In: Z. Oziewicz *et al* (eds.), **Spinors, Twistors, Clifford Algebras and Quantum Deformations**, Kluwer: Dordercht/Boston (1993), 203–214.

Hamiltonian Mechanics with Geometric Calculus

David Hestenes.

Abstract Hamiltonian mechanics is given an invariant formulation in terms of Geometric Calculus, a general differential and integral calculus with the structure of Clifford algebra. Advantages over formulations in terms of differential forms are explained.

1. INTRODUCTION.

In the recent renaissance of Analytic Mechanics, the calculus of differential forms has become the dominant mathematical language of practitioners. However, the physics community at large has been slow to adopt the language. This reluctance should not be attributed solely to the usual resistance of communities to innovation, for the calculus of forms has some serious deficiencies. For one thing, it does not articulate smoothly with vector calculus, and it is inferior to vector calculus for many applications to Newtonian mechanics. Another drawback is that the calculus of forms has accreted a veritable orgy of definitions and notations which make the preparation required to address even the simplest problems in mechanics inordinately excessive. This is evident, for example, in the pioneering textbook of Abraham and Marsden (1967), which provides nearly 200 pages of preparation before attacking any significant problem in mechanics. The same high ratio of formalism to results is characteristic of more recent books in the field, such as Libermann and Marle (1987). All this goes to show that the calculus of forms is not quite the right tool for mechanics.

Without denying that valuable insights have been gained with differential forms, the contention of this paper is that a better mathematical system is available for application to analytical mechanics; namely, the Geometric Calculus expounded by Hestenes and Sobczyk (1984, henceforth referred to as [GC]). In contrast to differential forms, this calculus includes and generalizes standard vector calculus with no need to change standard notation, and it has proven advantages in applications throughout Newtonian mechanics, most notably in rigid-body mechanics (Hestenes, 1985). Geometric Calculus also includes and generalizes the calculus of differential forms, as explained in [GC]. In particular, it embraces the quaternion theory of rotations and the entire theory of spinors, which are completely outside the purview of differential forms. This apparatus is crucial to the efficient development of rigid-body dynamics (Hestenes, 1985).

This paper shows how to employ Geometric Calculus in the formulation of Hamiltonian mechanics, though space limitations preclude the discussion of applications or advanced theory. However, the fundamentals are discussed in sufficient detail with supplementary references to make translation of standard results in symplectic geometry and Hamiltonian mechanics into the language of Geometric Calculus fairly straightforward.

2. VECTOR SPACE VERSION.

The reader is presumed to be familiar with Clifford algebra and Hamiltonian mechanics, but familiarity with [GC] will be needed for full comprehension of the ideas, as well as for their applications. Definitions, notations, and results from [GC] will be employed without explanation. Though Geometric Calculus makes a completely coordinate-free approach possible, it also facilitates computations with coordinates. Coordinates are employed here primarily to establish a relation to conventional formulations.

For a mechanical system described by coordinates $\{q_1, \dots, q_n\}$ and corresponding momenta $\{p_1, \dots, p_n\}$, we first define *configuration space* as an n-dimensional real vector space $\mathcal{R}^n$ spanned by an orthonormal basis $\{e_k\}$ with

$$e_j \cdot e_k = \tfrac{1}{2}(e_j e_k + e_k e_j) = \delta_{jk} \tag{1.1}$$

for $j, k = 1, 2, \dots, n$. The state of the system can then be represented by the pair of vectors

$$q = \sum_k q_k e_k, \quad p = \sum_k p_k e_k \,. \tag{1.2}$$

The vectors in configuration space generate a real Geometric Algebra, $\mathcal{R}_n = \mathcal{G}(\mathcal{R}^n)$, with geometric product

$$qp = q \cdot p + q \wedge p \,. \tag{1.3}$$

Differentiation with respect to vectors is defined in [GC, Chap.2] along with the necessary apparatus to perform computations without resorting to coordinates. However, it will suffice here to introduce the *vector derivative* ∂_q by specifying its relation to the coordinates:

$$\partial_q = \sum_k e_k \frac{\partial}{\partial q_k} \,. \tag{1.4}$$

Equation (1.2) can be solved to express the coordinates as *functions* of the vector q instead of as independent variables; thus

