

GEOMETRY OF THE DIRAC THEORY

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ABSTRACT. The Dirac wave function is represented in a form where all its components have obvious geometrical and physical interpretations. Six components compose a Lorentz transformation determining the electron velocity are spin directions. This provides the basis for a rigorous connection between relativistic rigid body dynamics and the time evolution of the wave function. The scattering matrix is given a new form as a spinor-valued operator rather than a complex function. The approach reveals a geometric structure of the scattering matrix and simplifies scattering calculations. This claim is supported by an explicit calculation of the differential cross-section and polarization change in Coulomb scattering. Implications for the structure and interpretation of relativistic quantum theory are discussed.

INTRODUCTION

The Dirac equation is one of the most well-established equations of physics, having led to a great variety of detailed predictions which have been experimentally confirmed with high precision. Yet the relativistic quantum theory based on the Dirac equation has never been given a single complete and selfconsistent physical interpretation which all physicists find satisfactory. Moreover, it is generally agreed that the theory must be modified to account for the electron mass, but there is hardly agreement on how to go about it.

This paper reviews and extends results from a line of research (Ref. [1-7]) aimed at clarifying the Dirac theory and simplifying its mathematical formulation. Of course, any such improvement in so useful a theory would be valuable in itself. But the ultimate goal is to achieve insight into the structure of the theory which identifies those features responsible for its amazing results, as well as features which might be modified to improve it.

The central result of this research is a formulation of the Dirac spinor wave function which reveals the geometrical and physical interpretation of all its components. This makes it possible to relate the time evolution of the wave function to relativistic rigid body mechanics, thus giving insight into the dynamics and establishing a connection with classical theories of spinning bodies. A fairly detailed review of these results is contained in this paper. In addition, the general solution of the Bargmann-Michel-Telegdi equation for constant fields is obtained in simple form from a spinor formulation of the theory.

Most of the new results in this paper arise from a reformulation of scattering theory in accord with the above ideas. A new spinor formulation of the S -matrix is obtained which combines the conventional spin scattering amplitudes into a meaningful unit. This makes it possible to relate the interpretation of the S -matrix to relativistic rigid body mechanics. Moreover, calculations are greatly simplified. Thus, the scattering cross section can be calculated directly without the usual sums over spin states, and the polarization change can be calculated without us using projection operators. These points are illustrated

by an explicit calculation for Coulomb scattering. The mathematical reason for these simplifications is the elimination of redundancy inherent in the conventional formulation. This redundancy is manifested in calculations by the appearance of terms with zero trace. Such terms never arise in the new approach.

The new spinor form of the S -matrix has a geometrical interpretation that arises from the elimination of imaginary numbers in its formulation. This suggests that it may provide physical insight into the formal analytic continuation of scattering amplitudes that plays such an important role in central scattering theory. Along a related line, the new approach may be expected to give insight into the “spin structure” of vertex functions. Quite generally, the approach promises to make explicit a geometrical structure of quantum electrodynamics arising from the geometrical structure of the wave function. Specifically, it produces a geometrical interpretation for the generator of electromagnetic gauge transformations which has implications for the Weinberg-Salam model.

The last section of this paper discusses a physical interpretation of the Dirac wave function consistent with its geometrical properties and the possibility that electrons are actually zero mass particles whose observed mass and spin arise from self-interactions in a unified theory of weak and electromagnetic interactions.

1. Spacetime Algebra.

We shall be concerned with flat spacetime, so each point in spacetime can be uniquely represented by an element x in a 4-dimensional vector space. We may define a *geometric product* of vectors so the vectors generate a real Clifford Algebra; this is simply an associative (but noncommutative) algebra distinguished by the property that the square x^2 of any vector x is a real scalar. The metric of spacetime is specified by the allowed values for x^2 . As usual, a vector x is said to be *timelike*, *lightlike* or *spacelike* if $x^2 > 0$, $x^2 = 0$ or $x^2 < 0$ respectively. I call the Clifford Algebra so defined the *Spacetime Algebra* (STA), because all its elements and algebraic operations have definite geometric interpretations, and, it suffices for the description of any geometric structure on spacetime.

The geometric product uv of vectors u and v can be decomposed into symmetric and antisymmetric parts defined by

$$u \cdot v = \frac{1}{2}(uv + vu) \quad (1.1)$$

and

$$u \wedge v = \frac{1}{2}(uv - vu) \quad (1.2)$$

so that

$$uv = u \cdot v + u \wedge v. \quad (1.3)$$

One can easily prove that the symmetric product $u \cdot v$ defined by (1.1) is scalar-valued. Thus, $u \cdot v$ is the usual *inner product* (or metric tensor) on spacetime. The quantity $u \wedge v$ is neither scalar nor vector, but a new entity called a *bivector* (or 2-vector). It represents an oriented segment of the plane containing u and v in much the same way that a vector represents a directed line segment.

Let $\{\gamma_\mu, \mu = 0, 1, 2, 3\}$ be a *righthanded orthonormal frame* of vectors; so

$$\gamma_0^2 = 1 \quad \text{and} \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1, \quad (1.4)$$

and it is understood that γ_0 points into the forward light cone. In accordance with (1.1), we can write

$$g_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu), \quad (1.5)$$

defining the components of the metric tensor $g_{\mu\nu}$ for the frame $\{\gamma_\mu\}$.

Representations of the vectors γ_μ by 4×4 matrices are called *Dirac matrices*. The *Dirac algebra* is the matrix algebra over the field of the complex numbers generated by the *Dirac matrices*. We shall see that the conventional formulation of the Dirac equation in terms of the Dirac algebra can be replaced by an equivalent formulation in terms of STA. This has important implications. First, a representation of the γ_μ by matrices is completely irrelevant to the Dirac theory; the physical significance of the γ_μ is derived entirely from their representation of geometrical properties of spacetime. Second, imaginaries in the complex number field of the Dirac algebra are superfluous, and we can achieve a geometrical interpretation of the Dirac wave function by eliminating them.

For these reasons we eschew the Dirac algebra and stick to further developments of STA until we are prepared to make contact with the Dirac theory.

A generic element of the STA is called a *multivector*. Any multivector M can be written in the *expanded form*

$$M = \alpha + a + F + bi + \beta i, \quad (1.6)$$

where α and β are scalars, a and b are vectors, and F is a bivector. The special symbol i will be reserved for the *unit pseudoscalar*, which has the following three basic algebraic properties:

- (a) it has negative square,

$$i^2 = -1, \quad (1.7a)$$

- (b) it anticommutes with every vector a ,

$$ia = -ai, \quad (1.7b)$$

- (c) it factors into the ordered product

$$i = \gamma_0 \gamma_1 \gamma_2 \gamma_3. \quad (1.7c)$$

Geometrically, the pseudoscalar i represents a unit oriented 4-volume for spacetime.

By multiplication the γ_μ generate a complete basis for the STA consisting of

$$1, \quad \gamma_\mu, \quad \gamma_\mu \wedge \gamma_\nu, \quad \gamma_\mu i, \quad i. \quad (1.8)$$

These elements comprise a basis for the 5 invariant components of M in (1.6), the scalar, vector, bivector, pseudovector and pseudoscalar parts respectively. Thus, they form a basis for the space of completely antisymmetric tensors on spacetime. It will not be necessary for us to employ a basis, however, because the geometric product enables us to carry out computations without it.

Computations are facilitated by the operation of *reversion*. For M in the expanded form (1.6), the reverse \tilde{M} is defined by

$$\tilde{M} = \alpha + a - F - bi + \beta i. \quad (1.9)$$

Note, in particular, the effect of reversion on scalars, vectors, bivectors and pseudoscalars:

$$\tilde{\alpha} = \alpha, \quad \tilde{a} = a, \quad \tilde{F} = -F, \quad \tilde{i} = i.$$

It is not difficult to prove that

$$(MN)^\sim = \tilde{N}\tilde{M} \tag{1.10}$$

for arbitrary multivectors M and N .

Any multivector M can be expressed as the sum of an even multivector M_+ and an *odd multivector* M_- . For M in the expanded form (1.6), we can write

$$M_+ = \alpha + F + i\beta, \tag{1.11a}$$

$$M_- = a + ib. \tag{1.11b}$$

The set $\{M_+\}$ of all even multivectors forms an important subalgebra of the STA called the *even subalgebra*. The odd multivectors do not form a subalgebra, but note that $M_- \gamma_0$ is even, and this defines a one-to-one correspondence between even and odd multivectors.

