is a detail that need not concern us here.

3. INVERSE KINEMATICS

A reaching movement takes the wrist from an initial position $\mathbf{r}_0 = \mathbf{a}_0 + \mathbf{b}_0$ to a target position $\mathbf{r} = \mathbf{a} + \mathbf{b}$. The inverse kinematics problem in this case is to solve for the joint movement spinors A and B in (2.4) in terms of the initial and final arm configurations.

The solution is to be expressed in terms of the most *behaviorally relevant* variables, namely, the *target direction* $\hat{\mathbf{r}} = \mathbf{r}/r$ and the *arm extension* $r = |\,\mathbf{r}\,|$. To apply to pointing gestures and target objects out of reach, the arm extension need not be identified with target distance. Note that for reaching with a rigid wrist, eqn (7) has $C = 1$ and takes the same form as eqn (4) with $\mathbf{b}_0$ replaced by a longer vector $\mathbf{b}_0 + \mathbf{c}_0$, so the analysis will be the same if $\mathbf{r}$ is regarded as a point in the hand instead of wrist position.

The shoulder joint has three degrees of freedom, while the elbow has two, but only the one affecting wrist position is relevant here. The target position $\mathbf{r}$ determines only three of these four degrees of freedom. The remaining parameter specifies the relative position of the elbow. Geometrically, a reaching movement can be described as an (internal) change of shape of the triangle in Figure 1 composed with an (external) rigid rotation about the shoulder vertex. Let us analyze the internal movement first.

The shape of the triangle in Figure 1 is completely determined by the lengths of its sides $r = |\,\mathbf{r}\,|$, $a = |\,\mathbf{a}\,|$, $b = |\,\mathbf{b}\,|$. Alternatively, the angles θ or φ can be employed. These angles and the unit bivector $\mathbf{i}$ for the plane are defined algebraically by the products

$$\hat{\mathbf{b}}\hat{\mathbf{a}} = e^{-\mathbf{i}\theta}\,, \tag{8}$$

$$\hat{\mathbf{r}}\hat{\mathbf{a}} = e^{-\mathbf{i}\varphi}\,, \tag{9}$$

where $\hat{\mathbf{a}} = \mathbf{a}/a$ and $\hat{\mathbf{b}} = \mathbf{b}/b$ are unit vectors. Later, we will want $\mathbf{i}$ expressed as the dual

$$\mathbf{i} = i\mathbf{e}_2\,, \tag{10}$$

where i, as always, is the unit pseudoscalar, and $\mathbf{e}_2$ is the direction of the elbow joint axis that is, of course, normal to the plane. The identification of $\mathbf{e}_2$ with elbow axis can be expressed algebraically by solving eqn (8) for

$$\mathbf{e}_2 = \frac{\mathbf{a}\times\mathbf{b}}{|\,\mathbf{a}\times\mathbf{b}\,|}\,. \tag{11}$$

The right side of this equation is undefined when the arm is fully extended so $\mathbf{a}\times\mathbf{b} = 0$, but $\mathbf{e}_2$ is well-defined nevertheless by the joint attitude spinor A. The sign in the exponential in eqn (8) has been chosen so that the *elbow rotation* from the extended arm position

$$\hat{\mathbf{b}} = e^{-\mathbf{i}\theta}\hat{\mathbf{a}} = \hat{\mathbf{a}}e^{\mathbf{i}\theta} \tag{12}$$

is a right-handed rotation about the axis $\mathbf{e}_2$ for positive θ. The *joint angle* θ is limited to the range $0 \le \theta < \pi$.

Angles and sides of the triangle are related by the constraint $\mathbf{r} = \mathbf{a} + \mathbf{b}$, from which, using eqns (8) and (9) we obtain

$$\mathbf{r}\hat{\mathbf{a}} = re^{-\mathbf{i}\varphi} = a + be^{-\mathbf{i}\theta} \,. \tag{13}$$

Note that this relation among the three alternative variables $\mathbf{r}$, θ, φ is independent of how the triangle is positioned in space. Thus, it provides an intrinsic description of triangle shape.

We can easily solve eqn (13) to get the joint angle θ as a function of arm extension r. However, we know that it is better to describe the elbow joint by a spinor E defined by

$$\hat{\mathbf{b}} = E\hat{\mathbf{a}}E^{\dagger} = E^2\hat{\mathbf{a}} \,. \tag{14}$$

so comparison with eqn (12) gives

$$E = e^{-\mathbf{i}\theta/2} = \alpha - \mathbf{i}\beta \doteq 1 + \hat{\mathbf{b}}\hat{\mathbf{a}} \,. \tag{15}$$

To express E as a function of r, we solve for scalars α and β as functions of r. Using the normalization

$$EE^{\dagger} = \alpha^2 + \beta^2 = 1 \tag{16}$$

from eqn (13), we obtain

$$r^2 = (a + bE^2)(a + bE^{-2}) = (a-b)^2 + 4ab\alpha^2 = (a+b)^2 - 4ab\beta^2 \,. \tag{17}$$