$$q_k = q_k(q) = q \cdot e_k \,. \tag{1.5}$$

Then the basis vectors e_k are given as gradients

$$e_k = \partial_q q_k \,. \tag{1.6}$$

The simple linear form (1.5) for the coordinate functions obtains only for orthogonal coordinates, but the general case is treated in [GC]. It should be noted, also, that the "inner product" in (1.1) and (1.5) has no physical significance as a "metric tensor." It is merely an algebraic mechanism for expressing functional relations. Among other things, it performs the role of *contraction* in the calculus of differential forms.

For a Hamiltonian, $H = H(q, p)$, Hamilton's equations of motion can be expressed in configuration space as the pair of equations

$$\dot{q} = \partial_p H \,, \tag{1.7}$$

$$\dot{p} = -\partial_q H \,. \tag{1.8}$$

Since p and q are independent variables, we can reduce this pair of coupled equations to a single equation in a space of higher dimension. However, to be useful, the extension to higher dimension must preserve the essential structure of Hamilton's equations in a way which facilitates computation. We now show how such a computationally efficient extension can be achieved with Geometric Calculus.

To that end, we define *momentum space* as an n-dimensional real vector space $\widetilde{R}^n$ spanned by an orthonormal basis $\{\widetilde{e}_k\}$ with

$$\widetilde{e}_j \cdot \widetilde{e}_k = \tfrac{1}{2}(\widetilde{e}_j\widetilde{e}_k + \widetilde{e}_k\widetilde{e}_j) = \delta_{jk}\,, \tag{1.9}$$

so the momentum of our mechanical system can be expressed as a vector

$$\widetilde{p} = \sum_k p_k \widetilde{e}_k. \tag{1.10}$$

Now we define *phase space* $\mathcal{R}^{2n}$ as the direct sum

$$\mathcal{R}^{2n} = \mathcal{R}^n \oplus \widetilde{R}^n. \tag{1.11}$$

This generates the *phase space* (geometric) *algebra* $\mathcal{R}_{2n} = \mathcal{G}(\mathcal{R}^{2n})$, which is completely defined by supplementing (1.1) and (1.9) with the orthogonality relations

$$e_j \cdot \widetilde{e}_k = \tfrac{1}{2}(e_j\widetilde{e}_k + \widetilde{e}_k e_j) = 0\,. \tag{1.12}$$

The *symplectic structure* of phase space is best described by introducing a *symplectic bivector*

$$J = \sum_k J_k \tag{1.13}$$

with component 2-blades

$$J_k = e_k\widetilde{e}_k = e_k \wedge \widetilde{e}_k\,. \tag{1.14}$$

The bivector J determines a unique pairing of directions in configuration space with directions in momentum space, as expressed by

$$\widetilde{e}_k = e_k \cdot J = e_k \cdot J_k = e_k J_k = -J_k e_k, \tag{1.15}$$

$$e_k = J \cdot \widetilde{e}_k = J_k \cdot \widetilde{e}_k = J_k\widetilde{e}_k = -\widetilde{e}_k J_k. \tag{1.16}$$

Each blade J_k pairs a coordinate q_k with its corresponding momentum p_k. Moreover, since each J_k satisfies

$$J_k^2 = -1, \tag{1.17}$$

it functions as a "unit imaginary" relating q_k to p_k. Thus, the bivector J determines a unique *complex structure* for phase space. The symplectic structure on phase space can be described without the reference (1.14) to basis vectors by defining the symplectic bivector J through a specification of its general properties. The symplectic bivector determines a skew-symmetric linear transformation $\underline{J}$ which maps each phase space vector x into a vector

$$\widetilde{x} = \underline{J}\,x = x \cdot J\,. \tag{1.18}$$

This, in turn, defines a skew-symmetric bilinear form

$$\widetilde{x} \cdot y = y \cdot (\underline{J}x) = x \cdot J \cdot y = J \cdot (y \wedge x) = -\widetilde{y} \cdot x\,. \tag{1.19}$$

