4. A Universal Model for Conformal Geometries of Euclidean, Spherical and Double-Hyperbolic Spaces†

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4.1 Introduction

The study of relations among Euclidean, spherical and hyperbolic geometries dates back to the beginning of last century. The attempt to prove Euclid's fifth postulate led C. F. Gauss to discover hyperbolic geometry in the 1820's. Only a few years passed before this geometry was rediscovered independently by N. Lobachevski (1829) and J. Bolyai (1832). The strongest evidence given by the founders for its consistency is the duality between hyperbolic and spherical trigonometries. This duality was first demonstrated by Lambert in his 1770 memoir [L1770]. Some theorems, for example the law of sines, can be stated in a form that is valid in spherical, Euclidean, and hyperbolic geometries [B1832].

To prove the consistency of hyperbolic geometry, people built various analytic models of hyperbolic geometry on the Euclidean plane. E. Beltrami [B1868] constructed a Euclidean model of the hyperbolic plane, and using differential geometry, showed that his model satisfies all the axioms of hyperbolic plane geometry. In 1871, F. Klein gave an interpretation of Beltrami's model in terms of projective geometry. Because of Klein's interpretation, Beltrami's model is later called Klein's disc model of the hyperbolic plane. The generalization of this model to n-dimensional hyperbolic space is now called the Klein ball model [CFK98].

In the same paper Beltrami constructed two other Euclidean models of the hyperbolic plane, one on a disc and the other on a Euclidean half-plane. Both models are later generalized to n-dimensions by H. Poincaré [P08], and are now associated with his name.

All three of the above models are built in Euclidean space, and the latter two are conformal in the sense that the metric is a point-to-point scaling of the Euclidean metric. In his 1878 paper [K1878], Killing described a hyperboloid

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model of hyperbolic geometry by constructing the stereographic projection of Beltrami's disc model onto the hyperbolic space. This hyperboloid model was generalized to n-dimensions by Poincaré.

There is another model of hyperbolic geometry built in spherical space, called hemisphere model, which is also conformal. Altogether there are five well-known models for the n-dimensional hyperbolic geometry:

- the half-space model,
- the conformal ball model,
- the Klein ball model,
- the hemisphere model,
- the hyperboloid model.

The theory of hyperbolic geometry can be built in a unified way within any of the models. With several models one can, so to speak, turn the object around and scrutinize it from different viewpoints. The connections among these models are largely established through stereographic projections. Because stereographic projections are conformal maps, the conformal groups of n-dimensional Euclidean, spherical, and hyperbolic spaces are isometric to each other, and are all isometric to the group of isometries of hyperbolic $(n+1)$-space, according to observations of Klein [K1872], [K1872].

It seems that everything is worked out for unified treatment of the three spaces. In this chapter we go further. We unify the three geometries, together with the stereographic projections, various models of hyperbolic geometry, in such a way that we need only one Minkowski space, where null vectors represent points or points at infinity in any of the three geometries and any of the models of hyperbolic space, where Minkowski subspaces represent spheres and hyperplanes in any of the three geometries, and where stereographic projections are simply rescaling of null vectors. We call this construction the *homogeneous model*. It serves as a sixth analytic model for hyperbolic geometry.

We constructed homogeneous models for Euclidean and spherical geometries in previous chapters. There the models are constructed in Minkowski space by projective splits with respect to a fixed vector of null or negative signature. Here we show that a projective split with respect to a fixed vector of positive signature produces the homogeneous model of hyperbolic geometry.

Because the three geometries are obtained by interpreting null vectors of the same Minkowski space differently, natural correspondences exist among geometric entities and constraints of these geometries. In particular, there are correspondences among theorems on conformal properties of the three geometries. Every algebraic identity can be interpreted in three ways and therefore represents three theorems. In the last section we illustrate this feature with an example.

The homogeneous model has the significant advantage of simplifying geometric computations, because it employs the powerful language of *Geometric*

Algebra. Geometric Algebra was applied to hyperbolic geometry by H. Li in [L97], stimulated by Iversen's book [I92] on the algebraic treatment of hyperbolic geometry and by the paper of Hestenes and Zielger [HZ91] on projective geometry with Geometric Algebra.

4.2 The hyperboloid model

In this section we introduce some fundamentals of the hyperboloid model in the language of Geometric Algebra. More details can be found in [L97].