The variable r is confined to the range $r_- < r \le r_+$, where

$$r_{\pm} = |\, a \pm b \,| \,. \tag{18}$$

Hence,

$$4ab = r_+^2 - r_-^2 , \qquad 2(a^2 + b^2) = r_+^2 + r_-^2 \,, \tag{19}$$

and eqn (17) can be written

$$\alpha^2 = \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \,, \tag{20a}$$

$$\beta^2 = \frac{r_+^2 - r^2}{r_+^2 - r_-^2} \,. \tag{20b}$$

Inserted into eqn (15), this gives the desired function

$$E(r) = \alpha - \mathbf{i}\beta \doteq (r^2 - r_-^2)^{1/2} - \mathbf{i}\,(r_+^2 - r^2)^{1/2} \,, \tag{21}$$

where the constant normalizing factor in eqns (20a,b) has been dropped on the right. Even so, the unnormalized spinor determines the r-dependence of the elbow joint angle θ by giving

$$\tan \tfrac{1}{2}\theta = \frac{\beta}{\alpha} = \left[\frac{r_+^2 - r^2}{r^2 - r_-^2}\right]^{\frac{1}{2}} \,. \tag{22}$$

The elbow joint angle θ, or better, the elbow spinor E correspond to muscle commands for holding the elbow in a particular posture. Accordingly, eqns (21) or (22) describe neural computations necessary to determine the muscle commands for a specified arm extension r. However, for computational efficiency the parameters α and β are more suitable for neural encoding than r itself. First, note that eqns (20a) or (20b) can be regarded as a rescaling of the variable r from the interval $[r_-, r_+]$ to a new variable α or β on the interval $[0, 1]$. Second, note that if both α and β are computed then they need not be normalized, because, as implied by eqn (22), their ratio determines the muscle command.

Comparing eqn (14) with eqn (2), we find that the spinor B describes a change in elbow extension from r_0 to r is given by

$$B = EE_0^\dagger = \alpha\alpha_0 + \beta\beta_0 - \mathbf{i}(\beta\alpha_0 - \alpha\beta_0)\,, \tag{23}$$

where $E = E(r)$ and $E_0 = E(r_0) = \alpha_0 - \mathbf{i}\beta_0$. This is a general result describing any (internal) change in the shape of the arm triangle in Figure 1.

The next task is to describe the (external) repositioning of the arm in space. First, to ascertain how changes in arm extension r are coupled to changes in target direction $\hat{\mathbf{r}}$, we examine movements confined to the $\mathbf{i}$-plane, that is, the plane in which the wrist, elbow, and shoulder lie. The relation of the wrist direction $\hat{\mathbf{r}}$ to the elbow direction $\hat{\mathbf{a}}$ can be described by a spinor defined by

$$\hat{\mathbf{r}} = U\hat{\mathbf{a}}U^\dagger = U^2\hat{\mathbf{a}}\,. \tag{24}$$

Comparison with eqns (13) and (15) shows that

$$U^2 = e^{-i\varphi} = \frac{1}{r}\,(a + bE^2)\,. \tag{25}$$

Whence, with the help of eqns (21) and (19), we obtain U as an explicit function of the arm extension r:

$$\begin{aligned} U = e^{-\mathbf{i}\varphi/2} &\doteq 1 + U^2 \doteq r + a + bE^2 \\ &\doteq 4a(r+a) + (r^2 - r_-^2)^{1/2} - \mathbf{i}(r_+^2 - r^2)^{1/2}\,. \end{aligned} \tag{26}$$

where inessential normalizing factors have been dropped in the alternative forms for U.

For a change in arm extension from r to r_0, the relative changes in wrist and elbow directions can be described by

$$\hat{\mathbf{r}}\hat{\mathbf{r}}_0 = U^2\hat{\mathbf{a}}U_0^2\hat{\mathbf{a}}_0 = U^2U_0^{-2}\hat{\mathbf{a}}\hat{\mathbf{a}}_0\,, \tag{27}$$

where the last equality depends on the assumption that initial and final arm configurations lie in the same plane. Now, if the movement in question is simply an elbow flexion, then $\hat{\mathbf{a}} = \hat{\mathbf{a}}_0$ and the change in wrist direction is given by

$$\hat{\mathbf{r}}\hat{\mathbf{r}}_0 = U^2U_0^{-2}\,. \tag{28}$$

However, we are more interested in an arm extension along a straight line, in which case $\hat{\mathbf{r}} = \hat{\mathbf{r}}_0$ and eqn (27) yields

$$\hat{\mathbf{a}}\hat{\mathbf{a}}_0 = U_0^2U^{-2}\,. \tag{29}$$

Comparing this with eqn (1) and writing $A = A_0$ for this special case gives us

$$A_0 = (\hat{\mathbf{a}}\hat{\mathbf{a}}_0)^{1/2} \doteq 1 + U_0^2 U^{-2} \doteq r_0/r + (a + bE_0^2)(a + bE^2)^{-1} . \tag{30}$$

This spinor describes the compensatory shoulder rotation that must accompany an elbow flexion to constrain the wrist motion to a straight line. Of course, it can be expressed as an explicit function of r by using eqn (21).

The expressions (23) and (30) for the elbow and shoulder movement spinors B and A_0 constitute the general solution of the inverse kinematics problem for straight line arm extension controlled by the single parameter r. The next step is to solve the problem for directional control by $\hat{\mathbf{r}}$.