This bilinear form is nondegenerate if and only if $\widetilde{x}$ is nonzero whenever x is nonzero, or, equivalently, if and only if $\langle J^n \rangle_{2n}$ is a nonvanishing pseudoscalar. With respect to the basis specified by (1.14),

$$\langle J^n \rangle_{2n} = \underbrace{J \wedge \ldots \wedge J}_{n \text{ times}} = n!(-1)^{[n/2]} E_n \widetilde{E}_n, \tag{1.20}$$

where $E_n = e_1 e_2 \ldots e_n = e_1 \wedge e_2 \wedge \ldots e_n$, $\widetilde{E}_n = \widetilde{e}_1 \widetilde{e}_2 \ldots \widetilde{e}_n$, and $[n/2]$ is the greatest integer in $n/2$. The "complex structure" expressed by (1.17) can be characterized more generally by

$$(\widetilde{x})^{\sim} = \underline{J}^2 x = -x. \tag{1.21}$$

It follows that

$$(\widetilde{x})^2 = x^2\,, \tag{1.22}$$

which can be regarded as a "hermitian form" associated with the complex structure.

The group of linear transformations on phase space which preserves symplectic structure is called the *symplectic group*. It has recently been shown that the symplectic group has a natural representation as a "spin group" (Doran *et. al.,* 1992). This promises to be the ideal vehicle for characterizing symplectic transformations.

Now, to define Hamiltonian mechanics on phase space, from the "position and momentum vectors" (1.2) we can describe the state of our physical system by a single point x in phase space defined by

$$x = \widetilde{q} + p = p + q \cdot J. \tag{1.23}$$

The derivative with respect to a phase space point is then given by

$$\partial \equiv \partial_x = \partial_{\tilde{q}} + \partial_p = \widetilde{\partial}_p, \tag{1.24}$$

and we have

$$\widetilde{\partial} = \widetilde{\partial}_x = -J \cdot \partial_x = -\partial_q + \widetilde{\partial}_p\,. \tag{1.25}$$

The Hamiltonian of the system is a scalar-valued function on phase space

$$H = H(x) = H(q, p)\,. \tag{1.26}$$

Accordingly, Hamilton's equation for a phase space trajectory, $x = x(t)$, of the system assumes the simple form

$$\dot{x} = \widetilde{\partial} H. \tag{1.27}$$

The transcription of the entire theory of Hamiltonian systems into this invariant formulation is now straightforward. For example, for any scalar-valued phase space function, $G = G(x)$, the *Poisson bracket* can be defined by

$$\{H, G\} = (\widetilde{\partial} H) \cdot \partial G = -\{G, H\}\,. \tag{1.28}$$

Its equivalence to the conventional definition in terms of coordinates is provided by

$$\begin{aligned} \{H, G\} &= (\widetilde{\partial}_p H - \partial_q H) \cdot (\partial_p + \widetilde{\partial}_q) G \\ &= (\partial_p H) \cdot \partial_q G - (\partial_q H) \cdot (\partial_p G) \\ &= \sum_k \left[\left(\frac{\partial H}{\partial p_k} \right) \left(\frac{\partial G}{\partial q_k} \right) - \left(\frac{\partial H}{\partial q_k} \right) \left(\frac{\partial G}{\partial p_k} \right) \right]. \end{aligned} \tag{1.29}$$

The definition (1.28) does not actually require that G be scalar-valued, so it can be applied to any multivector-valued function, $M = M(x)$, describing some "observable" property of the system. It follows that the equation of motion for the observable is given by

$$\dot{M} = \dot{x}\cdot\partial M = (\widetilde{\partial} H)\cdot\partial M = \{H, M\}\,. \tag{1.30}$$

For $M = x$ we have

$$\{H, x\} = (\widetilde{\partial} H)\cdot\partial x = \widetilde{\partial} H, \tag{1.31}$$

so Hamilton's equation (1.27) can be expressed in the form

$$\dot{x} = \{H, x\}\,. \tag{1.32}$$

According to (1.30), M is a *constant of the motion* if $\{H, M\} = 0$. It follows that H is a constant of the motion, since

$$\{H, H\} = (\widetilde{\partial} H)\cdot(\partial H) = J\cdot(\partial H \wedge \partial H) = 0. \tag{1.33}$$

Our next task is to generalize this approach to Hamiltonian mechanics on manifolds.