In the Minkowski space $\mathcal{R}^{n,1}$, the set

$$\mathcal{D}^n = \{x \in \mathcal{R}^{n,1} | x^2 = -1\} \tag{4.1}$$

is called an n-dimensional *double-hyperbolic space*, any element in it is called a *point*. It has two connected branches, which are symmetric to the origin of $\mathcal{R}^{n+1,1}$. We denote one branch by $\mathcal{H}^n$ and the other by $-\mathcal{H}^n$. The branch $\mathcal{H}^n$ is called the *hyperboloid model* of n-dimensional hyperbolic space.

4.2.1 Generalized points

Distances between two points

Let $\mathbf{p}, \mathbf{q}$ be two distinct points in $\mathcal{D}^n$, then $\mathbf{p}^2 = \mathbf{q}^2 = -1$. The blade $\mathbf{p} \wedge \mathbf{q}$ has Minkowski signature, therefore

$$0 < (\mathbf{p} \wedge \mathbf{q})^2 = (\mathbf{p} \cdot \mathbf{q})^2 - \mathbf{p}^2\mathbf{q}^2 = (\mathbf{p} \cdot \mathbf{q})^2 - 1. \tag{4.2}$$

From this we get

$$|\mathbf{p} \cdot \mathbf{q}| > 1. \tag{4.3}$$

Since $\mathbf{p}^2 = -1$, we can prove

Theorem 1. *For any two points* $\mathbf{p}, \mathbf{q}$ *in* $\mathcal{H}^n$ *(or* $-\mathcal{H}^n$*),*

$$\mathbf{p} \cdot \mathbf{q} < -1. \tag{4.4}$$

As a corollary, there exists a unique nonnegative number $d(\mathbf{p}, \mathbf{q})$ such that

$$\mathbf{p} \cdot \mathbf{q} = -\cosh d(\mathbf{p}, \mathbf{q}). \tag{4.5}$$

$d(\mathbf{p}, \mathbf{q})$ is called the *hyperbolic distance* between $\mathbf{p}, \mathbf{q}$.

Below we define several other equivalent distances. Let $\mathbf{p}, \mathbf{q}$ be two distinct points in $\mathcal{H}^n$ (or $-\mathcal{H}^n$). The positive number

$$d_n(\mathbf{p}, \mathbf{q}) = -(1 + \mathbf{p} \cdot \mathbf{q}) \tag{4.6}$$

is called the *normal distance* between $\mathbf{p}, \mathbf{q}$. The positive number

$$d_t(\mathbf{p}, \mathbf{q}) = |\mathbf{p} \wedge \mathbf{q}| \tag{4.7}$$

is called the *tangential distance* between $\mathbf{p}, \mathbf{q}$. The positive number

$$d_h(\mathbf{p}, \mathbf{q}) = |\mathbf{p} - \mathbf{q}| \tag{4.8}$$

is called the *horo-distance* between $\mathbf{p}, \mathbf{q}$. We have

$$\begin{array}{rcl} d_n(\mathbf{p}, \mathbf{q}) & = & \cosh d(\mathbf{p}, \mathbf{q}) - 1, \\ d_t(\mathbf{p}, \mathbf{q}) & = & \sinh d(\mathbf{p}, \mathbf{q}), \\ d_h(\mathbf{p}, \mathbf{q}) & = & 2 \sinh \dfrac{d(\mathbf{p}, \mathbf{q})}{2}. \end{array} \tag{4.9}$$

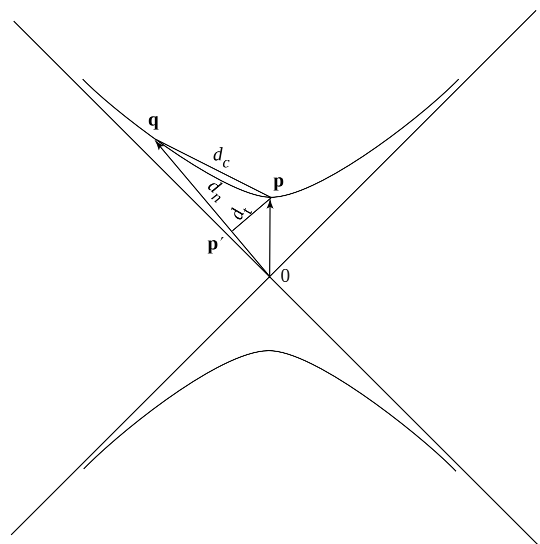

Figure 1: Distances in hyperbolic geometry.