Now we are prepared to state a powerful theorem of great utility: if $\{e_\mu\}$ and $\{\gamma_\mu\}$ are any pair of righthanded frames, then they are related by a Lorentz transformation which can be represented in the *spinor form*

$$e_\mu = R\gamma_\mu\tilde{R}, \tag{1.12}$$

where R is an *even* multivector satisfying

$$\tilde{R}R = 1. \tag{1.13}$$

Furthermore, this representation is unique except for the sign of R . Indeed, the 4 equations (1.12) can be solved for R in terms of the e_μ , and the γ_μ with the result

$$R = \pm(\tilde{A}A)^{1/2}A$$

where [8]

$$A = e_\mu\gamma^\mu = e_\mu \cdot \gamma_\nu\gamma^\nu\gamma^\mu.$$

This determines R explicitly in terms of the matrix elements $e_\mu \cdot \gamma_\nu$ for the Lorentz transformation. However, the spinor form (1.12) makes it possible to handle Lorentz transformations without using matrices. Use of the term “spinor” here will be justified when we relate it to spinors in the Dirac theory.

The set $\{R\}$ of all even multivectors R satisfying $\tilde{R}R = 1$ is a group under multiplication. In the theory of group representations it is called $SL(2, C)$ or “the spin-1/2 representation of the Lorentz group.” However, group theory alone does not specify its invariant imbedding in the STA. It is precisely this imbedding that makes it so useful in the applications to follow.

2. Space-Time Splits.

Using STA we can describe fields and particles by equations which are invariant in the sense that they are not referred to any inertial system. However, these equations must be related to any given inertial system used for observation and measurement.

An inertial system, the γ_0 -system say, is completely defined algebraically by a single future-pointing timelike unit vector γ_0 . This vector determines a split of spacetime and the elements of STA into space and time components. Our job now is to specify how this split is to be expressed algebraically.

Let p be the energy-momentum vector of a particle with (proper) mass m so that $p^2 = m^2$. The *space-time split* of p by γ_0 is expressed algebraically by the equation [9]

$$p\gamma_0 = p \cdot \gamma_0 + p \wedge \gamma_0 = E + \mathbf{p}, \quad (2.1)$$

where

$$E = p \cdot \gamma_0 \quad (2.2a)$$

is the *energy* and

$$\mathbf{p} = p \wedge \gamma_0 \quad (2.2b)$$

is the *relative momentum* in the γ_0 -system. Note that

$$p^2 = (p\gamma_0)(\gamma_0 p) = (E + \mathbf{p})(E - \mathbf{p}),$$

so the usual relation

$$p^2 = E^2 - \mathbf{p}^2 = m^2$$

is satisfied. We interpret \mathbf{p} as a vector in the 3-dimensional “space” of the γ_0 -system, but, according to (2.2b), it is a bivector in spacetime for the timelike plane containing p and γ_0 . We may refer to \mathbf{p} as a *relative vector* and to p as a *proper vector* to distinguish the two different uses of the term “vector,” but the adjectives “relative” and “proper” can be dropped when there is no danger of confusion. Of course, a space-time split similar to (2.1) can be made for any proper vector.

The expansion of a bivector F in a basis is given by

$$F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \quad (2.3)$$

where the $F^{\mu\nu}$ are its tensor components, but we will not need this expansion. The *space-time split* of F by γ_0 is obtained by decomposing F into a part

$$\mathbf{E} \equiv \frac{1}{2}(F - \gamma_0 F \gamma_0) \quad (2.4a)$$

$$i\mathbf{B} \equiv \frac{1}{2}(F + \gamma_0 F \gamma_0) \quad (2.4b)$$

which commutes with γ_0 , so

$$F = \mathbf{E} + i\mathbf{B} \quad (2.5)$$

If F is the electromagnetic field bivector, then this is exactly the split of F into an electric field \mathbf{E} and a magnetic field \mathbf{B} in the γ_0 -system. Of course, the split depends on γ_0 , and it applies to any bivector. From (2.4a) it follows that

$$\gamma_0 \mathbf{E} \gamma_0 = -\mathbf{E}. \quad (2.6)$$

This relation holds for any relative vector \mathbf{E} , so it can be interpreted as a *space inversion* in the γ_0 -system. *Relative reversion* of the bivector $F = \mathbf{E} + i\mathbf{B}$ is defined by

$$F^\dagger = \gamma_0 \tilde{F} \gamma_0 = \mathbf{E} - i\mathbf{B}. \quad (2.7a)$$

For an arbitrary multivector M we define M^\dagger by

$$M^\dagger = \gamma_0 \tilde{M} \gamma_0. \quad (2.7b)$$

The “dagger” symbol is appropriate because this operation corresponds exactly to hermitian conjugation in the Dirac algebra. It follows that hermitian conjugation, in contrast to reversion, is not an invariant operation; it is tacitly dependent on the choice of a particular inertial system. With respect to the γ_0 -system

$$(\mathbf{E}\mathbf{B})^\dagger = \mathbf{B}^\dagger \mathbf{E}^\dagger = \mathbf{B}\mathbf{E},$$

that is, relative reversion reverses the order of relative vectors.

The geometric product of relative vectors \mathbf{E} and \mathbf{B} can be decomposed into symmetric and antisymmetric parts in the same way that we decomposed the product of proper vectors. Thus, we obtain

$$\mathbf{E}\mathbf{B} = \mathbf{E} \cdot \mathbf{B} + i\mathbf{E} \times \mathbf{B}, \quad (2.8a)$$

$$\mathbf{E} \cdot \mathbf{B} = \frac{1}{2}(\mathbf{E}\mathbf{B} + \mathbf{B}\mathbf{E}) \quad (2.8b)$$

is the usual inner (or dot) product for Euclidean 3-space, and

$$\mathbf{E} \times \mathbf{B} = \frac{1}{2i}(\mathbf{E}\mathbf{B} - \mathbf{B}\mathbf{E}) \quad (2.8c)$$

is the usual *cross product* of conventional vector algebra. Strictly speaking, conventional vector algebra is not an algebra in the mathematical sense. Nevertheless, (2.8b) and (2.8c) show that it is completely contained within the STA. Therefore, translations from STA to vector algebra are effortless. They arise automatically from a spacetime split.

It is of interest to note that the three relative vectors

$$\boldsymbol{\sigma}_k = \gamma_k \wedge \gamma_0 = \gamma_k \gamma_0 \quad (\text{for } k = 1, 2, 3) \quad (2.9)$$

can be regarded as a *righthanded orthonormal* set of spatial directions in the γ_0 -system, and they generate a Clifford algebra with

$$\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 = i = \gamma_0 \gamma_1 \gamma_2 \gamma_3. \quad (2.10)$$

This algebra is just the even subalgebra of the STA. Also, it is isomorphic to the Pauli Algebra of 2×2 complex matrices with the $\boldsymbol{\sigma}_k$ corresponding to the Pauli matrices.

A *space-time split* of the Lorentz transformation (1.12) by γ_0 is accomplished by a split of the spinor R into the product

$$R = LU, \quad (2.11)$$

where $U^\dagger = \gamma_0 \tilde{U} \gamma_0 = \tilde{U}$ or

$$U \gamma_0 \tilde{U} = \gamma_0, \quad (2.12)$$

and $L^\dagger = \gamma_0 \tilde{L} \gamma_0 = L$ or

$$\gamma_0 \tilde{L} = L \gamma_0. \quad (2.13)$$

Equation (2.12) defines the “little group” of Lorentz transformations which leave γ_0 invariant; this is the group of spacial rotations in the γ_0 -system. Its “covering group” is the set $\{U\}$ of spinors satisfying (2.12); this is a representation in Clifford Algebra of the abstract group $SU(2)$.

Using (2.11), we can “split” the Lorentz transformation (1.12) into a sequence of two Lorentz transformations determined by the spinors U and L respectively; thus,

$$e_\mu = R \gamma_\mu \tilde{R} = L(U \gamma_\mu \tilde{U}) \tilde{L}. \quad (2.14)$$

The transformation $U \gamma_k \tilde{U}$ (for $k = 1, 2, 3$) is a spacial rotation of the proper vectors γ_k in the γ_0 -system. Multiplication by γ_0 expresses it as a rotation of relative vectors $\sigma_k = \gamma_k \gamma_0$ into relative vectors \mathbf{e}_k ; thus,

$$U \sigma_k \tilde{U} = U \sigma_k U^\dagger = \mathbf{e}_k. \quad (2.15)$$

From (2.12) it follows U can be written in the exponential form

$$U = e^{-\frac{1}{2} i \mathbf{a}}, \quad (2.16)$$

where \mathbf{a} is the relative vector specifying the axis and angle of the spacial rotation determined by U .