3. VECTOR MANIFOLD VERSION.

The initial characterization of configuration space in the preceding section depends on the choice of coordinates. There is a "canonical" choice, though. For a system of N particles a configuration space of dimension $n = 3N$ is naturally defined by

$$\mathcal{R}^{3N} = \underbrace{\mathcal{R}^3 \oplus \cdots \oplus \mathcal{R}^3}_{N \text{ times}}, \tag{2.1}$$

where a separate copy of the 3-dimensional "physical space" is allotted to each particle. Whatever the choice of "generalized coordinates," its relation to physical space must be maintained, so a mapping to the choice (2.1) must be specified. For many purposes, however, this mapping is not of interest, so we desire a formulation of mechanics where it can be suppressed or resurrected as needed.

For a system of particles or rigid bodies with constraints, the space of allowable states is a manifold of dimension $2n$ equal to the number of independent degrees of freedom. Although this manifold can be mapped locally into the vector space representation of phase space in the preceding section, this is awkward if the system has cyclic coordinates. Alternatively, we can describe here the representation of *phase space* as a $2n$-dimensional *vector manifold* $\mathcal{M}^{2n}$. The mathematical apparatus needed for differential and integral calculus on vector manifolds has already been developed in [GC]. The phase space manifold $\mathcal{M}^{2n}$ can be regarded as embedded in a vector space of higher dimension (e.g., of dimension $6N$ for an N particle system), but this is not required except, perhaps, to describe the relation to physical space expressed by (2.1).

The mathematical apparatus in [GC] enables us to adapt our vector space version of Hamiltonian mechanics to a vector manifold version with comparatively minor alterations.

The main difference is that the algebraic relations of interest will be defined on the tangent spaces of the manifold instead of on the manifold itself.

Each point x on the phase space manifold $\mathcal{M}^{2m}$ represents an allowable state of the system. The symplectic bivector J of the preceding section becomes a nondegenerate bivector field $J = J(x)$ on $\mathcal{M}^{2n}$ with values in the tangent algebra [GC, Chap.4]. For vector fields $v = v(x)$ and $u = u(x)$, in the tangent space at each point x, $J(x)$ determines a linear transformation

$$\widetilde{v} = \underline{J}v = v \cdot J \tag{2.2}$$

and a corresponding nondegenerate bilinear form

$$u \cdot \widetilde{v} = -v \cdot \tilde{u}\,, \tag{2.3}$$

just as in (1.18) and (1.19). However, a direct analogue of (1.21) is not feasible, because it may conflict with requirements on the derivatives of J. Instead, however, we can introduce another bivector field $K = K(x)$ with the property

$$\underline{K}\,\widetilde{v} = \widetilde{v} \cdot K = v. \tag{2.4}$$

Thus, $\underline{K} = \underline{J}^{-1}$ is the inverse of $\underline{J}$. Now the Jacobi identity [GC, p.14] implies that

$$\underline{K}\,\underline{J}v = K \cdot (v \cdot J) = (K \cdot v) \cdot J + v \cdot (K \times J)\,,$$

where $K \times J = \frac{1}{2}(KJ - JK)$ is the *commutator product*. So if $\underline{J}$ is to be the inverse of $\underline{K}$, we must have

$$K \times J = 0, \tag{2.5}$$

or the equivalent operator equation

$$\underline{K}\,\underline{J} = \underline{J}\,\underline{K} = 1\,. \tag{2.6}$$

To specify the relation of K to J more precisely, we note that, as in (1.13), they can each be expressed as a sum of n commuting blades.