The attitude of the arm in space can be described by the direction of the wrist $\hat{\mathbf{r}}$ and the elbow axis $\mathbf{e}_2$. These two vectors are necessarily orthogonal, so they determine a right-handed orthonormal frame $\{\mathbf{e}_k\}$ defined, for $k = 1, 2, 3$, by

$$\mathbf{e}_1 = \hat{\mathbf{r}}, \qquad \mathbf{e}_2, \qquad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 = i\mathbf{e}_2\mathbf{e}_1 . \tag{31}$$

For an arbitrary arm movement, these are time-dependent vector-valued functions $\mathbf{e}_k = \mathbf{e}_k(t)$. Let their initial values be

$$\boldsymbol{\sigma}_1 = \mathbf{e}_1(0) = \hat{\mathbf{r}}_0, \qquad \boldsymbol{\sigma}_2 = \mathbf{e}_2(0), \qquad \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2 = i\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 . \tag{32}$$

Then the movement can be described by an attitude spinor $S = S(t)$ determined by the equations

$$\mathbf{e}_k = S\boldsymbol{\sigma}_k S^\dagger . \tag{33}$$

These equations can be solved for S, with the result (Hestenes, 1986)

$$S \doteq 1 + \mathbf{e}_1\boldsymbol{\sigma}_1 + \mathbf{e}_2\boldsymbol{\sigma}_2 + \mathbf{e}_3\boldsymbol{\sigma}_3 . \tag{34}$$

Furthermore, S can be factored into the product

$$S = RT , \tag{35a}$$

where the spinor R is *defined* by

$$R = 1 + \mathbf{e}_1\boldsymbol{\sigma}_1 = 1 + \hat{\mathbf{r}}\hat{\mathbf{r}}_0 , \tag{35b}$$

and, after some nontrivial algebra, T can be expressed in the form

$$T = 1 + \mathbf{e}_1 \cdot \boldsymbol{\sigma}_1 + \mathbf{e}_2 \cdot \boldsymbol{\sigma}_2 + \mathbf{e}_3 \cdot \boldsymbol{\sigma}_3 - i\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2 \times \mathbf{e}_2) \cdot (\boldsymbol{\sigma}_1 + \mathbf{e}_1) . \tag{35c}$$

The change in wrist direction is completely determined by R, for $T\boldsymbol{\sigma}_1 T^\dagger = \boldsymbol{\sigma}_1$, so

$$\mathbf{e}_1 = \hat{\mathbf{r}} = S\hat{\mathbf{r}}_0 S^\dagger = R\hat{\mathbf{r}}_0 R^\dagger = R^2\hat{\mathbf{r}}_0 . \tag{36}$$

The spinor T describes a rotation of the elbow about the initial wrist direction $\boldsymbol{\sigma}_1 = \hat{\mathbf{r}}_0$. By analogy to the description of eye movement, let us refer to such a movement as reaching *torsion*. Torsion does not affect the wrist position, but it does affect the positioning of the hand for grasping. Accordingly, the factoring of S into R and T in eqn (35a) can be interpreted as a factoring of arm movement into reaching and grasping synergies. This mathematical representation should be helpful in empirical studies on the coupling of reaching and grasping movements. For example, eqn (35c) tells us immediately that the condition

$$(\boldsymbol{\sigma}_2 \times \mathbf{e}_2) \cdot (\boldsymbol{\sigma}_1 + \mathbf{e}_1) = 0 \tag{37}$$

is necessary and sufficient for pure reaching, so the empirical conditions under which it is fulfilled are worth studying. It might be expected to hold in pointing movements, for example. In combined reaching and grasping movements, it would be of interest to compare the temporal developments of R and T. There may be general neural rules governing torsion in reaching, just as there are in eye movement (Listing's law), but they will undoubtedly depend on what movement synergies are activated. Lacquanti and Soechting (1982) have already found invariants coupling shoulder and elbow movement in experimental studies of reaching.

The pieces can now be assembled to give the general solution to the inverse kinematics problem for reaching. For *any arm movement* from one posture to another, as described by eqn (4), the elbow flexion is described by the spinor $B = B(r)$ in eqn (23) and the shoulder rotation is described by the spinor

$$A = SA_0 = RTA_0\,, \tag{38}$$

where $A_0 = A_0(r)$ is given by eqn (30) and S, R, T are given by eqns (35a,b,c). This solution is expressed as a function of the target wrist position $\mathbf{r}$ factored into distance control expressed by $A_0 = A_0(r)$, $B = B(r)$ and direction control expressed by $R = R(\hat{\mathbf{r}})$. This factorization is of interest if (or when) the nervous system employs $r = |\,\mathbf{r}\,|$ and $\hat{\mathbf{r}}$ as movement control variables, and it should be helpful for studying that possibility experimentally. However, if the nervous system employs different control variables, geometric algebra should be just as helpful for analyzing their relation to the kinematics.

4. TRAJECTORY DESCRIPTION

In a reaching movement from an initial position $\mathbf{r}_0 = \mathbf{r}(t_0)$ to a (final) target position $\mathbf{r}_f = \mathbf{r}(t_f)$, the wrist follows a trajectory $\mathbf{r} = \mathbf{r}(t)$ where t is the time or any other convenient time parameter in the interval $[\,t_0, t_f\,]$ (Figure 2). The solution of the inverse problem in Section 3 determines the joint spinors A, B, and A_0 for each value of $\mathbf{r}(t)$ on the trajectory. The problem remains to describe the trajectory in terms of appropriate control parameters. The ultimate aim, of course, is to discover the control parameters employed by the central nervous system. This can be facilitated by comparing alternative descriptions of trajectory kinematics with experimental data. Two such alternatives will be considered here: first, the factorization of position into distance and direction; second, the factorization of velocity into speed and direction.