The spinor L determines a *boost* or “pure Lorentz transformation.” Thus (2.14) describes a split of the Lorentz transformation into a spacial rotation followed by a boost. The boost is completely determined by e_0 and γ_0 , for it follows from (2.12) and (2.13) that

$$e_0 = R \gamma_0 \tilde{R} = L \gamma_0 \tilde{L} = L^2 \gamma_0.$$

Hence,

$$L^2 = e_0 \gamma_0. \quad (2.17)$$

It is readily verified, then, that

$$L = (e_0 \gamma_0)^{\frac{1}{2}} = e_0 e_+ = e_+ \gamma_0 = \frac{1 + e_0 \gamma_0}{[2(1 + e_0 \cdot \gamma_0)]^{\frac{1}{2}}}, \quad (2.18)$$

where

$$e_+ = \frac{e_0 + \gamma_0}{|e_0 + \gamma_0|} = \frac{e_0 + \gamma_0}{[2(1 + e_0 \cdot \gamma_0)]^{\frac{1}{2}}}. \quad (2.19)$$

is the unit vector bisecting the angle between γ_0 and e_0 .

If $p = me_0$ is the proper momentum of a particle with mass m , then according to (2.1) we can write (2.17) and (2.18) in the forms

$$L^2 = \frac{p\gamma_0}{m} = \frac{E + \mathbf{p}}{m} \quad (2.20)$$

and

$$L = \frac{m + p\gamma_0}{[2m(1 + p \cdot \gamma_0)]^{\frac{1}{2}}} = \frac{m + E + \mathbf{p}}{[2m(1 + E)]^{\frac{1}{2}}}. \quad (2.21)$$

Then L describes a boost of a particle from rest to a relative momentum \mathbf{p} .

3. Relativistic Rigid Body Mechanics.

The equation

$$e_\mu = R\gamma_\mu\tilde{R} \quad (3.1)$$

can be used to describe the relativistic kinematics of a rigid body (with negligible dimensions) traversing a world line $x = x(\tau)$ with proper time τ , if we identify e_0 with the proper velocity v of the body (or particle), so that

$$\frac{dx}{d\tau} = v = e_0 = R\gamma_0\tilde{R} \quad (3.2)$$

Then $e_\mu = e_\mu(\tau)$ is a *comoving frame* traversing the world line along with the particle, and the spinor R must also be a function of proper time, so that, at each time τ , equation (3.1) describes a Lorentz transformation of some fixed frame $\{\gamma_\mu\}$ into the comoving frame $\{e_\mu(\tau)\}$. Thus, we have a spinor-valued function of proper time $R = R(\tau)$ determining a 1-parameter family of Lorentz transformations.

The spacelike vectors $e_k = R\gamma_k\tilde{R}$ (for $k = 1, 2, 3$) can be identified with the principal axes of the body, but for a particle with an intrinsic angular momentum or *spin*, it is most convenient to identify e_3 with the spin direction s ,

$$s = e_3 = R\gamma_3\tilde{R}. \quad (3.3)$$

In this case, we need not include the magnitude of the spin in our kinematics, because it is a constant of the motion.

From the fact that R is an even multivector satisfying $R\tilde{R} = 1$, it follows that $R = R(\tau)$ must satisfy a *spinor equation of motion* of the form

$$\dot{R} = \frac{1}{2}\Omega R, \quad (3.4)$$

where the dot represents the proper time derivative, and $\Omega = \Omega(\tau) = -\tilde{\Omega}$ is a bivector-valued function. Differentiating (3.1) and using (3.4), we see that the equations of motion for the comoving frame must be of the form

$$\dot{e}_\mu = \frac{1}{2}(\Omega e_\mu - e_\mu \Omega) \equiv \Omega \cdot e_\mu. \quad (3.5)$$

Clearly Ω can be interpreted as a generalized *rotational velocity* of the comoving frame.

The dynamics of a rigid body, that is, the action of external forces and torques on the body is completely described by specifying Ω as a specific function of the proper time. For a charged particle with an intrinsic magnetic moment in a constant (uniform) electromagnetic $F = \mathbf{E} + i\mathbf{B}$,

$$\Omega = \frac{e}{mc} \left[F + \frac{1}{2}(g-2)i\mathbf{B}' \right], \quad (3.6)$$

where m is the mass, e is the charge, g is the g -factor and B' is the magnetic field in the instantaneous rest frame, as defined by the *space-time split*.

$$i\mathbf{B}' \equiv \frac{1}{2}(F + vFv), \quad (3.7)$$

similar to (2.4b).

Substituting (3.6) into (3.5), we get

$$\dot{v} = \frac{e}{mc} F \cdot v. \quad (3.8)$$

as the equation of motion for the velocity, and

$$\dot{s} = \frac{e}{m} \left[F + \frac{1}{2}(g-2)i\mathbf{B}' \right] \cdot s. \quad (3.9)$$

as the equation of motion for the spin. The last equation (3.9) is the well-known Borgmann-Michel-Telegdi (BMT) equation, which has been applied to high precision measurements of the g -factor for the electron and the muon.

To apply the BMT equation, it must be solved to determine the rate of spin precession. To my knowledge the general solution for an arbitrary constant field F has not been published previously. However, the problem can be greatly simplified by replacing the BMT equation by the corresponding spinor equation. Substituting (3.6) into (3.4), the spinor equation can be put in the form

$$\dot{R} = \frac{e}{2m} FR + R \frac{1}{2}(g-2) \left(\frac{e}{2m} \right) i\mathbf{B}_0 \quad (3.10)$$

where, for initial energy E_0 , momentum \mathbf{p}_0 and $F = \mathbf{E} + i\mathbf{B}$ in the γ_0 -system, we have

$$i\mathbf{B}_0 = \tilde{R} i\mathbf{B} R = \frac{1}{2} \left[\tilde{L}_0 F L_0 - (\tilde{L}_0 F L_0)^\dagger \right] = i \left\{ \mathbf{B} + \frac{E_0}{m} \left(\mathbf{E} + \frac{\mathbf{p}_0 \times \mathbf{B}}{E_0 + m} \right) \right\} \quad (3.11)$$

with $L_0^2 = m^{-1}(E_0 + \mathbf{p}_0)$. Since F and \mathbf{B}_0 are constant, (3.10) has the general solution

$$R = \left\{ \exp \left[\frac{e}{2m} F \tau \right] \right\} L_0 \exp \left[\frac{1}{2}(g-2) \left(\frac{e}{2m} \right) i\mathbf{B}_0 \tau \right], \quad (3.12)$$

The last factor in (3.12) is especially significant, it gives $-\frac{e}{m} \left(\frac{g-2}{2} \right) \mathbf{B}_0 \tau$ immediately as the precession angle between the polarization vector (to be defined later) and the relative momentum \mathbf{p} ; this angle can be measured quite directly in experiments.

This result illustrates an important general fact, namely, that the four coupled vector equations (3.5) can be greatly simplified by replacing them by the single equivalent spinor

equation (3.4). The spinor solution is invariably simpler than a direct solution of the coupled equations.

Equation (3.10) is especially interesting because it is structurally related to the Dirac equation, as we shall see. Indeed, when radiative corrections are neglected, the Dirac equation implies $g = 2$ and (3.10) reduces to

$$\dot{R} = \frac{e}{2m} FR. \quad (3.13)$$

This was derived as an approximation of the Dirac equation in Ref. 5, and solutions when F is a plane wave or a Coulomb field were found in Ref. 3. The precise conditions under which (3.10) is a valid approximation of the Dirac equation have still not been determined.

4. Scattering of Polarized Particles.

A space-time split of the spin vector $s = R e_3 \tilde{R}$ can be made in two different ways. The most obvious approach is to make it in the same way as the split (2.1) of the momentum vector. But this approach has two serious disadvantages: the relative spin obtained in this way does not have a fixed magnitude, and it is awkward to compare spin directions of particles with different velocities. These disadvantages are eliminated by the alternative approach based on spinor split $R = LU$ established in Sec. 2.

Using $R = LU$ along with (2.12) and (2.13), we obtain

$$s\gamma_0 = LU\gamma_3\tilde{U}\tilde{L}\gamma_0 = LU\gamma_3\gamma_0\tilde{U}L.$$

Now we define the *relative spin vector* \mathbf{s} by

$$\mathbf{s} = U\boldsymbol{\sigma}_3U^\dagger = U\boldsymbol{\sigma}_3\tilde{U}, \quad (4.1)$$

where $\boldsymbol{\sigma}_3 = \gamma_3\gamma_0$. Then the *spacetime split* of the *proper spin vector* is given by

$$s\gamma_0 = L\mathbf{s}L. \quad (4.2)$$

By the way, for zero mass particles (see Appendix C), this approach to the spin split does not work, and one must revert to the first approach, which, however, becomes much simpler for this special case.