Points at infinity

A *point at infinity* of $\mathcal{D}^n$ is a one-dimensional null space. It can be represented by a single null vector uniquely up to a nonzero scale factor.

The set of points at infinity in $\mathcal{D}^n$ is topologically an $(n-1)$-dimensional sphere, called the *sphere at infinity* of $\mathcal{D}^n$. The null cone

$$\mathcal{N}^{n-1} = \{x \in \mathcal{R}^{n,1} | x^2 = 0, x \neq 0\} \tag{4.10}$$

of $\mathcal{R}^{n,1}$ has two branches. Two null vectors h_1, h_2 are on the same connected component if and only if $h_1 \cdot h_2 < 0$. One branch $\mathcal{N}^{n-1}_+$ has the property: for any null vector h in $\mathcal{N}^{n-1}_+$, any point $\mathbf{p}$ in $\mathcal{H}^n$, $h \cdot \mathbf{p} < 0$. The other branch of the null cone is denoted by $\mathcal{N}^{n-1}_-$.

For a null vector h, the *relative distance* between h and point $\mathbf{p} \in \mathcal{D}^n$ is defined as

$$d_r(h, \mathbf{p}) = |h \cdot \mathbf{p}|. \tag{4.11}$$

Imaginary points

An *imaginary point* of $\mathcal{D}^n$ is a one-dimensional Euclidean space. It can be represented by a vector of unit square in $\mathcal{R}^{n+1,1}$.

The dual of an imaginary point is a hyperplane. An *r-plane* in $\mathcal{D}^n$ is the intersection of an $(r+1)$-dimensional Minkowski space of $\mathcal{R}^{n,1}$ with $\mathcal{D}^n$. A *hyperplane* is an $(n-1)$-plane.

Let a be an imaginary point, $\mathbf{p}$ be a point. There exists a unique line, a 1-plane in $\mathcal{D}^n$, which passes through $\mathbf{p}$ and is perpendicular to the hyperplane $\tilde{a}$ dual to a. This line intersects the hyperplane at a pair of antipodal points $\pm\mathbf{q}$. The *hyperbolic, normal and tangent distances* between $a, \mathbf{p}$ are defined as the respective distances between $\mathbf{p}, \mathbf{q}$. We have

$$\begin{array}{lcl}
\cosh d(a, \mathbf{p}) & = & |a \wedge \mathbf{p}|, \\
d_n(a, \mathbf{p}) & = & |a \wedge \mathbf{p}| - 1, \\
d_t(a, \mathbf{p}) & = & |a \cdot \mathbf{p}|.
\end{array}$$

A *generalized point* of $\mathcal{D}^n$ refers to a point, or a point at infinity, or an imaginary point.

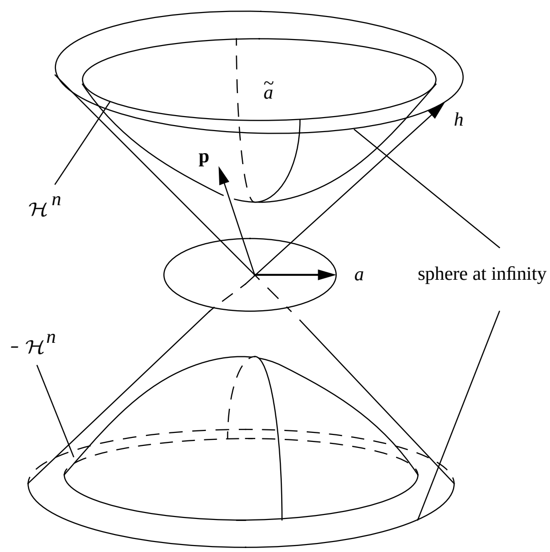

Figure 2: Generalized points in $\mathcal{D}^n$: $\mathbf{p}$ is a point, h is a point at infinity, and a is an imaginary point.

Oriented generalized points and signed distances

The above definitions of generalized points are from [L97], where the topic was $\mathcal{H}^n$ instead of $\mathcal{D}^n$, and where $\mathcal{H}^n$ was taken as $\mathcal{D}^n$ with antipodal points identified, instead of just a connected component of $\mathcal{D}^n$. When studying double-hyperbolic space, it is useful to distinguish between null vectors h and $-h$ representing the same point at infinity, and vectors a and $-a$ representing the same imaginary point. Actually it is indispensible when we study generalized spheres in $\mathcal{D}^n$. For this purpose we define oriented generalized points.