The solution of the inverse kinematics problem in Section 3 suggests that factorization of wrist position $\mathbf{r}$ into extension (or distance) $r = |\,\mathbf{r}\,|$ and direction $\hat{\mathbf{r}} = \mathbf{n}$ may be optimal for computational purposes, because control of $r = r(t)$ requires coordination of elbow and shoulder joints, but control of $\mathbf{n} = \mathbf{n}(t)$ involves only the shoulder. Accordingly, we write

$$\mathbf{r} = r\mathbf{n}\,. \tag{39}$$

In terms of the independent variables r and $\mathbf{n}$, the velocity $\mathbf{v}$ of the wrist is given by

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{n} + r\dot{\mathbf{n}}\,, \tag{40}$$

where the overdot signifies time derivative. The relation of $\mathbf{n}$ to $\mathbf{n}_0$ can be described by an angle ψ defined algebraically by

$$\mathbf{n} = \mathbf{n}_0 e^{\mathbf{I}\psi}, \tag{41}$$

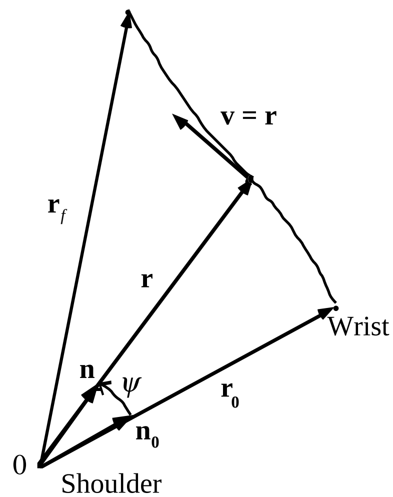

FIGURE 2. A *wrist trajectory* $\mathbf{r}=\mathbf{r}(t)$ with velocity $\mathbf{v} = \dot{\mathbf{r}}$.

where $\mathbf{I}$ is the unit bivector for the plane containing $\mathbf{n}$ and $\mathbf{n}_0$. If the wrist trajectory lies in a plane, then $\dot{\mathbf{n}}$ is constant and differentiation of eqn (41) gives

$$\dot{\mathbf{n}} = \mathbf{n}\mathbf{I}\dot{\psi} = -\dot{\psi}\,\mathbf{I}\mathbf{n}\,. \tag{42}$$

Then eqn (40) can be expressed in the form of a complex variable:

$$\mathbf{v}\mathbf{n} = \dot{r} + r\dot{\mathbf{n}}\mathbf{n} = \dot{r} - \mathbf{I}r\dot{\psi}\,. \tag{43}$$

Its modulus is

$$\mathbf{v}^2 = v^2 = \dot{r}^2 + r^2\dot{\mathbf{n}}^2 = \dot{r}^2 + r^2\dot{\psi}^2\,, \tag{44}$$

where $v = |\,\mathbf{v}\,|$ is the *speed* of the wrist. Of course, the last equality in eqns (43) and (44) applies only to planar trajectories. The elementary relation (44) should be compared with experimental data for possible evidence of independent control of r and $\mathbf{n}$.

Morasso's (1981) original finding that reaching trajectories are nearly straight lines with bell-shaped velocity profiles has been confirmed, extended, and refined by a number of investigators (Abend, Bizzi, & Morasso, 1982: Atkeson & Hollerbach, 1985: Uno, Kawato & Suzuki, 1989). A minimal neural network model for generating such trajectories, the VITE model, has been developed by Bullock and Grossberg (1988). It is a purely kinematic model with endpoint control of velocity direction and independent speed control by a *Go signal*. It is currently the most viable model of movement control because it is the simplest, and it accounts for the widest range of behavioral and neural data. The VITE model has its limitations, however, as Bullock and Grossberg fully realize, but that is not of concern here. Rather, we examine implications of the VITE model for coordinating direction and distance control.

For radial trajectories, eqn (44) reduces to $v^2 = \dot{r}^2$ so $\dot{r}$ must have a bell-shaped profile. From eqn (13), r is related to the elbow angle θ

$$r^2 = a^2 + b^2 + 2ab\cos\theta\,. \tag{45}$$

Differentiating this, we find

$$\dot{r} = \frac{-\dot{\theta} a \sin\theta}{[\, a^2 + b^2 + ab\cos\theta\,]^{1/2}}\,, \tag{46}$$

which shows that $\dot{\theta}$ is certainly not bell-shaped when $\dot{r}$ is. lf a bell-shaped profile is the signature of a movement control parameter, as the VITE model suggests, then $\dot{r}$ rather than $\dot{\theta}$ is a candidate.

In the more general case, eqn (42) implies that if v and $\dot{r}$ are both bell-shaped, then $r\dot{\psi}$ must be bell-shaped as well. This raises some interesting questions, because $r\dot{\psi}$ combines distance and direction control variables. Any failure of perfect coordination between distance and direction control would produce deviations from straight line trajectories as well as anomalies in the $r\dot{\psi}$ profile. The issue here is the expression of synergy formation in the kinematics of complex movements. Of course, there are other causes for deviations from straight line motion such as uncompensated forces and workspace boundaries. It should be possible, however, to distinguish them experimentally from imperfect coordination of distance and direction control if, indeed, the issue is relevant to neural control.