Equation (4.1) shows that the relative spin \mathbf{s} is a unit vector in the γ_0 -system (any definite inertial system chosen for convenience). According to (4.2), the relative spin \mathbf{s} is obtained from the proper spin by “factoring out” the velocity of the particle contained in L . We can interpret this by imagining the particle at any time suddenly brought to rest in the γ_0 -system by a “de-boost” specified by L . Then \mathbf{s} is the direction of the spin for this particle suddenly brought to rest. In this common rest system, the relative spin directions for different particles or the same particle at different times are readily compared, and the spin precession of a single particle is expressed as a precession $\mathbf{s} = \mathbf{s}(\tau)$ in this 3-dimensional space.

In quantum theory experiments are done on an ensemble of particles rather than a single system. The ensemble can be characterized by a *polarization vector* $\boldsymbol{\sigma}$ defined by

$$\boldsymbol{\sigma} = \epsilon \mathbf{s} \quad (\text{for } 0 \leq \epsilon \leq 1), \quad (4.3)$$

where ϵ is called the *degree of polarization*. In this case, we take the γ_0 -system to be the “lab system” in which measurements are made. The basic spin measurement consists of counting the number of particles $N_+(N_-)$ with spin *up* (*down*) along some direction specified by a unit vector \mathbf{n} . The measurements are related to the polarization vector by

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \frac{N_+ - N_-}{N_+ + N_-}, \quad (4.4)$$

and

$$\frac{N_+}{N_+ + N_-} = \frac{1}{2}(1 + \boldsymbol{\sigma} \cdot \mathbf{n}) \quad (4.5)$$

is interpreted as the probability that a single particle is polarized in the \mathbf{n} -direction. The term *polarization* is often used for the *helicity state* defined by

$$\boldsymbol{\sigma} \cdot \mathbf{p} = \frac{N_R - N_L}{N_R + N_L}, \quad (4.6)$$

which, of course, is a special case of (4.4) where \mathbf{n} is taken to be the direction of the momentum \mathbf{p} of particles in the ensemble. Then $N_R(N_L)$ refers to the number of particles with right (left)-handed polarization.

Scattering of a particle with spin (or a rigid body) takes the *initial velocity* $v_i = R_i \gamma_0 \tilde{R}_i$ into a *final velocity* $V_f = R_f \gamma_0 \tilde{R}_f$. The net effect is a rotation of the comoving frame from an initial value $e_\mu(\tau_i) = R_i \gamma_\mu \tilde{R}_i$ to a final value $e_\mu(\tau_f) = R_f \gamma_\mu \tilde{R}_f$, as expressed by the spinor equation

$$R_f = S_{fi} R_i. \quad (4.7)$$

The net effect of the scattering is thus completely described by the *spinor scattering operator*

$$S_{fi} = R_f \tilde{R}_i. \quad (4.8)$$

The problem, then, is to calculate the scattering operator from dynamical assumptions; for example, by integrating the spinor equation of motion (3.4).

This description of scattering applies to classical as well as quantum systems. According to (4.8) the scattering operator has unit modulus:

$$|S_{fi}|^2 = S_{fi} \tilde{S}_{fi} = 1. \quad (4.9)$$

We shall see that from the Dirac equation a scattering operator can be derived which differs from this one only by a modulus different from unity which determines the cross section for scattering into the final momentum state.

To determine the *spin precession* in scattering, we factor out the momenta of the initial and final states by making the split

$$L_f U_f = S_{fi} L_i U_i.$$

So we arrive at

$$U_f = U_{fi}U_i \quad (4.10)$$

where U_{fi} is the *spin scattering operator* defined by

$$U_{fi} = \tilde{L}_f S_{fi} L_i. \quad (4.11)$$

This operator rotates the initial spin into the final spin according to

$$\mathbf{s}_f = U_{fi} \mathbf{s}_i U_{fi}^\dagger. \quad (4.12)$$

Thus U_{fi} completely describes the effect of scattering, irrespective of the initial spin or polarization state.

5. The Real Dirac Theory.

To find a representation of the Dirac theory in terms of the STA, we begin with a Dirac spinor Ψ , a column matrix of 4 complex numbers. Let u be a fixed spinor with the properties

$$u^\dagger u = 1, \quad (5.1a)$$

$$\gamma_0 u = u, \quad (5.1b)$$

$$\gamma_2 \gamma_1 u = i' u. \quad (5.1c)$$

In writing this we must regard the γ_μ , for the time being, as 4×4 Dirac matrices, and i' as the unit imaginary in the complex number field of the Dirac algebra. Now we can write any Dirac spinor in the form

$$\Psi = \psi u, \quad (5.2)$$

where ψ is a matrix which can be expressed as a polynomial in the γ_μ . The coefficients in this polynomial can be taken as real, for if there is a term with an imaginary coefficient then (5.1c) enables us to make it real without altering (5.2; by replacing i' in the term by $\gamma_2 \gamma_1$ on the right. Furthermore, the polynomial can be taken to be an even multivector, for if any term is odd, then (5.1b) allows us to make it even by multiplying on the right by γ_0 . Thus, in (5.2) we may assume that ψ is a *real even multivector*. Now we may reinterpret the γ_μ in ψ as vectors in the STA instead of matrices. Thus, we have established a correspondence between Dirac spinors and even multivectors in the STA. The correspondence must be one-to-one, because the space of even multivectors (like the space of Dirac spinors) is exactly 8-dimensional, with 1 scalar, 1 pseudoscalar and 6 bivector dimensions.

There are other ways to represent a Dirac spinor in the STA, but all representations are, of course, mathematically equivalent. The representation chosen here has the advantages of simplicity and, as we shall see, ease of interpretation.

To distinguish a spinor ψ in the STA from its matrix representation Ψ in the Dirac algebra, let us call it a *real spinor* or a *real representation* of the *Dirac wave function* to emphasize the elimination of the uninterpreted imaginary i' , which is thereby shown to be irrelevant to the Dirac theory.

In terms of a real wave function ψ , the Dirac equation for an electron can be written in the form

$$\gamma^\mu (\partial_\mu \psi \gamma_2 \gamma_1 + e A_\mu \psi) = m \psi \gamma_0, \quad (5.3)$$

where m is the mass and $e = |e|$ is the charge of the electron, while the $A_\mu = A \cdot \gamma_\mu$ are components of the electromagnetic vector potential. To prove that this is equivalent to the standard matrix form of the Dirac equation, we simply interpret the γ_μ as matrices, multiply by u on the right, and use (5.1a, b, c) and (5.2) to get the standard form

$$\gamma^\mu (i' \partial_\mu + e A_\mu) \Psi = m \Psi. \quad (5.4)$$

This completes the proof.

Henceforth, we can work with the *real Dirac equation* (5.3) without reference to its matrix representation (5.4). We know that computations in STA can be carried out without introducing a basis, so let us write the real Dirac equation in the form

$$\square \psi \mathbf{i} + e A \psi = \psi p_0 \quad (5.5)$$

where $A = A_\mu \gamma_\mu$,

$$\square = \gamma^\mu \partial_\mu \quad (5.6)$$

is a vector differential operator, and $p_0 = m \gamma_0$ can be regarded as the momentum vector of a particle with mass m . The symbol

$$\mathbf{i} = \gamma_2 \gamma_1 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 = i \boldsymbol{\sigma}_3 \quad (5.7)$$

emphasizes that this bivector plays the role of the imaginary i' that appears explicitly in the matrix form (5.4) of the Dirac equation. To interpret the theory, it is crucial to note that the bivector \mathbf{i} has a definite geometrical interpretation while i' does not.

Now we must establish an interpretation for the wave function. Since ψ is even, $\psi \tilde{\psi}$ is even. But $\psi \tilde{\psi}$ is equal to its own reverse, so (1.9) implies that only its scalar and pseudoscalar parts can be nonzero. Therefore, we can write it in the form

$$\psi \tilde{\psi} = \rho e^{i\beta} \quad (5.8)$$

where ρ and β are scalars (real, of course) with $0 \leq \beta \leq \pi$. If $\rho = 0$, the wave function describes a zero mass particle, as explained in appendix C. If $\rho \neq 0$, then

$$R = (\rho e^{i\beta})^{-\frac{1}{2}} \psi \quad \text{satisfies} \quad R \tilde{R} = 1. \quad (5.9)$$

Thus, we can write ψ in the *canonical form*

$$\psi = \rho^{\frac{1}{2}} e^{\frac{1}{2} i\beta} R, \quad (5.10)$$

where R is an even multivector satisfying (5.9).

