

## Differential Forms in Geometric Calculus

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**Abstract** Geometric calculus and the calculus of differential forms have common origins in Grassmann algebra but different lines of historical development, so mathematicians have been slow to recognize that they belong together in a single mathematical system. This paper reviews the rationale for embedding differential forms in the more comprehensive system of Geometric Calculus. The most significant application of the system is to relativistic physics where it is referred to as Spacetime Calculus. The fundamental integral theorems are discussed along with applications to physics, especially electrodynamics.

### 1. Introduction

Contrary to the myth of the solitary scientific genius, science (including mathematics) is a social activity. We all feed on one another's ideas. Without our scientific culture the greatest mathematical genius among us could not progress beyond systematic counting recorded on fingers and toes. The primary means for communicating new ideas is the scientific literature. However, it is extremely difficult to read that literature without learning how through direct contact with others who already can. Even so, important ideas in the literature are overlooked or misconstrued more often than not. The history of mathematical ideas (especially those of Hermann Grassmann) shows this conclusively.

A workshop like this one, bringing together scientists with common interests but divergent backgrounds, provides a uniquely valuable opportunity to set the written record straight—to clarify and debate crucial ideas—to progress toward a consensus. We owe an immense debt of gratitude to the Workshop organizers who made this possible: *Professors Fred Brackx, Richard Delanghe, and Herman Serras*. This is also a good opportunity to pay special tribute to *Professor Roy Chisholm*, who, with uncommon insight into the social dimension of science, conceived, organized and directed the *First International Workshop on Clifford Algebras and Their Applications* in 1986. He set the standard for Workshops to follow. Without his leadership we would not be here today.

As in previous *Clifford Algebra Workshops* [1–4] my purpose here is to foment debate and discussion about fundamental mathematical concepts. This necessarily overflows into debate about the terminology and notations adopted to designate those concepts. At the outset, I want it understood that I intend no offense toward my esteemed colleagues who hold contrary opinions. Nevertheless, I will not mince words, as I could not take the subject more seriously. At stake is the very *integrity of mathematics*. I will strive to formulate and defend my position as clearly and forcefully as possible. At the same time, I welcome rational opposition, as I know that common understanding and consensus is forged in the dialectic struggle among incompatible ideas. Let the debate proceed!

I reiterate my contention that the subject of this conference should be called *Geometric Algebra* rather than *Clifford Algebra*. This is not a mere quibble over names, but a brazen claim to vast intellectual property. What's in these names? To the few mathematicians familiar with the term, “Clifford Algebra” refers to a minor mathematical subspecialty

concerned with quadratic forms, just one more algebra among many other algebras. We should not bow to such a myopic view of our discipline.

I invite you, instead, to join me in proclaiming that *Geometric Algebra* is no less than a universal mathematical language for precisely expressing and reasoning with geometric concepts. “Clifford Algebra” may be a suitable term for the grammar of this language, but there is far more to the language than the grammar, and this has been largely overlooked by the strictly formal approach to Clifford Algebra.

Let me remind you that Clifford himself suggested the term *Geometric Algebra*, and he described his own contribution as an application of Grassmann’s *extensive algebra* [3]. In fact, all the crucial geometric and algebraic ideas were originally set forth by Grassmann. What is called “Grassmann Algebra” today is only a fragment of Grassmann’s system. His entire system is closer to what we call “Clifford Algebra.” Though we should remember and admire the contributions of both Grassmann and Clifford, I contend that the conceptual system in question is too universal to be attached to the name of any one individual. Though Grassmann himself called it the *Algebra of Extension*, I believe he would be satisfied with the name *Geometric Algebra*. He was quite explicit about his intention to give geometry a suitable mathematical formulation.

Like the real number system, Geometric Algebra is our common heritage, and many individuals besides Grassmann and Clifford have contributed to its development. The system continues to evolve and has expanded to embrace differentiation, integration, and mathematical analysis. No consensus has appeared on a name for this expanded mathematical system, so I hope you will join me in calling it *Geometric Calculus*.

Under the leadership of *Richard Delanghe*, mathematical analysis with Clifford Algebra has become a recognized and active branch of mathematics called *Clifford Analysis*. I submit, though, that this name fails to do justice to the subject. Clifford analysis should not be regarded as just one more branch of analysis, along side real and complex analysis. Clifford analysis, properly construed, generalizes, subsumes, and unifies all branches of analysis; it is the whole of analysis. To proclaim that fact, workers in the field should set modesty aside and unite in adopting a name that boldly announces claim to the territory. At one time I suggested the name *Geometric Function Theory* [5], but I am not particularly partial to it. However, I insist on the term *Geometric Calculus* for the broader conceptual system which integrates analysis with the theory of manifolds, differential geometry, Lie groups, and Lie algebras.

The proclamation of a universal *Geometric Calculus* [1,5] has met with some skepticism [3], but the main objection has now been decisively answered in [6], which shows that, not only does embedding a vector space with its dual into a common geometric algebra not suffer a loss of generality, but there are positive advantages to it as well. Indeed, physicists and mathematicians have been doing just that for some time without recognizing the fact. I believe that the remaining barriers to establishing a consensus on *Geometric Calculus* are more psychological or sociological than substantive. My intention in this article is to keep hammering away at those barriers with hope for a breakthrough.

The literature relating Clifford algebra to fiber bundles and differential forms is rapidly growing into a monstrous, muddled mountain. I hold that the muddle arises mainly from the convergence of mathematical traditions in domains where they are uncritically mixed by individuals who are not fully cognizant of their conceptual and historical roots. As I have noted before [1], the result is a highly redundant literature, with the same results appearing

over and over again in different notational guises. The only way out of this muddle, I think, is to establish a consensus on the issues. Toward that end, I now present my own views on the issues. I include some personal history on the evolution of my views with the hope that it will highlight the most important ideas. I will presume that the reader has some familiarity with the notation and nomenclature I use from my other publications.