Now we turn to the factorization of velocity into speed and direction and its kinematic implications. This is an old topic that was thoroughly analyzed by differential geometers more than a century ago. The aim here is to show how it can be simplified using geometric algebra. This should be of practical value because the factorization is often employed in the analysis of experimental data, and it may have theoretical significance.

The description of a smooth trajectory $r = r(t)$ can be decomposed into a description of the geometrical path traversed and the distance traversed along the path by introducing a path length variable $s = s(t)$. Accordingly, the velocity can be factored into

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{s}\mathbf{r}' = v\hat{\mathbf{v}}\,, \tag{47a}$$

where the speed is given by

$$v = |\,\dot{\mathbf{r}}\,| = \dot{s}\,, \tag{47b}$$

and the velocity direction is given by

$$\hat{\mathbf{v}} = \mathbf{r}' = \frac{d\mathbf{r}}{ds}\,, \tag{47c}$$

the prime denoting differentiation with respect to path length.

The geometry of the path can be described by specifying the derivatives of $\hat{\mathbf{v}}$ in the following systematic way. A path-dependent orthonormal frame of vectors $\{\mathbf{e}_k = \mathbf{e}_k(s),\quad k = 1,2,3\}$, called a *Frenet frame* is introduced by identifying $\mathbf{e}_1$ with the path tangent vector

$$\mathbf{e}_1 = \hat{\mathbf{v}} = \mathbf{r}'\,, \tag{48}$$

and defining the other vectors by the system of equations

$$\begin{aligned} \mathbf{e}_1' &= \kappa\,\mathbf{e}_2, \\ \mathbf{e}_2' &= -\kappa\,\mathbf{e}_1, +\tau\,\mathbf{e}_3, \\ \mathbf{e}_3' &= -\tau\,\mathbf{e}_2, \end{aligned} \tag{49}$$

where κ is a nonnegative scalar called the *curvature* of the path, and the scalar τ is called the *torsion*. By direct computation it can be shown that

$$\kappa = |\,\mathbf{r}''\,| = \frac{|\,\dot{\mathbf{r}} \times \ddot{\mathbf{r}}\,|}{|\,\dot{\mathbf{r}}\,|^3} = \frac{|\,\mathbf{v} \times \dot{\mathbf{v}}\,|}{|\,\mathbf{v}\,|^3}\,. \tag{50}$$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}'''}{|\,\mathbf{r}''\,|^3} = \frac{(\mathbf{v} \times \dot{\mathbf{v}})\cdot\ddot{\mathbf{v}}}{|\,\mathbf{v} \times \dot{\mathbf{v}}\,|^3}\,. \tag{51}$$

The eqns (49) are the famous *Frenet* equations of classical differential geometry (Goetz, 1970).

For a planar trajectory the torsion vanishes, and the geometrical shape of the path can be described by the curvature $\kappa = \kappa(t)$. From experimental data the curvature can be measured by evaluating the right side of (50). Then the trajectory can be described by exhibiting the speed and curvature profiles $v = v(t)$ and $\kappa = \kappa(t)$. This method for describing curved arm trajectories has been employed by Abend *et al.* (1982) and many other researchers, particularly in the study of complex movements such as handwriting.

The description of arm trajectories by velocity and curvature profiles is certainly convenient for experimental data analysis, but its theoretical significance is open to question. The basic theoretical question it raises is whether speed and velocity direction are subject to independent control by the CNS. The VITE model asserts that they are for straight line trajectories. However, for curved trajectories they cannot be independently controlled because they are coupled dynamically. This suggests an important possibility that does not seem to have been mentioned in the literature heretofore, namely, that the *tendency toward straight line trajectories in reaching is a consequence of attempts by the CNS to factor speed and direction control*. This idea may be helpful in generalizing the adaptive features of the VITE model.

Geometric algebra makes it possible to simplify the Frenet description of path geometry with a method we have already employed for a different purpose (Hestenes, 1992). The Frenet frame $\{e_k\}$ is completely determined by a spinor (quaternion)-valued function of the path length $F = F(s)$ through

$$\mathbf{e}_k = F\boldsymbol{\sigma}_k F^\dagger\,, \tag{52}$$

where F is normalized to $FF^\dagger = 1$ and, as before, $\{\boldsymbol{\sigma}_k\}$ is a conveniently chosen fixed orthonormal frame. The Frenet spinor must satisfy a differential equation of the form

$$F' = -\tfrac{1}{2} i\omega_D F\,, \tag{53}$$

so differentiation of (52) yields

$$\mathbf{e}_k' = \omega_D \times \mathbf{e}_k\,. \tag{54}$$

The rotational velocity vector ω_D for a Frenet frame is called the *Darboux vector*. To express it in terms of curvature and torsion, we solve the three eqns (54) to get

$$\omega_D = \frac{1}{2}\sum_{k=1}^{3} \mathbf{e}_k \times \mathbf{e}_k' = \frac{1}{2}\sum_{k=1}^{3} \mathbf{e}_k \times \mathbf{e}_k'\,. \tag{55}$$