Since ψ is a solution of the Dirac equation, it is a spinor field. At every spacetime point it determines a *unique* timelike vector field

$$\psi \gamma_0 \tilde{\psi} = \rho R \gamma_0 \tilde{R} = \rho v. \quad (5.11)$$

This is exactly the *Dirac current* of the conventional Dirac theory. From the Dirac equation it follows that

$$\square \cdot (\rho v) = 0, \quad (5.12)$$

as required for the interpretation of the Dirac current as a *probability current*. Thus, we can interpret v as the local (proper) velocity of the electron and ρ as the *proper probability density*, that is, the probability density in the local rest frame determined by v . The quantity

$$\rho_0 = \gamma_0 \cdot (\rho v) = \rho v \cdot \gamma_0 \quad (5.13)$$

is the probability density in the γ_0 -system. The wave function also determines a spin vector field

$$\psi \gamma_3 \tilde{\psi} = \rho R \gamma_3 \tilde{R} = \rho R \gamma_3 \tilde{R}. \quad (5.14)$$

The interpretation of $s = R \gamma_3 \tilde{R}$ as a *spin vector* can be justified by using the Dirac equation (Ref. 5), but it is enough simply to assert it here.

The wave function determines, in fact, a complete frame of vector fields

$$\psi \gamma_\mu \tilde{\psi} = \rho e_\mu, \quad (5.15)$$

where

$$e_\mu = R \gamma_\mu \tilde{R}, \quad (5.16)$$

with $v = e_0$ and $s = e_3$. We have encountered such equations already in our formulation of relativistic rigid body mechanics. The only difference is that here the e_μ are vector fields whereas before they composed a comoving frame attached to a single world line. However, the equation of continuity (5.12) implies the existence of streamlines with the velocity v as tangent vector. Therefore, along any streamline the e_μ of (5.16) can be regarded as a *comoving frame* exactly as before.

We have arrived, now, at a physical and geometrical interpretation of the 8 real parameters in the Dirac wave function by putting it in the canonical form (5.10). One parameter ρ specifies the probability density. Six parameters in R specify a Lorentz transformation determining a field of comoving frames e_μ . The identification of $s = e_3$ as a spin vector relates the frame to relativistic rigid body mechanics and so gives it a physical as well as a geometrical interpretation. An interpretation of the remaining invariant parameter β will arise from a study of the plane wave solutions, where it will be associated with a rotation sense of the comoving frame. Applied to motion along a streamline, equation (3.4) becomes

$$\dot{R} = v \cdot \square R = \frac{1}{2} \Omega R, \quad (5.17)$$

where the bivector Ω is the *rotational velocity*. The properties of Ω have been studied in Ref. 5 but we need not go into such detail here. The point of greatest interest here concerns the interpretation of Ω . If Ω can be given a purely mechanical interpretation, then the Dirac theory can be unified with the rigid body theory. Otherwise, the theories merely stand in strong mathematical analogy with one another. We shall return to this issue later on.

It should be realized that the index μ on the e_μ is a *free index*, unrelated to any coordinate system. The e_μ are invariant vector fields on spacetime. The equation $e_\mu = R \gamma_\mu \tilde{R}$ relates the e_μ to some arbitrarily chosen fixed frame γ_μ , so the set of e_μ is determined by a single

TABLE I: BILINEAR COVARIANTS

Scalar	$\tilde{\Psi}\Psi = \Psi^\dagger\gamma_0\Psi = \rho \cos \beta = \langle \psi\tilde{\psi} \rangle_s$
Vector	$\tilde{\Psi}\gamma_\mu\Psi = \psi^\dagger\gamma_0\gamma_\mu\psi = \rho v \cdot \gamma_\mu = \langle \gamma_\mu\psi\gamma_0\tilde{\psi} \rangle_s$
Bivector	$\tilde{\Psi}\gamma_\mu \wedge \gamma_\nu\Psi = \Psi^\dagger\gamma_0\frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\Psi$ $= \rho(ie^{i\beta}sv) \cdot (\gamma_\mu \wedge \gamma_\nu) = \langle \gamma_\mu\gamma_\nu\psi\gamma_2\gamma_1\tilde{\psi} \rangle_s$
Pseudovector	$i'\tilde{\Psi}\gamma_5\gamma_\mu\Psi = \rho s \cdot \gamma_\mu = \langle \gamma_\mu\psi\gamma_3\tilde{\psi} \rangle_s$
Pseudoscalar	$\tilde{\Psi}\gamma_5\Psi = -\rho \sin \beta = \langle i\psi\tilde{\psi} \rangle_s$

spinor R . Of course, if we wish to change the reference frame $\{\gamma_\mu\}$ to some other fixed frame $\{\gamma'_\mu\}$, then, the spinor R must be changed to R' so $e_\mu = R\gamma_\mu\tilde{R} = R'\gamma'_\mu R'$ remains invariant. This change corresponds to a change of matrix representation in the matrix form of the Dirac theory. Its physical significance is evidently trivial.

Equations (5.8), (5.11) and (5.14) relating the real wave function ψ to invariant scalar and vector fields on spacetime correspond to real bilinear covariants in the Dirac theory, as shown in Table 1. The correspondence can be proved by using (5.1 a,b,c) and (5.2). Of course, the γ_μ in the table are to be interpreted as matrices when they operate on Ψ and as vectors in the STA when they multiply ψ . The bracket $\langle \dots \rangle_s$ used in the table means “scalar part,” and the fairly standard symbol $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ is employed in the matrix expressions.

Since there are $1 + 4 + 6 + 4 + 1 = 16$ distinct bilinear covariants but only 8 parameters in the wave function, the various covariants are not independent of one another. The table shows exactly how they are interrelated by expressing them in terms of ρ , β , $v \cdot \gamma_\mu$ and $s \cdot \gamma_\mu$, which constitutes a set of 7 independent parameters since the four components of the velocity and spin vectors are constrained by the three conditions $v^2 = -s^2 = 1$ and $v \cdot s = 0$. This makes it possible to write down by inspection a variety of identities, such as

$$\rho^2 = (\tilde{\Psi}\Psi)^2 + (\tilde{\Psi}\gamma_5\Psi)^2 = (\tilde{\Psi}\gamma_\mu\Psi)(\tilde{\Psi}\gamma^\mu\Psi) = -(\tilde{\Psi}\gamma_\mu\gamma_5\Psi)(\tilde{\Psi}\gamma^\mu\gamma_5\Psi),$$

which are derived more or less laboriously in the literature.

The most significant thing to note about the table is that only 7 of the 8 parameters in the wave function appear. The missing parameter is the *phase* of the wave function, and its absence is tied up with the absence of e_1 and e_2 , as we shall see in the next section.

The correspondence between the *matrix* formulation and the *real* formulation of the Dirac theory has now been sufficiently established, so we can henceforth work with the real theory

alone with the assurance that it is mathematically equivalent to the matrix theory. We shall see, however, that the real formulation will continue to reveal a geometric structure of the Dirac theory which is not at all apparent in the matrix formulation.

6. Real Plane Waves.

For a free particle, the Dirac equation (5.5) reduces to

$$\square\psi\mathbf{i} = \psi p_0, \quad (6.1)$$

where $p_0 = m\gamma_0$ and $\mathbf{i} = \gamma_2\gamma_1 = i\sigma_3$.

This equation admits the “plane wave” solution

$$\psi = \rho^{\frac{1}{2}} e^{\frac{1}{2}i\beta} R = \rho^{\frac{1}{2}} e^{\frac{1}{2}i\beta} R_0 e^{-i p \cdot x}, \quad (6.2)$$

where the spinor R satisfying $R\tilde{R} = 1$ has been decomposed into $R = R_0 e^{-i p \cdot x}$ to exhibit its spacetime dependence. Inserting (6.2) into (6.1) and using $\square p \cdot x = p$, we obtain

$$p\psi = \psi p_0.$$

Solving for p we get

$$p = e^{i\beta} R p_0 \tilde{R}.$$

This implies $e^{i\beta} = \pm 1$, or equivalently,

$$e^{\frac{1}{2}i\beta} = 1 \quad \text{or} \quad i. \quad (6.3)$$

Thus we obtain two distinct solutions characterizing a particle with momentum

$$p = \pm R p_0 \tilde{R}.$$

This equation implies $p^2 = p_0^2 = m^2$. With the positive sign it describes a boost from momentum p_0 to momentum p . With the negative sign it describes a particle with negative energy $E = \gamma_0 \cdot p < 0$, which is inadmissible on physical grounds. But this can easily be rectified by changing the sign in the phase of the wave function (6.2).

Thus, we obtain two distinct kinds of plane wave solutions with positive energy $E = p \cdot \gamma_0$:

$$\psi_- = \rho^{\frac{1}{2}} R_0 e^{-i p \cdot x}, \quad (6.4a)$$

$$\psi_+ = \rho^{\frac{1}{2}} i R_0 e^{+i p \cdot x}. \quad (6.4b)$$

We can identify these as *electron* and *positron* wave functions respectively. The two wave functions describe particles with identical velocity and spin directions:

$$\psi_{\pm} \gamma_0 \tilde{\psi}_{\pm} = \rho R_0 \gamma_0 \tilde{R}_0 = \rho v, \quad (6.5a)$$

$$\psi_{\pm} \gamma_3 \tilde{\psi}_{\pm} = \rho R_0 \gamma_3 \tilde{R}_0 = \rho s. \quad (6.5b)$$

Note that it is unnecessary to decompose + into “spin up” and “spin down” solutions. The spin state is completely described by (6.5b), which is independent of any chosen “axis of quantization.”