## 2. What is a manifold?

The formalism for modern differential geometry (as expounded, for example, by O’Neill [7]) was developed without the insights of Geometric Algebra, except for a fragment of Grassmann’s system incorporated into the calculus of differential forms. Can the formalism of differential geometry be improved by a *new synthesis* which incorporates Geometric Algebra in a fundamental way? My answer is a resounding YES! Moreover, I recommend the Geometric Calculus found in [5] as the way to do it. I am afraid, however, that the essential reasons for this new synthesis have been widely overlooked, so my purpose is to emphasize them today. Readers who want more mathematical details can find them in [5].

Everyone agrees, I suppose, that the concept of a (*differentiable*) *manifold* is the foundation for differential geometry. However, the very definition of “manifold” raises a question. In the standard definition [7] coordinates play an essential role, but it is proved that the choice of well-defined coordinates is arbitrary. In other words, the concept of a manifold is really independent of its representation by coordinates. Why, then, is the clumsy apparatus of coordinate systems used to define the concept? The reason, I submit, is historical: no better means for describing the structure of a manifold was available to the developers of the concept. Futhermore, I claim that Geometric Algebra alone provides the complete system of algebraic tools needed for an intrinsic characterization of manifolds to replace the extrinsic characterization with coordinates. This is not to say that coordinates are without interest. It merely displaces coordinates from a central place in manifold theory to the periphery where they can be employed when convenient.

Now to get more specific, let  $x$  be a generic point in a  $m$ -dimensional manifold  $\mathcal{M}$ , and suppose that a patch of the manifold is parameterized by a set of coordinates  $\{x^\mu\}$ , as expressed by

$$x = x(x^1, x^2, \dots, x^m). \quad (2.1)$$

If the manifold is embedded in a vector space, so  $x$  is vector-valued, then the vector fields  $e_\mu = e_\mu(x)$  of tangent vectors to the coordinate curves parameterized by  $x_\mu$  are given by

$$e_\mu = \partial_\mu x = \frac{\partial x}{\partial x^\mu}. \quad (2.2)$$

I recall that when I was a graduate student reading Cartan’s work on differential geometry, I was mystified by the fact that Cartan wrote down (2.2) for any manifold without saying anything about the values of  $x$ . This violated the prohibition against algebraic operations among different points on a general manifold which I found in all the textbooks; for the very meaning of (2.2) is supplied by its definition as the limit of a difference quotient:

$$\partial_\mu x = \lim \frac{\Delta x}{\Delta x^\mu}. \quad (2.3)$$

Certainly  $\Delta x^\mu$  is well defined as a scalar quantity, but what is the meaning of  $\Delta x$  if it is not a “difference vector,” and what meaning can be attributed to the limit process if no measure  $|\Delta x|$  of the magnitude of  $\Delta x$  is specified? I concluded that (2.2) was merely a heuristic device for Cartan, for he never appealed to it in any arguments.

Evidently, others came to the same conclusion, for in modern books on differential geometry [7] the mysterious  $x$  has been expunged from (2.2) so  $e_\mu$  is identified with  $\partial_\mu$ ; in other words, *tangent vectors are identified with differential operators*. I think this is a *bad idea* which has complicated the subject unnecessarily. It is all very well to treat differential operators abstractly and express some properties of manifolds by their commutation relations, but this does not adequately characterize the properties of tangent vectors. The usual way to remedy this is to impose additional mathematical structure, for example, by defining a metric tensor by

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu). \quad (2.4)$$

Geometric algebra gives us another option which I maintain is more fundamental. As has been explained many times elsewhere, the very meaning of being a vector entails defining the *geometric product*

$$e_\mu e_\nu = e_\mu \cdot e_\nu + e_\mu \wedge e_\nu. \quad (2.5)$$

The inner product defines a metric tensor by

$$g_{\mu\nu} = e_\mu \cdot e_\nu \quad (2.6)$$

This has the huge advantage over (2.4) of integrating the metric tensor into algebraic structures at the ground floor. Of course, the geometric product (2.5) is incompatible with the identification  $e_\mu = \partial_\mu$  of vectors with differential operators. This led me eventually to what I believe is a deeper approach to differentiation as explained below.

Adopting (2.5) requires that we regard  $e_\mu$  as a vector, so (2.2) and (2.3) are meaningful only if the point  $x$  is a vector so  $\Delta x$  is a vector difference. I call such a manifold, whose points are vectors, a *vector manifold*. Now this seems to subvert our original intention of developing a general theory of manifolds by limiting us to a special case. It took me many years to realize that this is not the case, so I am sympathetic of colleagues who are skeptical of my claim that *the theory of vector manifolds is a general theory of manifolds*, especially since all details of the theory are not fully worked out. I would like to convince some of you, at least, that the claim is plausible and invite you to join me working out the details. I believe the payoff will be great, because the effort has been very productive already, and I believe the work is essential to establishing a truly Universal Geometric Calculus.

As explained in [3], I believe that skepticism about Geometric Calculus in general and vector manifolds in particular can be attributed to the prevalence of certain *mathematical viruses*, beliefs that limit or otherwise impair our understanding of mathematics. These include the beliefs that a vector manifold cannot be well defined without embedding it in a vector space, and it is necessarily a metric manifold, thus being too specialized for general manifold theory. As I have treated these viruses in [3] and [5], I will not address them here. I merely wish to describe my own struggle with these viral infections in the hope that it will motivate others to seek treatment. Let me mention, though, that [6] contains some potent new medicine for such treatment.