$$J = \sum_k J_k, \quad K = \sum_k K_k. \tag{2.7}$$

Moreover, we can select each K_k to be proportional to J_k. Then the condition $\underline{K} = \underline{J}^{-1}$ can be expressed by the more specific condition

$$K_k = J_k^{-1} \tag{2.8}$$

for each k. This generalizes the condition $J_k^{-1} = -J_k$ in (2.8). Incidentally, we note that

$$J \cdot K = \sum_k K_k \cdot J_k = n. \tag{2.9}$$

Modern approaches to Hamiltonian mechanics (Abraham and Marsden, 1967; Libermann and Marle, 1987) begin with symplectic manifolds. A manifold $\mathcal{M}^{2n}$ is said to be *symplectic* if it admits a *closed*, *nondegenerate* 2-form ω. As shown in [GC], this is equivalent to admitting a closed, nondegenerate bivector field K on the vector manifold. Indeed, the 2-form can be defined by

$$\omega = K \cdot (dx \wedge dy), \tag{2.10}$$

where dx and dy are tangent vectors. The 2-form is said to be closed if its "exterior differential" vanishes, that is, if

$$d\omega = (dx \wedge dy \wedge dz) \cdot (\partial \wedge K) = 0\,. \tag{2.11}$$

This condition is obviously satisfied if K has vanishing *curl:*

$$\partial \wedge K = 0. \tag{2.12}$$

Actually, though, (2.11) implies only the weaker condition of vanishing *cocurl:*

$$\nabla \wedge K = \underline{P}(\partial \wedge K) = 0, \tag{2.13}$$

where $\underline{P}$ is the projection into the *tangent algebra* of $\mathcal{M}^{2n}$ (see [GC, p.140]). The tangent algebra is essentially the same thing as the "Clifford bundle" which "pastes" Clifford algebras on manifolds, instead of generating them from a vector manifold as in [GC]. The *coderivative* ∇ as well as the *derivative* ∂ is an essential concept for calculus on vector manifolds, and its properties are thoroughly discussed in [GC, Chapt.4], so we can exploit some of its properties without establishing them here.

Instead of translating the "differential forms approach" into geometric algebra, it is more enlightnening to ascertain directly what condition on the bivector field $J = J(x)$ are required to ensure the essential features of Hamiltonian mechanics on $\mathcal{M}^{2n}$. Hamilton's equation (1.27) can be adopted without change. The Hamiltonian $H(x)$ determines a vector field $\widetilde{h} = \widetilde{h}(x)$ on $\mathcal{M}^{2n}$ given by

$$\widetilde{h} = \widetilde{\partial} H = (\partial H) \cdot J. \tag{2.14}$$

Hamilton's equation

$$\dot{x}(t) = \widetilde{h}\big(x(t)\big) \tag{2.15}$$

determines *integral curves* of this vector field. This condition that these curves describe an "incompressible flow" is given by *Liouvilles Theorem*

$$\nabla \cdot \widetilde{h} = \partial \cdot \widetilde{h} = 0\,. \tag{2.17}$$

Since

$$\partial \cdot \widetilde{h} = \partial \cdot (-J \cdot h) = -(\partial \cdot J) \cdot h + J \cdot (\partial \wedge h).$$

and $\partial \wedge h = \partial \wedge \partial H = 0$, the condition

$$\nabla \cdot J = \underline{P}(\partial \cdot J) = 0 \tag{2.18}$$

suffices to imply Liouvilles Theorem. We adopt (2.18) instead of the weaker condition $h \cdot (\partial \cdot J) = (h \wedge \partial) \cdot J = 0$, because it appears to be essential for the theory of canonical transformations outlined below.

The definition (1.28) for the Poisson bracket can be taken over to $\mathcal{M}^{2n}$ without change. However, the role of J in determining its properties must be examined. Scalar-valued functions $F = F(x)$, $G = G(x)$, $H = H(x)$ determine vector fields

$$\widetilde{f} = \widetilde{\partial} F, \qquad \widetilde{g} = \widetilde{\partial} G, \qquad \widetilde{h} = \widetilde{\partial} H. \tag{2.19}$$

Let us refer to such fields as *symplectic vector fields*. It follows from (2.19) that

$$\widetilde{\partial} \cdot f = -J \cdot (\partial \wedge f) = 0\,, \tag{2.20}$$

but (2.18) implies the stronger condition

$$\partial \cdot \widetilde{f} = -\widetilde{\partial} \cdot f = 0. \tag{2.21}$$

Therefore, all nonvanishing symplectic vector fields generate incompressible flows on (or automorphisms of) $\mathcal{M}^{2n}$.