If the wave functions are normalized to one particle per unit volume V in the γ_0 -system, then

$$\rho = \frac{m}{EV} = \frac{1}{v \cdot \gamma_0 V}, \quad (6.6)$$

so $\rho_0 = \gamma_0 \cdot (\rho v) = V^{-1}$. We use this in Appendix B.

In accordance with (5.17), for the wave functions ψ_{\pm} , the *rotational velocities* Ω_{\pm} along a streamline is given by

$$\frac{d\psi_{\pm}}{d\tau} = v \cdot \square \psi_{\pm} = \frac{1}{2} \Omega_{\pm} \psi_{\pm}. \quad (6.7)$$

From this we obtain

$$\Omega_{\pm} = \pm 2p \cdot v R \mathbf{i} \tilde{R} = \pm 2m R \gamma_2 \gamma_1 \tilde{R} = \pm 2m e_2 e_1. \quad (6.8)$$

Thus, the comoving frame rotates “about the spin axis $s = e_3$ ” in the $e_2 e_1$ plane with an angular speed $|\Omega_{\pm}| = 2m$ equal to twice the rest energy. The angle of rotation is given by the phase of the wave function. The electron and positron wave functions ψ_{\pm} differ only by ascribing opposite senses to this rotation.

We have here a geometrical distinction between electron and positron wave functions which is not apparent in the matrix formulation, because it uses an imaginary i' which has no geometric interpretation. The question is: Does this tell us something new and fundamental about the theory? I will return to this question after we examine how to handle scattering with the real theory.

7. Scattering Theory

At first sight it appears that the real Dirac equation

$$\square \psi \mathbf{i} - \psi p_0 = -e A \psi \quad (7.1)$$

might be more difficult to solve than the matrix version, because conventional methods based on Fourier transforms make extensive use of a unit imaginary i' that commutes with everything. It turns out, however, that conventional methods can be employed with only a slight modification to account for noncommutivity of the bivector $\mathbf{i} = \gamma_2 \gamma_1$ and, remarkably, certain simplifications actually occur. The discussion here follows the approach and notation of Bjorken and Drell [10] as closely as possible, so the reader can see where the essential differences appear and similarities can be passed over quickly.

For scattering problems it is most convenient to convert the Dirac equation to an integral equation

$$\psi(x) = \psi_i(x) - e \int d^4 x' S_F(x - x') A(x') \psi(x'). \quad (7.2)$$

This provides a solution of the Dirac equation (7.1) if the Green’s function $S_F(x - x')$ satisfies the equation

$$\square S_F(X - x') M(x') \mathbf{i} - S_F(x - x') M(x') p_0 = \delta^4(x - x') M(x'), \quad (7.3)$$

where $M = M(x')$ is an arbitrary multivector valued function of x' .

Equation (7.3) has the *causal solution*

$$\begin{aligned} S_F(x - x')M = & -\frac{\theta(t - t')}{(2\pi)^3} \int \frac{d^3p}{2E} (pM + Mp_0) \mathbf{i} e^{-\mathbf{i}p \cdot (x - x')} \\ & + \frac{\theta(t - t')}{(2\pi)^3} \int \frac{d^3p}{2E} (pM + Mp_0) \mathbf{i} e^{\mathbf{i}p \cdot (x - x')}, \end{aligned} \quad (7.4)$$

where $E = p \cdot \gamma_0 > 0$. Note that $S_F(x - x')$ is a linear operator on M here. In general M does not commute with p or p_0 , so it cannot be pulled from under the integral.

On the right side of (7.2) the first term $\psi_i = \psi_i(x)$ can be regarded as an initial incoming plane wave while the second term, call it $\psi_\infty(x)$, is the scattered wave. Inserting Feynmann's causal Green's function (7.4) into (7.2), we find for the scattered wave

$$\psi_\infty(x) = e \int \frac{d^4x'}{(2\pi)^3} \int \frac{d^3p}{2E} [pA(x')\psi(x') + A(x')\psi(x')p_0] \mathbf{i} e^{-\mathbf{i}p \cdot (x - x')}.$$

The scattered wave can be expressed as superposition

$$\psi_\infty(x) = \int \frac{d^3p_f}{(2\pi)^3} \psi_f(x)$$

of *final state plane waves* ψ_f . Hence,

$$\psi_\infty(x) = e \int \frac{d^4x'}{2E_f} [p_f A(x')\psi(x') + A(x')\psi(x')p_0] \mathbf{i} e^{-\mathbf{i}p \cdot (x - x')} \quad (7.5)$$

where p_f is the final state momentum.

In the *Born Approximation* we approximate $\psi(x')$ under the integral in (7.5) by the initial plane wave ψ_i . Using $\psi_i p_0 = p_i \psi_i$,

where p_i is the initial momentum, we obtain from (7.5)

$$\psi_f(x) = \frac{e}{2E_f} \int d^4x' [p_f A(x') + A(x')p_i] \psi_i \mathbf{i} e^{-\mathbf{i}p \cdot (x - x')}. \quad (7.6)$$

Now, for any given vector potential $A(x')$ we can perform the integral in (7.6) to obtain a solution to the scattering problem. The quantity in square brackets in (7.6) is especially significant, for, as we shall see, it determines the general algebraic structure of the scattering matrix.

8. Coulomb Scattering.

The Coulomb potential can be written in the form

$$A(x') = \frac{Ze}{|x'|} \gamma_0, \quad (8.1)$$

where $\mathbf{x}' = x' \wedge \gamma_0$. Inserting this into (7.6) and carrying out the integral we obtain

$$\psi_f = S_{fi} \psi e^{-i\mathbf{q}\cdot\mathbf{x}} \frac{(2\pi)^3}{E_f} \delta(E_f - E_i) \quad (8.2)$$

where $q = p_f - p_i$ and

$$S_{fi} = \frac{Ze^2}{\mathbf{q}^2} (p_f \gamma_0 + \gamma_0 p_i), \quad (8.3)$$

with $\mathbf{q} = \mathbf{p}_f - \mathbf{p}_i$. The delta function in (8.2) implies $E_f = E_i \equiv E$, so

$$p_f \gamma_0 = E - \mathbf{p}_f \quad \text{and} \quad \gamma_0 p_i = E - \mathbf{p}_i, \quad (8.4)$$

$$S_{fi} = \frac{Ze^2}{\mathbf{q}^2} (2E + \mathbf{q}).$$

Let us call S_{fi} the *S-matrix*, because it plays the role of the usual scattering matrix.

The big difference is that here the values of the *S-matrix* are real spinors, whereas the values are complex numbers in the conventional matrix theory. For any S_{fi} , it follows from the form of (8.1) that the differential scattering cross section is given by

$$\frac{d\sigma}{d\Omega_f} = |S_{fi}|^2 = S_{fi} \tilde{S}_{fi}. \quad (8.5)$$

This is a general result, so its proof has been relegated to Appendix B.

The formula (8.5) is for elastic scattering from an initial state with momentum p_i to a final state with momentum p_f , irrespective of the spin state. The formula could be modified to get the scattering cross section for a particular initial spin state, but the result is easily obtained from the more elegant and useful polarization operator calculated below.