Though we want a coordinate-free theory, it is worth noting that the geometric product (2.5) facilitates calculations with coordinates. For example, it enables the construction of

the pseudoscalar for the coordinate system:

$$e_{(m)} = e_1 \wedge e_2 \wedge \dots \wedge e_m. \quad (2.7)$$

For a metric manifold we can write

$$e_{(m)} = |e_{(m)}| I_m, \quad (2.8)$$

where  $I_m = I_m(x)$  is a unit pseudoscalar for the manifold, and its modulus

$$|e_{(m)}| = |\det g_{\mu\nu}|^{1/2} \quad (2.9)$$

can be calculated from (2.7) using (2.6).

Instead of beginning with coordinate systems, the coordinate-free approach to vector manifolds in [5] begins by assuming the existence of a pseudoscalar field  $I_m = I_m(x)$  and characterizing the manifold by specifying its properties. At each point  $x$ ,  $I(x)$  is a pseudoscalar for the tangent space. If the manifold is smooth and orientable, the field  $I_m(x)$  is single-valued. If the manifold is not orientable,  $I$  is double-valued. Self-intersections and discontinuities in a manifold can be described by making  $I_m$  and its derivatives multivalued. This brings us back to the question of how to define differentiation without using coordinates. But let us address it first by reconsidering coordinates.

The inverse of the mapping (2.1) is a set of scalar-valued functions

$$x^\mu = x^\mu(x) \quad (2.10)$$

defined on the manifold  $\mathcal{M}$ . The gradients of these functions are vector fields

$$e^\mu = \partial x^\mu \quad (2.11)$$

on  $\mathcal{M}$ , and this entails the existence of a “vectorial” *gradient operator*  $\partial = \partial_x$ . But how to define it? If we take the  $e^\mu$  as given, then it can be defined in terms of coordinates by

$$\partial = e^\mu \partial_\mu, \quad (2.12)$$

where

$$\partial_\mu = e_\mu \cdot \partial \quad (2.13)$$

provided

$$e^\mu \cdot e_\nu = \delta_\nu^\mu. \quad (2.14)$$

But how can we define  $\partial$  without using coordinates?

Before continuing, I want to make it clear that I do not claim that vector manifolds are the only manifolds of interest. My claim is that *every manifold is isomorphic to a vector manifold*, so any manifold can be handled in a coordinate-free way by defining its relation to a suitable vector manifold instead of defining a coordinate covering for it. Of course, coordinate coverings have the practical value that they have been extensively developed and applied in the literature. We should take advantage of this, but my experience suggests that new insight can be gained from a coordinate-free approach in nearly every case.

It is often of interest to work directly with a given manifold instead of indirectly with a vector manifold isomorph. For example, the spin groups treated in [6] are multivector manifolds, so if (2.1) is applied directly, the point  $x$  is a spinor not a vector. In that case, it is easily shown that the tangents  $e_\mu$  defined by (2.2) are not vectors but, when evaluated at the identity, they are bivectors comprising a basis for the Lie algebra of the group. This is good to know, but the drawback to working with  $e_\mu$  which are bivectors or multivectors of other kind is that the pseudoscalar (2.7) is not defined, and that complicates analysis. The advantage of mapping even such well-behaved entities as spin groups into vector manifolds is that it facilitates differential and integral calculus on the manifold.

### 3. What is a derivative?

The differential operator defined by (2.12), where the  $e^\mu$  are tangent vectors generating a Clifford algebra on the manifold, is often called the *Dirac operator*. With no offence intended to my respected colleagues, I think that name is a *bad choice!*—not in the least justified by the fact that it has been widely used in recent years. Worse, it betrays a failure to understand what makes that operator so significant, not to mention its insensitivity to the historical fact that the idea for such an operator originated with Hamilton nearly a century before Dirac.

Whether they recognize it or not, everyone using the Dirac operator is working directly with functions defined on a vector manifold or indirectly with some mapping into a vector manifold. I hold that the *Dirac operator is a vectorial operator precisely because it is the derivative with respect to a vector*. It is the derivative with respect to a vector variable, so I propose to call it simply the *derivative* when the variable is understood, or the *vector derivative* when emphasis on the vectorial nature of the variable is appropriate. This is to claim, then, that the operator has a universal significance transcending applications to relativistic quantum mechanics where Dirac introduced it.

The strong claim that the operator  $\partial = \partial_x$  is *the derivative* needs justification. If it is so fundamental, why is this not widely recognized and accepted as such? My answer is: Because the universality of Geometric Algebra and the primacy of vector manifolds have not been recognized. When Geometric Calculus is suitably formulated, the conclusion is obvious. Let me describe how I arrived at a formulation. At the same time we will learn how to define the vector derivative without resorting to coordinates, something that took me some years to discover.

The fundamental significance of the vector derivative is revealed by Stokes' theorem. Incidentally, I think the only virtue of attaching Stokes' name to the theorem is brevity and custom. His only role in originating the theorem was setting it as a problem in a Cambridge exam after learning about it in a letter from Kelvin. He may, however, have been the first person to demonstrate that he did not fully understand the theorem in a published article: where he made the blunder of assuming that the double cross product  $v \times (\partial \times v)$  vanishes for any vector-valued function  $v = v(x)$ . The one-dimensional version of Stokes' theorem is widely known as the *fundamental theorem of integral calculus*, so it may be surprising that this name is not often adopted for the general case. I am afraid, though, that many mathematicians have not recognized the connection. Using different names for theorems differing only in dimension certainly doesn't help. I suggest that the

*Boundary Theorem of Calculus* would be a better name, because it refers explicitly to a key feature of the theorem. Let me use it here.