The Poisson bracket can be written in a variety of forms, including

$$\begin{aligned}\{F, G\} &= -J \cdot (f \wedge g) = \widetilde{f} \cdot g = -\widetilde{g} \cdot f \\ &= \widetilde{\partial} \cdot (Fg) = -\widetilde{\partial} \cdot (Gf)\,.\end{aligned} \tag{2.22}$$

Alternatively, using (2.4), one can write

$$\{F, G\} = K \cdot (\widetilde{f} \wedge \widetilde{g})\,, \tag{2.23}$$

which, according to (2.10), expresses the bracket as a 2-form evaluated on symplectic vector fields. This is closer to conventional formulations in terms of differential forms. However, (2.22) is simpler because K is not involved.

An essential property of the Poisson bracket is the *Jacobi identity*

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0\,. \tag{2.24}$$

Using (2.22) to express the left side of (2.24) in terms of vector fields, we obtain

$$\begin{aligned}&-\widetilde{\partial} \cdot [(\widetilde{g} \cdot h)f + (\widetilde{h} \cdot f)g + (\widetilde{f} \cdot g)h] \\ &\quad = \widetilde{\partial} \cdot [J \cdot (f \wedge g \wedge h)] \\ &\quad = (f \wedge g \wedge h) \cdot (\widetilde{\partial} \wedge J) + (J \wedge \widetilde{\partial}) \cdot (f \wedge g \wedge h) \\ &\quad = (f \wedge g \wedge h) \cdot (\widetilde{\partial} \wedge J) - \tfrac{1}{2}(J \wedge J) \cdot [\partial \wedge (f \wedge g \wedge h)].\end{aligned} \tag{2.25}$$

This computation employed the algebraic identities

$$\begin{aligned}J \cdot (f \wedge g \wedge h) &= J \cdot (f \wedge g)h - J \cdot (f \wedge h)g + J \cdot (g \wedge h)f \\ &= (\widetilde{g} \cdot f)h + (\widetilde{f} \cdot h)g + (\widetilde{h} \cdot g)f\end{aligned} \tag{2.26}$$

[GC, eqn.(1-1.40)], and

$$(J \wedge J) \cdot \partial = 2J \wedge (J \cdot \partial) = -2J \wedge \widetilde{\partial}\,, \tag{2.27}$$

$$(J \wedge J) \cdot [\partial \wedge (f \wedge g \wedge h)] = [(J \wedge J) \cdot \partial] \cdot (f \wedge g \wedge h) \tag{2.28}$$

[GC, eqn. (1-1.25b) or (1-4.6)].

The last term in (2.25) vanishes identically when f, g and h are gradients. Therefore, from (2.25) it follows that the Jacobi identity (2.24) obtains if and only if

$$\widetilde{\nabla} \wedge J = \underline{P}(\widetilde{\partial} \wedge J) = 0\,. \tag{2.29}$$

This condition is not independent of the "incompressibility condition" (2.18), for from (2.27) we obtain the relation

$$\begin{aligned} \tfrac{1}{2}\nabla \cdot (J \wedge J) &= (\nabla \cdot J) \wedge J - (J \cdot \nabla) \wedge J \\ &= J \wedge (\nabla \cdot J) + \widetilde{\nabla} \wedge J\,. \end{aligned} \tag{2.30}$$

Thus, (2.24) and (2.29) together imply

$$\nabla \cdot (J \wedge J) = 0\,. \tag{2.31}$$

In analogy with (1.30), a multivector field $M = M(x)$ which is invariant under the flow generated by a symplectic vector field $\widetilde{f} = \widetilde{\partial} F$ satisfies

$$\{F, M\} = \widetilde{f} \cdot \nabla M = 0\,. \tag{2.32}$$

Note the use of $\widetilde{f} \cdot \nabla$ instead of $\widetilde{f} \cdot \partial$ when M is not scalar-valued. A flow is said to be a *canonical transformation* when it leaves the symplectic bivector J invariant, that is, when

$$\{F, J\} = \widetilde{f} \cdot \nabla J = 0\,. \tag{2.33}$$

The differentiable vector fields on a manifold compose a Lie algebra under the *Lie bracket* defined for vector fields $u = u(x)$ and $v = v(x)$ by

$$[u, v] = u \cdot \partial v - v \cdot \partial u = \nabla \cdot (u \wedge v) + u \nabla \cdot v - v \nabla \cdot u\,. \tag{2.34}$$