Inserting eqn (49) into eqn (55) and using $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 = i\mathbf{e}_2\mathbf{e}_1$, we obtain

$$\omega_D = \kappa\,\mathbf{e}_3 + \tau\,\mathbf{e}_1 \tag{56}$$

or

$$\omega_D = \frac{\mathbf{v} \times \dot{\mathbf{v}}}{|\,\mathbf{v}\,|^3} + \frac{(\mathbf{v} \times \dot{\mathbf{v}}) \times (\mathbf{v} \times \ddot{\mathbf{v}})}{(\mathbf{v} \times \dot{\mathbf{v}})^2 |\,\mathbf{v}\,|} \,. \tag{57}$$

Substituting eqn (56) into eqn (53), we obtain

$$F' = -\tfrac{1}{2} i F(\kappa\,\mathbf{e}_3 + \tau\,\mathbf{e}_1)\,. \tag{58}$$

The advantage of this approach is that the set of three Frenet eqns (49) is reduced to a single spinor eqn (58). Integration of eqn (58) is trivial when $\tau = 0$ and κ is a specified function of path length. However, particular examples will not be discussed here. Our concern here is with general method.

This section has discussed two different factorizations of trajectory kinematics: the factorization of position into distance and direction, and the factorization of velocity into speed and direction. These factorizations may be valuable for analyzing experimental data by exposing regularities in the data that might otherwise go unnoticed. However, the fundamental theoretical question is whether one or both of these factorizations is employed by the CNS in movement control. No attempt was made to answer this question here; rather, the aim has been to sharpen the issue and the mathematical tools to be used in its analysis.

5. NEUROGEOMETRY OF MOTOR CONTROL

The invariant formulation of vertebrate body kinematics developed in the present article and its companion is intended as a contribution to a research program on the neurogeometry of sensory-motor control. Though the term neurogeometry suggests a certain approach to sensory-motor research. it is not a widely accepted term. To place the present work within the neurogeometry program, therefore, let me describe some of the most relevant ideas and issues as I see them. The account is necessarily; incomplete because neurogeometry is only a nascent theory. For the sake of brevity, the account is framed in a dogmatic mode without nuances or caveats.

A primary problem of neuroscience is to decipher the neural codes employed by the CNS. In the sensorymotor subsystem, the key to the neural code is kinematics. Body kinematics is the geometry of movement, and to achieve accurate motor control this geometrymust be expressed in the neural codes at every stage of sensory-motor processing. In other words, the *external geometry* of body movement must be expressed in an *internal geometry* of sensory-motor control. To characterize this internal geometry mathematically is the avowed purpose of neurogeometry.

Neurogeometry is thus a mathematical theory of sensory-motor control, with the name giving explicit recognition to the primacy of geometry (*qua* kinematics) in the theory. The development of motor control theory necessarily begins with a description of motor behavior. The first level of description is qualitative, as is evident in Rosenbaum's (1991) introduction to the subject. As the subject matured, it became increasingly quantitative (Jeannerod, 1988), and formal kinematics is increasingly employed. For a fully quantitative theory of motor control, a complete mathematical description of motor behavior is essential. A contention of the present article is that geometric algebra is the best available tool for this purpose.

A quantitative theory of motor control requires a quantitative model of what is controlled, namely, the vertebrate skeleto-muscular svstem. For kinematic purposes, this system can be accurately modeled as system of N-linked rigid bodies. Because each rigid body has three rotational and three translational degrees of freedom, the configuration space of the entire system has dimension $6N$. But if the linkages entail K constraints, then any possible configuration is represented by a point on a $(6N - K)$-dimensional kinematic manifold (or surface) in configuration space. Moreover, any body movement is a change in configuration, which can be described quantitatively as a curve on the kinematic manifold. This configuration space description of body kinematics is an alternative to the more direct physical space description employed in this article, but it can also be efficiently expressed in terms of geometric algebra. However the body kinematics is described, it is an essential prelude to neurogeometry.

Neurogeometry begins with a description of the motor neuron signals for kinematic states (postures and/or movements) of the body. The description must be geometrical because body kinematics is geometrical. A valid description should make it possible to decipher empirically measured motor neuron signals, to interpret them geometrically as commands to the muscles to produce particular kinematic states of the body.

Neurogeometry goes on to produce a neural network theory of sensory-motor control in which every processing stage has geometrical interpretation specifying its relation to body kinematics. Much neural network modeling that has already been published can be reinterpreted neurogeometrically. The most extensive and profound neural network theory of sensory-motor control has been developed by Grossberg and his coworkers, Bullock and Kuperstein. Their theory goes far beyond others in analyzing the implications of adaptive constraints and kinematic invariants. Moreover, it is compatible with the perspective of neurogeometry.

Neural modeling is done at several different levels of biological organization (Shepherd, 1990). The level at which a geometric interpretation is most appropriate could well be called the *psychophysical level*. At that level sensory-motor information is encoded in activity patterns across neuronal populations (Hestenes, 1991b). This is presumed in the following discussion.