My first formulation of the Boundary Theorem [8] entirely in the language of Geometric Calculus had the form

$$\int d\omega \cdot \partial A = \oint d\sigma A, \quad (3.1)$$

where the integral on the left is over an  $m$ -dimensional oriented vector manifold  $\mathcal{M}$  and the integral on the right is over its boundary  $\partial\mathcal{M}$ . The integrand  $A = A(x)$  has values in the Geometric Algebra, and  $\partial = \partial_x$  is the derivative with respect to the vector variable  $x$ .

The most striking and innovative feature of (3.1) is that the differential  $d\omega = d\omega(x)$  is  $m$ -vector-valued; in other words, it is a pseudoscalar for the tangent space of  $\mathcal{M}$  at  $x$ . Likewise,  $d\sigma = d\sigma(x)$  is an  $(m-1)$ -vector-valued pseudoscalar for  $\partial\mathcal{M}$ . Later I decided to refer to  $d\omega$  as a *directed measure* and call the integrals with respect to such a measure *directed integrals*. In formulating (3.1) it became absolutely clear to me that *it is the use of directed integrals along with the vector derivative that makes the Boundary Theorem work*. This fact is thoroughly disguised in other formulations of Stokes' Theorem. As far as I know it was first made explicit in [8]. It seems to me that hardly anyone else recognizes this fact even today, and the consequence is unnecessary redundancy and complexity throughout the literature.

When I showed in [8] that the scalar part of (3.1) is fully equivalent to the standard formulation of the "Generalized Stokes' Theorem" in terms of differential forms, I wondered if (3.1) is a genuine generalization of that theorem. It took me several years to decide that, properly construed, this is so. I was impressed in [8] with the fact that (3.1) combined nine different integral theorems of conventional vector calculus into one, but I haven't seen anyone take note of that since. In any case, the deeper significance of directed measure appears in the definition of the derivative.

For a long time I was bothered by the appearance of the inner product on the left side of (3.1). I thought that in a fundamental formulation of the Boundary Theorem only the geometric product should appear. I recognized in [8], though, that if  $d\omega \wedge \partial = 0$  then  $d\omega \cdot \partial = d\omega\partial$ , and, with the appropriate limit process, the *vector derivative can be defined by*

$$\partial A = \lim_{d\omega \rightarrow 0} \frac{1}{d\omega} \oint d\sigma A. \quad (3.2)$$

This definition is indeed coordinate-free as desired, but considerable thinking and experience was required to see that it is *the best way to define the vector derivative*. The clincher was the fact that it simplifies the proof of the Boundary Theorem almost to a triviality. The Boundary Theorem is so fundamental that we should design the vector derivative to make it as simple and obvious as possible. The definition (3.2) does just that! The answer to the question of when the inner product  $d\omega \cdot \partial$  in eqn. (3.1) can be dropped in favor of the geometric product  $d\omega\partial$  is inherent in what has already been said. Those who want it spelled out should refer to [5] or [10].

I should say that the general idea of an integral definition is an old one—I do not know how old—I learned about it from [9], where it is used to define gradient, divergence, and curl. The standard definition of a derivative is so heavily emphasized that few mathematicians seem to realize the advantages of an integral definition. The fact that the right side of

(3.2) reduces to a difference quotient in the one-dimensional case supports the view that the integral definition is the best one.

The next advance in my understanding of the vector derivative and the Boundary Theorem began in 1966 when I started teaching graduate electrodynamics entirely in Geometric Algebra. As I reformulated the subject in this language, I was delighted to discover fresh insights at every turn. There is no substitute for detailed calculation and problem solving to deepen and consolidate mathematical and physical understanding. During this period I developed the necessary techniques for performing *completely coordinate-free calculations* with the vector derivative. The basic ideas were published in two brief papers which I still consider as among my best work. The first paper [10] refined, expanded and generalized my formulations of the vector derivative, directed integration, and the Boundary Theorem. It was there that I was finally convinced that the integral definition for the vector derivative is fundamental.

The second paper [11] derived a generalization of Cauchy's integral formula for  $n$  dimensions. I believe that this is one of the most important results in mathematics —so important that it has been independently discovered by several others, most notably Richard Delanghe [12] because he, with the help of brilliant students like Fred Brackx and Frank Sommen, has been responsible for energetically developing the implications of this result into the rich new mathematical domain of *Clifford Analysis*. As my paper is seldom mentioned in this domain, perhaps you will forgive me for pointing out that it contains significant features which are not appreciated in most of the literature even today. Besides the fact that the formulation and derivations are completely coordinate-free, my integral formula is actually more general than usual one, because it applies to any differentiable function or distribution, not just monogenic functions. That has too many consequences to discuss here.

In these two brief papers [10,11] on the foundations of Geometric Calculus, I made the mistake of not working out enough examples. There were so many applications to choose from that I naively assumed that anyone could generate examples easily. Subsequent years teaching graduate students disabused me of that assumption. I found that it was not an inherent difficulty of the subject so much as misconceptions from prior training that limited their learning [3].

My work on the foundations of Geometric Calculus continued into 1975, though the resulting manuscript was not published as a book [5] until 1984. That book includes and extends the previous work. It contains many other new developments in Geometric Calculus, but let me point out what is most relevant to the topics of present interest. In my previous work I restricted my formulation of the Boundary Theorem (3.1) and the vector derivative (3.2) to manifolds embedded in a vector space, though I had the strong belief that the restriction was unnecessary. It was primarily to remove that restriction that I developed the concept of vector manifolds in [5]. I was still not convinced that (3.2) applies without modification to such general vector manifolds until the relation between the vector derivative  $\partial$  and the coderivative  $\nabla$  was thoroughly worked out in [5]. The operator  $\partial$  can be regarded as a coordinate-free generalization of the "partial derivative," while  $\nabla$  is the same for the "covariant derivative." Though the Boundary Theorem is formulated for general vector manifolds in [5], and its scalar part is shown to be equivalent to Stokes' Theorem in terms of differential forms, most of its applications are restricted to manifolds in a vector space, because it's only for that case that explicit Green's functions are known.