The properties of the Lie bracket are studied at length in [GC]. For symplectic fields we derive the identity

$$\begin{aligned} [\widetilde{f}, \widetilde{g}] &= \widetilde{f} \cdot \partial \widetilde{g} - \widetilde{g} \cdot \partial \widetilde{f} = \{F, \widetilde{\partial} G\} - \{G, \widetilde{\partial} F\} \\ &= \widetilde{\partial}\{F, G\} + f \cdot (\widetilde{g} \cdot \partial J) - g \cdot (\widetilde{f} \cdot \partial J)\,. \end{aligned} \tag{2.35}$$

According to (2.33), the last two terms in (2.35) vanish for canonical transformations. Therefore, the canonical transformations compose a closed Lie algebra on $\mathcal{M}^{2n}$, and the Poisson bracket of "canonical generators" F and G is also a canonical generator. This should suffice to show how the general theory of canonical transformations can be developed on vector manifolds.

As a final point, the crucial role of the symplectic bivector J in canonical transformations suggests that it should be more intimately linked with the Hamiltonian H in the theory. One attractive possibility for linking them is to introduce a bivector field Ω given by

$$\Omega = HJ\,. \tag{2.36}$$

Then (2.18) implies

$$\widetilde{h} = (\partial H) \cdot J = \nabla \cdot (HJ) = \nabla \cdot \Omega \,, \tag{2.37}$$

and Hamilton's equation (2.15) takes the form

$$\dot{x} = \nabla \cdot \Omega \,. \tag{2.38}$$

Thus, Ω is a *bivector potential* for Hamiltonian flow, and H plays the role of an integrating factor for this bivector field. This is very suggestive!

4. CONCLUSIONS.

Experts will have noted that phase space is identified with its own dual space in the preceding formulation of Hamiltonian mechanics. Some may claim that the conventional formulation in terms of differential forms is preferable because it does not make that identification. On the contrary, it can be argued that such generality is excessive, contributing little if anything to deepening analytical mechanics, while introducing unnecessary complications. Be that as it may, it should be recognized that the identification of phase space with its dual is a deliberate choice and not an intrinsic limitation of geometric algebra. Indeed, the geometric algebra apparatus needed to separate phase space from its dual is available in Doran *et. al.* (1992) and ready to be applied to mechanics. Ironically, that apparatus automatically produces a kind of quantization, something which can only be imposed artificially in conventional approaches. It remains to be seen if that fact has significant physical import.

The purpose of this short paper has been to lay the foundation for a reformulation of analytical mechanics in the language of geometric calculus. Translation of standard results into this language is not difficult, but it will not be without surprises and new insights as the treatment above already suffices to show. Though the emphasis here has been on an invariant methodology, a powerful apparatus for dealing with coordinates is available in [GC]. One especially promising possibility is an extention of the invariant formulation for rigid-body mechanics in Hestenes (1985) to a phase space formulation for systems of linked rigid bodies. That is likely to have important applications to robotics.

REFERENCES

R. Abraham and J. Marsden (1967), *Foundations of Mechanics*, W.A.Benjamin, New York.

C. Doran, D. Hestenes, F. Sommen, and N. VanAcker (1992), Lie Groups as Spin Groups, *J. Math. Phys.* (submitted).

D. Hestenes (1985), *New Foundations for Classical Mechanics*, Kluwer, Dordrecht/Boston; paperback ed. (1987); fourth printing with corrections (1992).

D. Hestenes and G. Sobczyk (1984), *Clifford Algebra to Geometric Calculus*, Kluwer, Dordrecht/ Boston; paperback ed. (1987); third printing with corrections (1992).

P. Libermann and C-M Marle (1987), *Symplectic Geometry and Analytical Mechanics*, Reidel, Dordrecht/Boston.