Any null vector in $\mathcal{R}^{n,1}$ represents an *oriented point at infinity* of $\mathcal{D}^n$. Two null vectors in $\mathcal{D}^n$ are said to represent the same oriented point at infinity if and only they differ by a positive scale factor; in other words, null vectors f and $-f$ represent two antipodal oriented points at infinity.

Any unit vector in $\mathcal{R}^{n,1}$ of positive signature represents an *oriented imaginary point* of $\mathcal{D}^n$. Two unit vectors a and $-a$ of positive signature represent two antipodal oriented imaginary points. The dual of an oriented imaginary point is an oriented hyperplane of $\mathcal{D}^n$.

A point in $\mathcal{D}^n$ is already oriented.

We can define various *signed distances* between two oriented generalized points, for example,

- the signed normal distance between two points $\mathbf{p}, \mathbf{q}$ is defined as
$$-\mathbf{p} \cdot \mathbf{q} - 1, \tag{4.12}$$
which is nonnegative when $\mathbf{p}, \mathbf{q}$ are on the same branch of $\mathcal{D}^n$ and ≤ -2 otherwise;

- the signed relative distance between point $\mathbf{p}$ and oriented point at infinity h is defined as
$$-h \cdot \mathbf{p}, \tag{4.13}$$
which is positive for $\mathbf{p}$ on one branch of $\mathcal{D}^n$ and negative otherwise;

- the signed tangent distance between point $\mathbf{p}$ and oriented imaginary point a is defined as
$$-a \cdot \mathbf{p}, \tag{4.14}$$
which is zero when $\mathbf{p}$ is on the hyperplane $\tilde{a}$, positive when $\mathbf{p}$ is on one side of the hyperplane and negative otherwise.

4.2.2 Total spheres

A *total sphere* of $\mathcal{D}^n$ refers to a hyperplane, or the sphere at infinity, or a generalized sphere. An *r-dimensional total sphere* of $\mathcal{D}^n$ refers to the intersection of a total sphere with an $(r+1)$-plane.

A *generalized sphere* in $\mathcal{H}^n$ (or $-\mathcal{H}^n$, or $\mathcal{D}^n$) refers to a sphere, or a horosphere, or a hypersphere in $\mathcal{H}^n$ (or $-\mathcal{H}^n$, or $\mathcal{D}^n$). It is defined by a pair (c, ρ), where c is a vector representing an oriented generalized point, and ρ is a positive scalar.

1. When $c^2 = -1$, i.e., c is a point, then if c is in $\mathcal{H}^n$, the set

$$\{\mathbf{p} \in \mathcal{H}^n | d_n(\mathbf{p}, c) = \rho\} \tag{4.15}$$

is the *sphere* in $\mathcal{H}^n$ with center c and normal radius ρ; if c is in $-\mathcal{H}^n$, the set

$$\{\mathbf{p} \in -\mathcal{H}^n | d_n(\mathbf{p}, c) = \rho\} \tag{4.16}$$

is a *sphere* in $-\mathcal{H}^n$.

2. When $c^2 = 0$, i.e., c is an oriented point at infinity, then if $c \in \mathcal{N}_+^{n-1}$, the set

$$\{\mathbf{p} \in \mathcal{H}^n | d_r(\mathbf{p}, c) = \rho\} \tag{4.17}$$

is the *horosphere* in $\mathcal{H}^n$ with center c and relative radius ρ; otherwise the set

$$\{\mathbf{p} \in -\mathcal{H}^n | d_r(\mathbf{p}, c) = \rho\} \tag{4.18}$$

is a *horosphere* in $-\mathcal{H}^n$.

3. When $c^2 = 1$, i.e., c is an oriented imaginary point, the set

$$\{\mathbf{p} \in \mathcal{D}^n | \mathbf{p} \cdot c = -\rho\} \tag{4.19}$$

is the *hypersphere* in $\mathcal{D}^n$ with center c and tangent radius ρ; its intersection with $\mathcal{H}^n$ (or $-\mathcal{H}^n$) is a *hypersphere* in $\mathcal{H}^n$ (or $-\mathcal{H}^n$). The hyperplane $\tilde{c}$ is called the *axis* of the hypersphere.

A hyperplane can also be regarded as a hypersphere with zero radius.

4.3 The homogeneous model

In this section we establish the homogeneous model of the hyperbolic space. Strictly speaking, the model is for the double-hyperbolic space, as we must take into account both branches.