Nevertheless, I am convinced that there are beautiful applications waiting to be discovered in the general case. This is especially relevant to cohomology theory which has not yet been fully reformulated in terms of Geometric Calculus, though I am confident that it will be enlightening to do so.

For a final remark about foundations, let me call your attention to the article [13] by Garret Sobczyk. Triangulation by simplexes is an alternative to coordinates for a rigorous characterization of manifolds, and it is especially valuable as an approach to calculations on vector manifolds. Garret and I talked about this a lot while preparing [5], so I am glad he finally got around to writing out the details and illustrating the method with some applications. I believe this method is potentially of great value for treating finite difference equations with Geometric Algebra. Anyone who wants to apply Geometric Calculus should put it in his tool box.

#### 4. What is a differential form?

The concept of differential needs some explication, because it comes with many guises in the literature. I believe that the concept is best captured by defining a *differential of grade  $k$*  to be a  $k$ -blade in the tangent algebra of a given vector manifold. Recall from [5] that a  $k$ -blade is a simple  $k$ -vector. Readers who are unfamiliar with other technical terms in this article will find full explanations in [5]. Of course, differentials have usually been employed without any reference to Geometric Algebra or vector manifolds, but I maintain that they can always be reformulated to do so. The point of the present formulation is that the property of a direction in a tangent space is inherent in the concept of a differential, and this property should be given an explicit formulation by representing the differential as a blade.

For the differential in a directed integral such as (3.1), I often prefer the notation

$$d\omega = d^m x, \tag{4.1}$$

because it has the advantage of designating explicitly both the differential's grade and the point to which it is attached. The differential of a coordinate curve through  $x$  is a tangent vector which, using (2.2), can be expressed in terms of the coordinates by

$$d_\mu x = e_\mu dx^\mu \tag{4.2}$$

(no sum on  $\mu$ ). Note the placement of the subscript on the left to avoid confusion between  $dx^\mu$ , a scalar differential for the scalar variable  $x^\mu$ , and the vector differential  $d_\mu x$  for the vector variable  $x$ . We can use (4.2) to express (4.1) in terms of coordinates:

$$d^m x = d_1 x \wedge d_2 x \wedge \dots \wedge d_m x = e_1 \wedge e_2 \wedge \dots \wedge e_m dx^1 dx^2 \dots dx^m. \tag{4.3}$$

This is appropriate when one wants to reduce a directed integral to an iterated integral on the coordinates. However, it is often simpler to evaluate integrals directly without using coordinates. (Examples are given in [5].)

On a metric manifold, a differential  $d^m x$  can be resolved into its magnitude  $|d^m x|$  and its direction represented by a unit  $m$ -blade  $I_m$ :

$$d^m x = I_m |d^m x|. \tag{4.4}$$

Then, according to (4.3) and (2.9),

$$|d^m x| = |\det g_{\mu\nu}|^{1/2} dx^1 dx^2 \dots dx^m. \quad (4.5)$$

This is a familiar expression for the “volume element” in a “multiple integral,” and it is really all one needs to establish my contention that any integral can be reformulated as a directed integral, for

$$|d^m x| = I_m^{-1} d^m x, \quad (4.6)$$

so we can switch from one integral with the “scalar measure”  $|d^m x|$  to one with “directed measure”  $d^m x$  simply by inserting  $I_m^{-1}(x)$  in the integrand. Of course, this is not always desirable, but you may be surprised how often it is when you know about it!

A *differential k-form*

$$L = L(d^k x) = L(x, d^k x) \quad (4.7)$$

can be defined on a given vector manifold as a linear function of a differential of grade  $k$  with values in the Geometric Algebra. To indicate that its values may vary over the manifold, dependence on the point  $x$  is made explicit on the right side of (4.7). As explained in [5], the *exterior differential* of  $L$  can be defined in terms of the vector derivative  $\partial = \partial_x$  by

$$dL = \dot{L}(d^k x \cdot \dot{\partial}) = L(\dot{x}, (d^k x) \cdot \dot{\partial}), \quad (4.8)$$

where the accent on  $\dot{\partial}$  indicates that it differentiates the variable  $\dot{x}$ .

Now we can write down the *Boundary Theorem* in its most general form:

$$\int dL = \oint L. \quad (4.9)$$

This generalizes (3.1), to which it reduces when  $L = d^{m-1}x A$ . The formulation (4.9) has been deliberately chosen to look like the standard “Generalized Stokes’ Theorem,” but it is actually more general because  $L$  is not restricted to scalar values, and this, as has been mentioned, leads to such powerful new results as the “generalized Cauchy integral formula.”

Equally important, (4.7) makes the fundamental dependence of a  $k$ -form on the  $k$ -vector variable explicit, and (4.8) shows how *the exterior derivative derives from the vector derivative* (or Dirac operator, if you will). All this is hidden in the abbreviated formulation (4.9) and, in fact, throughout the standard calculus of differential forms. A detailed discussion and critique of this standard calculus is given in [5]. A huge literature has arisen in recent years combining differential forms with Clifford algebras and the Dirac operator. By failing to understand how all these things fit together in a unified Geometric Calculus, this literature is burdened by a *gross excess of formalism*, which, when stripped away, reveals much of it as trivial.