Now, by substituting (8.4) into (8.5) we immediately obtain the *Mott* scattering cross section

$$\frac{d\sigma}{d\Omega_f} = \frac{Z^2 e^4}{\mathbf{q}^4} (4E^2 - \mathbf{q}^2) = \frac{Z^2 e^4 (1 - \mathbf{v}^2 \sin^2 \frac{1}{2}\theta)}{4E^2 \mathbf{v}^4 \sin^4 \frac{1}{2}\theta}, \quad (8.6)$$

where we have used

$$\mathbf{q}^2 = (\mathbf{p}_f - \mathbf{p}_i)^2 = 2\mathbf{p}^2(1 - \cos\theta) = 4E^2 \mathbf{v}^2 \sin^2 \frac{1}{2}\theta.$$

To determine the polarization change in Coulomb scattering, we merely need to evaluate the *spin scattering operator*

$$U_{fi} = \tilde{L}_f \hat{S}_{fi} L_i \quad (8.7)$$

previously defined by equation (4.11). Here $\hat{S}_{fi} = |S_{fi}|^{-1} S_{fi}$. We are not interested in the modulus of the operator here, so we can ignore all scalar factors in the calculation and then normalize the end product. From (8.3) we have

$$S_{fi} \sim p_f \gamma_0 + \gamma_0 p_i = L_f^2 + L_i^2.$$

Substituting this into (8.7) we get

$$U_{fi} \sim \tilde{L}_f (L_f^2 + L_i^2) \tilde{L} = L_f L_i + \tilde{L}_f \tilde{L}_i.$$

Then using (2.21) to write

$$\begin{aligned} L_f L_i &\sim (m + E + \mathbf{p}_f)(m + E + \mathbf{p}_i), \\ \tilde{L}_f \tilde{L}_i &\sim (m + E - \mathbf{p}_f)(m + E - \mathbf{p}_i), \end{aligned}$$

and, recalling (2.16) and (2.6), we obtain

$$\begin{aligned} U_{fi} &= e^{-\frac{1}{2}i\mathbf{a}} \sim (m + E)^2 + \mathbf{p}_f \mathbf{p}_i \\ &= (E + m)^2 + \mathbf{p}_f \cdot \mathbf{p}_i + i\mathbf{p}_f \times \mathbf{p}_i. \end{aligned} \tag{8.8}$$

Therefore,

$$\tan \frac{1}{2}\mathbf{a} = \frac{\mathbf{p}_i \times \mathbf{p}_f}{(E + m)^2 + \mathbf{p}_f \cdot \mathbf{p}_i} = \frac{\mathbf{n} \sin \theta}{\left(\frac{E + m}{E - m}\right) + \cos \theta}. \tag{8.9}$$

Thus the *relative spin or polarization vector* (defined in Sec. 4) is rotated about the normal \mathbf{n} to the scattering plane, through an angle $|\mathbf{a}|$ given by (8.9).

The result (8.9) agrees with equation (7.97) of Bjorken and Drell [10] and with equation (29) of Krase and Good [11]. The reader is invited to compare their methods with the present one. Every step in the calculations of the cross section and the polarization change has been included here to show how simple these calculations really are in the real formulation of scattering theory.

The example of Coulomb scattering illustrates some general features of the real scattering theory. The scattering is completely described by a single real spinor-valued S -matrix $S_{fi} = |S_{fi}| \hat{S}_{fi}$. Its modulus $|S_{fi}|$ determines the scattering cross section while its “direction” \hat{S}_{fi} determines the polarization change.

9. The Substructure and Interpretation of the Dirac Theory.

The recent success of the Weinberg-Salam model unifying weak and electromagnetic interactions supports the longstanding belief of many physicists that the electron is really a massless particle whose apparent mass arises from self-interactions, though all theories fall short of calculating that mass. A more general argument for a massless electron is based on the fact that in mass-energy conversion we can often identify the converted mass completely with kinetic and potential energy of a previously bound system. This strongly suggests that the rest mass of any object can be attributed entirely to internal kinetic and potential energy, in other words, that the ultimate constituents of matter all have zero mass.

I wish to propose an interpretation of the Dirac theory consistent with the idea that the electron is a massless particle. To do this it is necessary to attribute a substructure to theory, but this substructure must be such that it explains or, at least, provides a unified interpretation of the mathematical structure of the Dirac theory.

The main idea underlying my proposal is that a free electron is a massless particle bound by self-interactions to some inertial system; call it the *rest system* of the electron. The electron’s mass m is, of course, to be identified with the energy of the self-interaction. But

this energy is composed of kinetic and potential energies. The kinetic self-energy must be associated with an internal angular momentum since the electron is bound. So it is natural to identify this angular momentum with the electron spin. Thus, we arrive at the idea that the spin is an angular momentum arising from self-interaction, while the mass can be identified with the kinetic energy of the internal motion.

These ideas can be quantified with a simple model. We take the electron to be a massless particle with proper momentum P so $P^2 = 0$. Self-interaction confines it to a *rest system* with proper velocity v , where $v^2 = 1$. In this system the electron has *relative momentum* \mathbf{p} and *kinetic energy*

$$E = P \cdot v = mc^2. \quad (9.1)$$

Thus,

$$Pv = P \cdot v + P \wedge v = E + c\mathbf{p}. \quad (9.2)$$

(Natural units will not be used in this Section.)

The orbit of the electron in the rest system is a circle with *spin* (angular momentum) $\mathbf{s} = \mathbf{r} \times \mathbf{p}$. According to the Dirac theory (Ref. 5), the magnitude of the spin is $|\mathbf{s}| = \frac{1}{2}\hbar$. But

$$|\mathbf{s}| = r|\mathbf{p}| = r \frac{E}{c} = rmc = \frac{\hbar}{2}. \quad (9.3)$$

Therefore, the radius of the orbit is of the order of the Compton wavelength

$$r = \frac{\hbar}{2mc} = 1.0 \times 10^{-13} \text{ cm}, \quad (9.4)$$

and the circular frequency is

$$|\Omega| = \frac{|\mathbf{v}|}{r} = \frac{c}{r} = \frac{2mc^2}{\hbar} = 3 \times 10^{23} \text{ sec}^{-1}. \quad (9.5)$$

The world line of this model electron is a helical curve with tangent vector P winding around a straight timelike line with tangent vector v . Averaged over one cycle the proper momentum is

$$\langle P \rangle_{\text{ave}} = mc^2 v = p. \quad (9.6)$$

We can interpret the Dirac theory in terms of this model of a “spinning electron” by assuming that the wave function describes a statistical ensemble of such particles, and the velocity v in the Dirac current $\psi\gamma_0\tilde{\psi} = \rho v$ is the average velocity of the ensemble. Then the free electron plane wave function (6.4a) describes an ensemble of particles with average momentum p given by (9.6); the particles of the ensemble are uniformly distributed in the electron rest system, so ρ is constant; and the spin axes of all particles are collinear, so the spin current $\psi\gamma_3\tilde{\psi} = \rho s$ is constant and uniform.

Our model provides a coherent interpretation of every feature of the free particle wave function. It relates the mass to the spin to the phase of the wave function. The phase simply describes the angle through which the electron has passed in its circular orbit. Even the circular frequency (9.5) required by the model is identical to the frequency (5.8) of the comoving frame determined by the wave function. Thus, our model takes the analogy between rigid body mechanics and the Dirac wave function quite literally. It tells us that

the Dirac wave function represents a locally circulating particle motion as a local relativistic rigid body motion described by a rotating frame.

The idea that the electron spin arises from a helical orbit in spacetime has been suggested by too many authors to mention. (See Ref. 5 for some references.) But no one to my knowledge has related the idea to the Dirac wave function in the manner suggested here. Many have noted that the so called *zitlerbewegung* of a Dirac electron can be interpreted as a kind of helical motion or circulation (Ref. 5). But the zbw appears only in wave packets, not in plane waves. This has long made it difficult to understand the zbw as an interpretation or explanation of spin, for the plane wave solutions certainly describe a particle with spin. The model suggested here puts the matter in a new light. The Dirac current is supposed to be an average over an ensemble of locally circulating particles, much like a system of *Amperian currents*. The plane wave describes a uniform distribution of currents, so the currents cancel out locally and there is no net circulation. However, a wave packet describes a nonuniform distribution, so the cancellation is imperfect and there is a net circulation manifested in a nonvanishing curl $\square \wedge v$ in the relativistic case (Ref. 5) and $\nabla \times v$ in the nonrelativistic case (Ref. 12).

The model suggested here has many implications and raises many questions, such as, how does one account for diffraction with the model? Many of these issues have been discussed at some length elsewhere [12], so we will not confront them here.

Nothing has been said here about the nature of the self-interaction and how it determines the electron mass and spin. The position taken here is that the Dirac theory tacitly takes this self-interaction for granted, and new physical assumptions will be required to explain it. But the Dirac theory is not without clues to a deeper theory. Perhaps the most significant clue brought to light by the real formulation of the Dirac theory is the fact that the *generator of electromagnetic gauge transformations* in the Dirac equation (5.5) is the bivector

$$\mathbf{i} = \gamma_2 \gamma_1 = i \sigma_3. \quad (9.7)$$

Therefore, it has a definite geometrical interpretation. Now, the Weinberg-Salam model extends the electromagnetic gauge group to a larger gauge group that includes weak interactions. Therefore, the generators of this larger group should also have interpretation linked to the geometry of the Dirac theory. This idea has been elaborated elsewhere [13], but the theory is still far from complete.

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APPENDIX A: ALGEBRAIC STRUCTURE OF THE S -MATRIX.

For a large class of problems, the spinor S -matrix for *elastic scattering* has the algebraic form

$$S_{fi} = p_f M + M p_i, \tag{A.1}$$

where $p_f^2 = p_i^2 = m^2$, and M is an odd multivector. Since M is odd, it has the decomposition into vector and pseudovector parts

$$M = a + ib, \tag{A.2}$$

where a and b are vectors.