5.1. PPC and Self-Calibration

When the body is maintaining a particular posture, the pattern of neural command signals to the muscles constitute a representation of posture called the *present position command* (PPC) by Bullock and Grossberg (1988). It has often been argued that, for rapid and accurate movement, the PPC must also be employed as an *efferent copy* (or corollary discharge) in internal computations. One reason for this is that afferent measurements of posture are too slow to track rapid movements, so the best available alternative is to employ the PPC predictively during movement. However, that raises a serious *self-calibration* problem, first subjected to a detailed analysis by Grossberg and Kuperstein (1989). They note that muscle length is a highly nonlinear function of the neural command signals and argue that this functional relation must be *linearized* by recalibration to compensate for inevitable changes in muscle performance due to injury, growth, and so on. However, they tacitly assume that muscle length is the preferred parametrization of posture, though their argument suffices to prove that muscle contraction can be linearly calibrated to any convenient parameter. Neurogeometry suggests alternatives. For example, our kinematic analysis of reaching suggests that arm extension is a behaviorally more significant variable than elbow angle (which corresponds closely to muscle length). Moreover, it is quite possible, if not likely, that the contractions of a given muscle have different calibrations depending on which movement synergy is engaged. Note that such adaptive rescaling changes the neural code or, as Pellionisz and Linas (1985) would say, changes the *metric* of the neurogeometry.

5.2. Kinematic Invariants and Motor Synergies

A *motor* (or *muscle*) *synergy* is a coordinated action of muscle groups to produce a single class of gestures. Synergy formation reduces the number of degrees of freedom among the muscles involved, and this is reflected in the existence of kinematic invariants of the gestures. For example, Listing's law describes an invariant of saccadic eye movements. As we have noted, such invariants characterize the constraints on a synergy and so provide clues about the neural variables controlling the synergy. Neurogeometry seeks to describe the neural control variables geometrically to make the connection with the kinematics of movement explicit. Synergy formation is a self-organization process that can be described geometrically as the formation of a kinematic manifold in configuration space. A gesture can then be described as a curve on that manifold, a curve with specified endpoints. According to neurogeometry, for given endpoints the curve is determined by the geometry of the manifold, in much the same way as particle histories are determined by spacetime geometry in Einstein's General Theory of Relativity. However, neurogeometry is not developed sufficiently to supply an equally complete mathematical formulation.

Bullock and Grossberg (1991) propose a kinematic invariant of reaching gestures called *position code invariance*, which asserts that the path of a given gesture is invariant under both speed and stiffness rescaling. This is to say that the same gesture can be performed at different speeds and muscle stiffness levels. As noted in the preceding section, the VITE model is designed to exhibit speed rescaling invariance. but that is difficult to achieve for curved trajectories. The outstanding feature of the VITE model is that the movement control variables are purely kinematical. To preserve this feature when generalizing the VITE model to accommodate independent muscle stiffness control. Grossberg and Bullock (1991) developed the FLETE model, which explains how stiffness rescaling invariance can be implemented in a biologically plausible neural network. Muscle stiffness control can be interpreted geometrically as sculpting a valley around the movement trajectory to stabilize it against perturbing external forces. A detailed review of experimental evidence for such trajectory stabilization is given by Bizzi and Mussa-Ivaldi (1990). The main idea in all this is that neural motor control can be described geometrically even under the influence of perturbing forces. Thus, in motor control, dynamics is reduced to kinematics!

These qualitative remarks serve only to provide a neurogeometric perspective for integrating the present work on body kinematics with the neural motor control theory of Bullock, Grossberg and Kuperstein. Their rich theory contains a mathematical formulation and analysis of many more important network principles and designs, such as designs for *invariant target maps*. The emphasis here is on the primacy of perceptual geometry and body kinematics for determining the constraints on neural network designs. More specifically, the present work is intended to demonstrate the value of geometric algebra for mathematical formulation and analysis of these design constraints.

It is enlightening to compare the neurogeometry research program with Newton's original program for developing mechanics (Hestenes, 1992). In the preface to his great *Principia*, Newton concluded that his approach to research can be reduced to this: from the motions of bodies (kinematics) infer the forces, and from the forces deduce the motions (dynamics). This iterative cycle of quantitative kinematical description coupled with dynamical modeling and testing has been employed by physicists for three centuries with incredible success to discover the fundamental forces of nature.

Similarly, the neurogeometry approach is to quantify the kinematics of body movement (including adaptive changes in kinematics) to ascertain constraints on motor control designs, then to develop neurally plausible models of sensory-motor control that can be tested empirically. The goal of neurogeometry can be described as deciphering the neural sensory-motor code. Analogously, Newton's goal could be described as deciphering the code of physical motions to discover hidden physical forces. Indeed, Galileo described the goal of science (philosophy) in precisely these terms (translation from Burtt, 1932):

"Philosophy is written in that great book which ever lies before our eyes—I mean the Universe—but we cannot understand it if we do not first learn the language and grasp the symbols in which it is written. This book is written in the mathematical language, and the symbols are triangles, circles, and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth."

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APPENDIX: EUCLIDEAN HYPERSPINORS FOR KINEMATIC CHAINS

As shown in several references, the geometric algebra employed in this article has rich generalizations to multilinear spaces and manifolds of arbitrary dimension. This appendix calls attention to one little-known generalization that may prove to be especially valuable for describing and analyzing complex body kinematics.

A sequence of rotations and translations is called a *kinematic chain* in robotic theory (Spong & Vidyasagar, 1989). The variable position of any point on the body can be described by a parametrized kinematic chain. Thus, the position vector $\mathbf{f}$ of a finger tip is described by a kinematic chain in eqn (7). We have noted how such a description is simplified by the spinor representation of rotations. Further simplification can be achieved by introducing a similar representation for translations.