There is an alternative formulation of the Boundary Theorem which is often more convenient in physics and Clifford analysis. We use (4.4) and the fact that on the boundary the interior pseudoscalar  $I_m$  is related to the boundary pseudoscalar  $I_{m-1}$  by

$$I_m = I_{m-1} n, \quad (4.10)$$

where  $n = n(x)$  is the *unit* outward normal (null vectors not allowed here). Indeed, (4.10) can be adopted as a definition of the outward normal. We define a tensor field  $T(n) = T(x, n(x))$ , by

$$T(n) = L(I_m n), \quad (4.11)$$

and its *divergence* by

$$\dot{T}(\dot{\partial}) = \dot{L}(I_n \dot{\partial}) + L(\dot{I}_n \cdot \dot{\partial}). \quad (4.12)$$

The last term vanishes if

$$\partial \cdot I_m = 0, \quad (4.13)$$

in which case, using (3.4), the Boundary Theorem can be rewritten in the form

$$\int \dot{T}(\dot{\partial}) |d^m x| = \oint T(n^{-1}) |d^{m-1} x|. \quad (4.14)$$

This version can fairly be called *Gauss' Theorem*, since it includes theorems with that name as a special case. It has the advantage of exhibiting the role of the vector derivative explicitly. This theorem applies to spaces of any signature, including the indefinite signature of spacetime. The effect of signature in the theorem is incorporated in the  $n^{-1}$ , which becomes  $n^{-1} = n$  if  $n^2 = 1$  or  $n^{-1} = -n$  if  $n^2 = -1$ .

As an application of great importance, suppose we have a Green's function  $G = G(y, x)$  defined on our manifold  $\mathcal{M}$  and satisfying the differential equation

$$\partial_y G(y, x) = -G(y, \dot{x}) \dot{\partial}_x = \delta^m(y - x), \quad (4.15)$$

where the right side is the  $m$ -dimensional delta function. Let  $T(n)$  be given by

$$T(n) = GnF, \quad (4.16)$$

where  $F = F(x)$  is any differentiable function. If  $y$  is an interior point of  $\mathcal{M}$ , substitution of (4.16) into (4.14) yields

$$F(y) = \int G(y, x) \partial F(x) |d^m x| - \oint G(y, x) n^{-1} F(x) |d^{m-1} x|. \quad (4.17)$$

This great formula allows us to calculate  $F$  inside  $\mathcal{M}$  from its derivative  $\partial F$  and its values on the boundary if  $G$  is known.

The specific form of the function  $G$ , when it can be found, depends on the manifold. If  $\mathcal{M}$  is embedded in an  $m$ -dimensional vector space,  $G$  is the so-called *Cauchy Kernel*:

$$G(y, x) = \frac{\Gamma(m/2)}{2\pi^{m/2}} \frac{x - y}{|x - y|^m}, \quad (4.18)$$

and (4.17) yields the generalization of Cauchy's Integral formula originally found in [11]. The  $\Gamma$  in (4.18) denotes the gamma function. The function  $F = F(x)$  is said to be *monogenic* if  $\partial F = 0$ , in which case the first term on the right side of (4.17) vanishes. It is a good exercise for beginners to show that, in this case, (4.17) really does reduce to the famous Cauchy integral when  $m = 2$ .

## 5. Spacetime Calculus

When applied to a *spacetime manifold*, that is, a 4-dimensional vector manifold modeling physical spacetime, the Geometric Algebra is called *Spacetime Algebra* [8], and Geometric

Calculus is called *Spacetime Calculus*. The preceding results have many applications to *spacetime physics*. Note that I did not say “relativistic physics,” because the spacetime calculus provides us with an *invariant* (coordinate-free) formulation of physical equations generally, and it enables us to calculate without introducing inertial frames and Lorentz transformations among them. True, it is important to relate invariant physical quantities to some reference frame in order to interpret experimental results, but that is done better with a *spacetime split* [14] than with Lorentz transformations. An example is given below.

We limit our considerations here to *Minkowski spacetime*, modeled with the elements  $\{x\}$  of a 4-dimensional vector space. Let  $u$  be a constant, unit, timelike vector (field) directed in the forward light cone. The assumption  $u^2 = 1$  fixes the signature of the spacetime metric. The vector  $u$  determines a 1-parameter family of spacetime hyperplanes  $\mathcal{S}(t)$  satisfying the equation

$$u \cdot x = t. \quad (5.1)$$

The vector  $u$  thus determines an inertial frame with time variable  $t$ , so  $\mathcal{S}(t)$  is a surface of simultaneous  $t$ .

Let  $\mathcal{V}(t)$  be a convex 3-dimensional region (submanifold) in  $\mathcal{S}(t)$  which sweeps out a 4-dimensional region  $\mathcal{M}$  in the time interval  $t_1 \leq t \leq t_2$ . In this interval the 2-dimensional boundary  $\partial\mathcal{V}(t)$  sweeps out a 3-dimensional wall  $\mathcal{W}$ , so  $\mathcal{M}$  is bounded by  $\partial\mathcal{M} = \mathcal{V}(t_1) + \mathcal{V}(t_2) + \mathcal{W}$ . We can use the integral formula (4.17) to solve *Maxwell’s equation*

$$\partial F = J \quad (5.2)$$

in the region  $\mathcal{M}$  for the *electromagnetic field*  $F = F(x)$  “produced by” the *charge current* (density)  $J = J(x)$ . The field  $F$  is bivector-valued while the current  $J$  is vector-valued. For simplicity, let us enlarge  $\mathcal{V}(t)$  to coincide with  $\mathcal{S}(t)$  and assume that the integral of  $F$  over  $\partial\mathcal{V}$  is vanishingly small at spatial infinity. Then  $\mathcal{M}$  is the entire region between the hyperplanes  $\mathcal{S}_1 = \mathcal{S}(t_1)$  and  $\mathcal{S}_2 = \mathcal{S}(t_2)$ , and (4.17) gives us