Now a fundamental property of the S -matrix is that it must rotate the initial momentum into the final momentum. This is assured by the theorem

$$S_{fi} p_i \tilde{S}_{fi} = |S_{fi}|^2 p_f, \tag{A.3}$$

which obtains if S_{fi} has the general form (A.1).

Theorem (A.1) is an immediate consequence of the corollary

$$S_{fi} p_i = p_f S_{fi}. \tag{A.4}$$

The corollary is proved by

$$(p_f M + M p_i) p_i = p_f M p_i + M m^2 = p_f (M p_i + p_f M).$$

To evaluate $|S_{fi}|^2$, we use (A.2) in (A.1) to decompose S_{fi} into its scalar, bivector and pseudoscalar parts, obtaining

$$S_{fi} = a \cdot p + a \wedge q - i(p \wedge b + q \cdot b), \quad (\text{A.5})$$

where $p = p_f + p_i$ and $q = p_f - p_i$. From this we obtain

$$|S_{fi}|^2 \equiv S_{fi} \tilde{S}_{fi} = (a \cdot p)^2 - (b \cdot q)^2 + |a \wedge q|^2 - |b \wedge p|^2 + 2ia \wedge b \wedge p \wedge q. \quad (\text{A.6})$$

This can be further expanded by using

$$|a \wedge q|^2 = (a \wedge q) \cdot (q \wedge a) = a^2 q^2 - (a \cdot q)^2,$$

but it is easier to interpret as it is.

The interpretation of $|S_{fi}|^2$ as a scattering cross section in Appendix B requires that $|S_{fi}|^2$ be a positive scalar. Equation (A.6) shows that it is a scalar, but it is positive only for certain values of a and b . For Coulomb scattering $b = 0$, so

$$|S_{fi}|^2 = (a \cdot p)^2 + |a \wedge q|^2$$

The first term is always positive, and the second term is positive if a is a timelike vector, as it is in the Coulomb case. On the other hand, the term $-(b \cdot q)^2$ in (A.6) is, of course, always negative while $-|b \wedge p|^2$ is positive for spacelike b . The last term in (A.6) can have either sign for any a and b unless they have some special relation to p and q .

The general point of interest here is that the study of the scattering matrix has been reduced to the study of the quantity $M = a + ib$ which must describe the dynamics of the scattering completely. Hopefully, this will help us attain a deeper insight into the general theory.

APPENDIX B: SCATTERING CROSS SECTION.

The probability current for the initial (plane wave) state is

$$J_i \equiv \psi_i \gamma_0 \tilde{\psi}_i = \rho_i \frac{p_i}{m} = \left(\frac{m}{E_i V} \right) \frac{p_i}{m}. \quad (\text{B.1})$$

Recalling the mnemonic

$$[2\pi\delta(E_f - E_i)]^2 = 2\pi T \delta(E_f - E_i)$$

from Bjorken and Drell, we obtain from equation (7.2) the final state probability current

$$J_f \equiv \psi_f \gamma_0 \tilde{\psi}_f = S_{fi} J_i \tilde{S}_{fi} \frac{(2\pi)^3}{E_f^2} T \delta(E_f - E_i). \quad (\text{B.2})$$

Now, if we assume that the scattering matrix S_{fi} has the general structure (A.1) so theorem (A.3) applies, then (B.2) gives us

$$J_f = |S_{fi}|^2 p_f \frac{(2\pi)^3}{V E_i E_f^2} T \delta(E_f - E_i). \quad (\text{B.3})$$

Therefore, the final state probability density in the γ_0 - system is

$$\rho_{f0} \equiv J_f \cdot \gamma_0 = |S_{fi}|^2 \frac{(2\pi)^3}{V E_i E_f} T \delta(E_f - E_i). \quad (\text{B.4})$$

The differential scattering cross section is defined by

$$\frac{d\sigma}{d\Omega_f} = \frac{1}{|\mathbf{J}_i|} \int \frac{\mathbf{p}_f \mathbf{d}|\mathbf{p}_f|}{(2\pi)^3} \frac{\rho_{f0}}{T} = \frac{V}{|\mathbf{v}_i|} \int \frac{|v_f| E_f^2}{(2\pi)^3} dE_f \frac{\rho_{f0}}{T}, \quad (\text{B.5})$$

where, from (B.1), $\mathbf{J}_i = J_i \wedge \gamma_0 = \mathbf{p}_i/E_i V = \mathbf{v}_i/V$ is the incident flux.

Substituting (B.4) into (B.5), we obtain at once the general scattering theorem

$$\frac{d\sigma}{d\Omega_f} = |S_{fi}|^2 = S_{fi} \tilde{S}_{fi}. \quad (\text{B.6})$$

Our proof of this theorem is limited by assuming the energy delta function $\delta(E_f - E_i)$. Of course, a more general proof can be achieved with a more general delta function.

APPENDIX C: WAVE FUNCTIONS FOR MASSLESS PARTICLES.

For completeness a few words should be said about *real* wave functions for massless particles, especially since they are so important in the theory of weak interactions.

Let ψ be a real Dirac spinor with the canonical form (5.10). We can decompose ψ into the orthogonal components

$$\psi_{\pm} = \psi \frac{1}{2} (1 \pm \boldsymbol{\sigma}_3) \quad (\text{C.1})$$

so

$$\psi = \psi_+ + \psi_-. \quad (\text{C.2})$$

These components correspond to the *helicity components*.

$$\psi_{\pm} = \frac{1}{2} (1 \pm i' \gamma_5) \psi$$

in the matrix formulation, to prove this, use (5.2) and (5.1c) to obtain

$$i' \gamma_5 \psi = \gamma_5 \Psi i' u = \gamma_5 \psi \gamma_2 \gamma_1 u.$$

Therefore, since $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = i$ and $\gamma_2 \gamma_1 = i \gamma_3 \gamma_0 = i \boldsymbol{\sigma}_3$, the expression $i' \gamma_5 \Psi$ in the matrix formulation corresponds to $i \psi i \boldsymbol{\sigma}_3 = -\psi \boldsymbol{\sigma}_3$ in the real formulation. Hence, Ψ_{\pm} corresponds to ψ_{\pm} .

The properties of ψ_{\pm} can be determined without reference to the matrix formulation. First of all, from the identity $(1 + \boldsymbol{\sigma}_3)(1 - \boldsymbol{\sigma}_3) = 0$ it follows that

$$\psi_{\pm} \tilde{\psi}_{\pm} = 0. \quad (\text{C.3})$$

This is the same as equation (5.8) for the case $\rho = 0$. Next, from

$$(1 \pm \boldsymbol{\sigma}_3)^2 = 2(1 \pm \boldsymbol{\sigma}_3)$$

and

$$(1 \pm \boldsymbol{\sigma}_3)\gamma_0 = \gamma_0 \pm \gamma_3 = \pm(1 \pm \boldsymbol{\sigma}_3)\gamma_3,$$

along with (5.11) and (5.14), we obtain the *probability current*

$$J_{\pm} \equiv \psi_{\pm} \gamma_0 \tilde{\psi}_{\pm} = \frac{1}{2} \psi (\gamma_0 + \gamma_3) \tilde{\psi} = \frac{1}{2} \rho (v + s), \quad (\text{C.4})$$

and the spin current

$$\psi_{\pm} \gamma_3 \tilde{\psi}_{\pm} = \pm J_{\pm} \quad (\text{C.5})$$

Actually, the scale factor ρ in (C.4) cannot be interpreted as a proper probability density like it was for a Dirac electron, because

$$J_{\pm}^2 = 0, \quad (\text{C.6})$$

so J_{\pm} does not determine a local rest system. However, the probability density in any inertial system, the γ_0 -system say, is still given by

$$\rho_0 = \gamma_0 \cdot J_{\pm}. \quad (\text{C.7})$$

For zero mass, the free-particle Dirac equation (6.1) becomes

$$\square \psi_{\pm} = 0. \quad (\text{C.8})$$

For a plane wave of momentum p this implies $p\psi_{\pm} = 0$, so using (C.4) we obtain

$$pJ_{\pm} = p \cdot J_{\pm} + p \wedge J_{\pm} = 0.$$

This tells us that p is collinear with J_{\pm} , so, of course, $p^2 = 0$.

According to (C.5) the spin vector is collinear with J_{\pm} , so it does not describe an independent degree of freedom. The two signs show that the two solutions ψ_{\pm} are distinguished by spin parallel or antiparallel with J_{\pm} . These are the so-called left (right)-handed wave functions. Of course, the space-time split of the spin vector in Sec. 4 does not apply to massless particles.

Finally, it might be noted that

$$\psi_{\pm} \gamma_1 \tilde{\psi}_{\pm} = 0 = \psi_{\pm} \gamma_2 \tilde{\psi}_{\pm}$$

does not give us any useful information.