Let $\mathbf{x}$ be the position vector for a generic point on a body in Euclidean 3-space. With geometric algebra, any *rigid displacement* f of the body can be written in the form

$$\mathbf{x} \to f(\mathbf{x}) = R(\mathbf{x} + \mathbf{a}_0)R^\dagger = R\mathbf{x}R^\dagger + \mathbf{a}\,, \tag{A.1}$$

expressing it as a *rotation* determined by a unimodular spinor R preceded by a translation by the vector $\mathbf{a}_0$ or followed by a translation by

$$\mathbf{a} = R\mathbf{a}_0R^\dagger\,. \tag{A.2}$$

The drawback with eqn (A.1) is that rotations combine multiplicatively and translations combine additively. Translations as well as rotations can be expressed multiplicatively by the following artifice.

We introduce a new algebraic entity ϵ, which has the null property

$$\epsilon^2 = 0 \tag{A.3}$$

and commutes with vectors, that is,

$$\epsilon\mathbf{a} = \mathbf{a}\epsilon \tag{A.4}$$

for every vector $\mathbf{a}$. The translation by $\mathbf{a}$ can be represented by a *hyperspinor* $T_\mathbf{a}$ defined by

$$T_\mathbf{a} = e^{\frac{1}{2}\mathbf{a}\epsilon} = 1 + \tfrac{1}{2}\mathbf{a}\epsilon\,. \tag{A.5}$$

Note that the power series expansion of the exponential function in eqn (A.5) is terminated by the fact that $\epsilon^2 = 0$.

The rigid displacement f defined by eqn (A.1) can now be represented more compactly by a hyperspinor F defined by

$$F = T_\mathbf{a}R = RT_{\mathbf{a}_0}\,. \tag{A.6}$$

Assuming that

$$\epsilon^\dagger = \epsilon \tag{A.7}$$

so

$$T_\mathbf{a}T_\mathbf{a}^\dagger = T_\mathbf{a}^2 = T_{2\mathbf{a}} = 1 + \epsilon\mathbf{a}\,, \tag{A.8}$$

the relation of f to F can be described by the equation

$$F(1 + \mathbf{x}\epsilon)F^\dagger = 1 + f(\mathbf{x})\epsilon\,. \tag{A.9}$$

That is all there is to it!

The advantage of using hyperspinors to represent rigid displacements is that the composition of arbitrary displacements is reduced to multiplication. Thus, for displacements represented by F_1 and F_2, the composite is given by

$$F_3 = F_2F_1\,. \tag{A.10}$$

Each F_i can be factored into a rotation followed by a translation, as expressed by

$$F_i = T_{\mathbf{a}_i} R_i \,. \tag{A.11}$$

Accordingly, from eqn (A.10) it follows easily that

$$R_3 = R_2 R_1 \tag{A.12a}$$

and

$$\mathbf{a}_3 = \mathbf{a}_2 + R_2 \mathbf{a}_1 R_2^\dagger \,. \tag{A.12b}$$

Thus, the translational and rotational factors of the composite transformation can be computed directly without referring to the action on a body point $\mathbf{x}$ as eqn (A.1) does.

The set of all rigid displacements form a mathematical group sometimes called the *Euclidean group*. Equation (A.9) establishes that hyperspinors defined by eqn (A.6) are *representations* of the Euclidean group. [Actually, the representations are double-valued, because eqn (A.9) is invariant under replacement of F by $-F$. The significance of this detail is explained in Section 5-3 of Hestenes (1986).] Equation (A.10) is the basis group property expressed in terms of hyperspinors. The special power of this representation comes from the structure of geometric algebra.

As an illustrative application to body kinematics, the kinematic chain determining the vector $\mathbf{f}$ in eqn (7) can now be represented by a hyperspinor

$$F = A T_{\mathbf{a}_0} B T_{\mathbf{b}_0} C T_{\mathbf{c}_0} \,. \tag{A.13}$$

It follows from eqn (A.9) that $\mathbf{f}$ is determined by

$$F F^\dagger = 1 + \mathbf{f}\epsilon \,, \tag{A.14}$$

from which eqn (7) can be derived. This suffices to show how any kinematic chain can be represented multiplicatively by hyperspinors and related to the vectorial form.

It remains to be seen how valuable hyperspinors will be for modeling complex body kinematics. Though some evidence has been noted in the preceding paper (Hestenes, 1992b) that the CNS may, in effect, employ spinors in its computations, the possibility that it also employs hyperspinors seems utterly remote. That should not gainsay, however, the value of hyperspinors for the mathematical description and analysis of motor behavior. Howeover, the efficiency of designs for robotic control might be optimized by incorporating hyperspinors.

The basic eqns (A.6) and (A.9) for the hyperspinor representation of the Euclidean group were derived by Hestenes (1991a) from a much more general mathematical content with a richer group structure—including reflections, inversions, and dilatations in spaces of arbitrary dimension and signature. It is shown there that the quantity in this appendix can be interpreted geometrically as a null vector in a space of higher dimension. Though all of that supplies a powerful mathematical perspective with many other applications, it is not clear that it can enhance the mathematical treatment of body kinematics.