$$F(y) = \int_{\mathcal{M}} G(y, x) J(x) |d^4x| + F_1 - F_2, \quad (5.3)$$

where

$$F_k(y) = \int_{\mathcal{S}_k} G(y, x) u F(x) |d^3x|. \quad (5.4)$$

Because of the condition (4.15) on the Green’s function, the  $F_k$  satisfy the homogeneous equation

$$\partial F_k = 0. \quad (5.5)$$

A *retarded Green’s function*  $G_k$  can be found which vanishes on  $\mathcal{S}_2$ , in which case  $F_1$  solves the *Cauchy problem* for the homogeneous Maxwell equation (5.5).

Green’s functions for spacetime have been extensively studied by physicists and the results, contained in many books, are easily adapted to the present formulation. Thus, from [15] we find the following Green’s function for (5.3) and (5.4):

$$G(r) = \frac{1}{4\pi} \partial_r \delta(r^2) = \frac{1}{2\pi} r \delta'(r^2), \quad (5.6)$$

where  $r = x - y$  and  $\delta$  denotes a 1-dimensional delta function with derivative  $\delta'$ . The analysis of retarded and advanced parts of  $G$  and their implications is standard, so it need not be discussed here.

Taking  $\mathcal{M}$  to be all of spacetime so  $F_1$  and  $F_2$  can be set to zero, equation (5.3) with (5.6) can be integrated to get the field produced by point charge. For a particle with charge  $q$  and world line  $z = z(\tau)$  with proper time  $\tau$ , the charge current can be expressed by

$$J(x) = q \int_{-\infty}^{\infty} d\tau v \delta^4(x - z(\tau)), \quad (5.7)$$

where  $v = v(\tau) = dz/d\tau$ . Inserting this into (5.3) and integrating, we find that the retarded field can be expressed in the following explicit form

$$F(x) = \frac{q}{4\pi} \frac{r \wedge [v + r \cdot (v \wedge \dot{v})]}{(r \cdot v)^3} = \frac{q}{4\pi(r \cdot v)^2} \left[ \frac{r \wedge v}{|r \wedge v|} + \frac{1}{2} \frac{r \dot{v} v r}{r \cdot v} \right], \quad (5.8)$$

where  $r = x - z$  satisfies  $r^2 = 0$  and  $z$ ,  $v$ ,  $\dot{v} = dv/d\tau$  are all evaluated at the intersection of the backward light cone with vertex at  $x$ . This elegant invariant form for the classical *Lienard-Wiechart field* has been found independently by Steve Gull.

As another important example, we show that (4.14) gives us an immediate integral formulation of any physics conservation law with a suitable choice of  $T(n)$ . Introducing the notations

$$\dot{T}(\dot{\partial}) = f \quad (5.9)$$

and

$$\mathcal{I} = \int_{\mathcal{M}} f |d^4x| = \int_{t_1}^{t_2} dt \int_{\mathcal{V}(t)} f |d^3x|, \quad (5.10)$$

for the region  $\mathcal{M}$  defined above, we can write (4.14) in the form

$$\mathcal{I} = P(t_2) - P(t_1) - \int_{t_1}^{t_2} dt \oint_{\partial\mathcal{V}(t)} T(n) |d^2x|, \quad (5.11)$$

where

$$P(t) = \int_{\mathcal{V}(t)} T(u) |d^3x|. \quad (5.12)$$

Now for some applications.

### Energy-Momentum Conservation:

We first suppose that  $T(n)$  is the energy-momentum tensor for some *physical system*, which could be a material medium, an electromagnetic field, or some combination of the two, and it could be either classical or quantum mechanical. For example, the usual energy-momentum tensor for the electromagnetic field is given by

$$T(n) = -\frac{1}{2} F n F. \quad (5.13)$$

In general, the tensor  $T(n)$  represents the flux of energy-momentum through a hypersurface with normal  $n$ .

For the vector field  $f = f(x)$  specified independently of the tensor field  $T(n) = T(x, n(x))$ , equation (5.9) is the *local energy-momentum conservation law*, where the *work-force density*  $f$  characterizes the effect of external influences on the system in question. Equation (5.11) is then the *integral energy-momentum conservation law* for the system. The vector  $P(t)$  given by (5.12) is the *total energy-momentum* of the system contained in  $\mathcal{V}(t)$  at time  $t$ . The quantity  $\mathcal{I}$  is the total *Impulse* delivered to the system in the region  $\mathcal{M}$ .

In the limit  $t_2 \rightarrow t_1 = t$ , the conservation law (5.11) can be written

$$\frac{dP}{dt} = \mathcal{F} + \oint_{\partial\mathcal{V}} T(n) |d^2x|, \quad (5.14)$$

where

$$\mathcal{F}(t) = \int_{\mathcal{V}(t)} f |d^3x| \quad (5.15)$$

is the total work-force on the system. We can decompose (5.14) into separate energy and momentum conservation laws by using a spacetime split: we write

$$Pu = E + \mathbf{p}, \quad (5.16)$$

where  $E = P \cdot u$  is the energy and  $\mathbf{p} = P \wedge u$  is the momentum of the system. Similarly we write

$$\mathcal{F}u = W + \mathbf{F}, \quad (5.17)$$

where  $W = \mathcal{F} \cdot u$  is the work done on the system and  $\mathbf{F} = \mathcal{F} \wedge u$  is the force exerted on it. We write

$$T(n)u = \mathbf{n} \cdot \mathbf{s} + \mathbf{T}(\mathbf{n}), \quad (5.18)$$

where  $\mathbf{n} \cdot \mathbf{s} = u \cdot T(n)$  is the *energy flux*,  $\mathbf{T}(\mathbf{n}) = T(n) \wedge u$  is the *stress tensor*, and  $\mathbf{n} = n \wedge u = nu$  represents the normal as a “relative vector.” We also note that

$$xu = t + \mathbf{x} \quad (5.19)$$

splits  $x$  into a time  $t = x \cdot u$  and a position vector  $\mathbf{x} = x \wedge u$ . Finally, we multiply (5.14) by  $u$  and separate scalar and relative vector parts to get the *energy conservation law*

$$\frac{dE}{dt} = W + \oint \mathbf{s} \cdot \mathbf{n} |d^2\mathbf{x}| \quad (5.20)$$

and the *momentum conservation law*

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} + \oint \mathbf{T}(\mathbf{n}) |d^2\mathbf{x}|. \quad (5.21)$$

These are universal laws applying to all physical systems.

### Angular Momentum Conservation:

The “generalized orbital angular momentum tensor” for the system just considered is defined by

$$L(n) = T(n) \wedge x. \quad (5.22)$$

With (4.9), its divergence is

$$\dot{L}(\partial) = f \wedge x + T(\partial) \wedge \dot{x}. \quad (5.23)$$

For a symmetric tensor such as (5.13) the last term vanishes. But, in general, there exists a bivector-valued tensor  $S(n)$ , the *spin tensor* for the system, which satisfies

$$\dot{S}(\partial) = \dot{x} \wedge T(\partial). \quad (5.24)$$

Now define the *total angular momentum tensor*

$$M(n) = T(n) \wedge x + S(n). \quad (5.25)$$

Then the *local angular momentum conservation law* for the system is

$$\dot{M}(\partial) = f \wedge x. \quad (5.26)$$

Replacing (5.9) by (5.26), we can reinterpret (5.11) as an integral law for angular momentum and analyze it the way we did energy-momentum.

### Charge Conservation:

From Maxwell's equation we derive the local charge conservation law

$$\partial \cdot J = \partial \cdot (\partial \cdot F) = (\partial \wedge \partial) \cdot F = 0. \quad (5.27)$$

Now write  $T(n) = n \cdot J$  and change the notion of (5.12) to

$$Q(t) = \int_{\mathcal{V}(t)} u \cdot J |d^3x|, \quad (5.28)$$

an expression for the total charge contained in  $\mathcal{V}(t)$ . Then (5.11) becomes

$$Q(t_2) - Q(t_1) = \int_{t_1}^{t_2} dt \oint_{\partial\mathcal{V}(t)} n \cdot J |d^2x|. \quad (5.29)$$

This is the charge conservation equation, telling us that the total charge in  $\mathcal{V}(t)$  changes only by flowing through the boundary  $\partial\mathcal{V}(t)$ .

To dispel any impression that only the Gaussian form (4.14) of the Boundary Theorem is of interest in spacetime physics, I present one more important example: an integral formulation of Maxwell's equation (5.2), which can be decomposed into trivector and vector parts:

$$\partial \wedge F = 0, \quad (5.30)$$

$$\partial \cdot F = J. \quad (5.31)$$

Using the algebraic identity  $(d^3x) \cdot (\partial \wedge F) = (d^3x) \cdot \partial F$ , we deduce immediately from (3.1) that

$$\oint d^2x \cdot F = 0 \quad (5.32)$$

for any closed 2-dimensional submanifold  $\mathcal{B}$  in spacetime. A spacetime split shows that this integral formula is equivalent to Faraday's Law or "the absence of magnetic poles," or a mixture of the two, depending on the choice of  $\mathcal{B}$ .

To derive a similar integral formula for the vector part (5.31) of Maxwell's equation, in analogy to (4.10), define a unit normal  $n$  by writing

$$d^3x = in |d^3x|, \quad (5.33)$$

where  $i$  is the unit dextral pseudoscalar for spacetime, and the use of the identity  $(\partial \cdot F)i = \partial \wedge (Fi)$  to establish

$$d^3x \cdot (\partial \wedge (Fi)) = d^3x \cdot (Ji) = J \cdot n |d^3x|, \quad (5.34)$$

Insertion of this into (3.1) yields the integral equation

$$\oint d^2x \cdot (Fi) = \int J \cdot n |d^3x|, \quad (5.35)$$

which, like (5.32), holds for any closed 2-manifold  $\mathcal{B}$ , where the integral on the right is over any 3-manifold with boundary  $\mathcal{B}$ . Again a spacetime split reveals that (5.35) is equivalent to Ampere's Law, Gauss' Law, or a combination of the two, depending on the choice of  $\mathcal{B}$ .

The two integral equations (5.32) and (5.35) are fully equivalent to the two parts of Maxwell's equations (5.30) and (5.31). They can be combined into a single equation. First multiply (5.35) by  $i$  and use (5.34) to put it in the less familiar form

$$\oint (d^2x) \wedge F = \int (d^3x) \wedge J. \quad (5.36)$$

Adding (5.32) to (5.36), we can write the integral version of the whole Maxwell's equation (5.2) in the form

$$\oint \langle d^2x F \rangle_I = \int \langle d^3x J \rangle_I, \quad (5.37)$$

where  $\langle \dots \rangle_I$  selects only the "invariant (= scalar+pseudoscalar) parts." I have not seen Maxwell's equation in the form (5.37) before. I wonder if this form has some slick physical applications.

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