Fixing a vector a_0 of positive signature in $\mathcal{R}^{n+1,1}$, assuming $a_0^2 = 1$, we get

$$\mathcal{N}_{a_0}^n = \{x \in \mathcal{R}^{n+1,1} | x^2 = 0, x \cdot a_0 = -1\}. \tag{4.20}$$

Applying the orthogonal decomposition

$$x = P_{a_0}(x) + P_{\tilde{a}_0}(x) \tag{4.21}$$

to vector $x \in \mathcal{N}_{a_0}^n$, we get

$$x = -a_0 + \mathbf{x} \tag{4.22}$$

where $\mathbf{x} \in \mathcal{D}^n$, the negative unit sphere of the Minkowski space represented by $\tilde{a}_0$. The map $i_{a_0} : \mathbf{x} \in \mathcal{D}^n \mapsto x \in \mathcal{N}_{a_0}^n$ is bijective. Its inverse map is $P_{a_0}^{\perp}$.

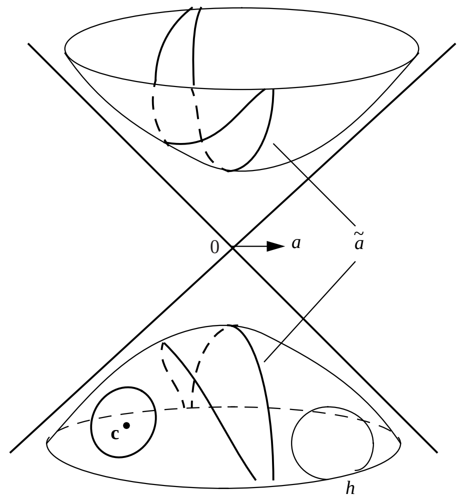

Figure 3: Generalized spheres in $\mathcal{D}^n$: $\mathbf{p}$ the center of a sphere, h the center of a horosphere, a the center of a hypersphere.

Theorem 2.

$$\mathcal{N}_{a_0}^n \simeq \mathcal{D}^n. \tag{4.23}$$

We call $\mathcal{N}_{a_0}^n$ the *homogeneous model* of $\mathcal{D}^n$. Its elements are called *homogeneous points*.

We use $\mathcal{H}^n$ to denote the intersection of $\mathcal{D}^n$ with $\mathcal{H}^{n+1}$, and $-\mathcal{H}^n$ to denote the intersection of $\mathcal{D}^n$ with $-\mathcal{H}^{n+1}$. Here $\pm\mathcal{H}^{n+1}$ are the two branches of $\mathcal{D}^{n+1}$, the negative unit sphere of $\mathcal{R}^{n+1,1}$.

4.3.1 Generalized points

Let $\mathbf{p}, \mathbf{q}$ be two points in $\mathcal{D}^n$. Then for homogeneous points $\mathbf{p}, \mathbf{q}$

$$p \cdot q = (-a_0 + \mathbf{p}) \cdot (-a_0 + \mathbf{q}) = 1 + \mathbf{p} \cdot \mathbf{q}. \tag{4.24}$$

Thus the inner product of two homogeneous points "in" $\mathcal{D}^n$ equals the negative of the signed normal distance between them.

An oriented point at infinity of $\mathcal{D}^n$ is represented by a null vector h of $\mathcal{R}^{n+1,1}$ satisfying

$$h \cdot a_0 = 0. \tag{4.25}$$

For a point $\mathbf{p}$ of $\mathcal{D}^n$, we have

$$h \cdot p = h \cdot (-a_0 + \mathbf{p}) = h \cdot \mathbf{p}. \tag{4.26}$$

Thus the inner product of an oriented point at infinity with a homogeneous point equals the negative of the signed relative distance between them.

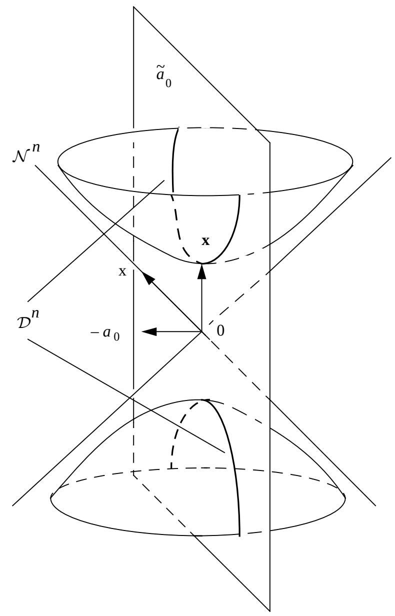

Figure 4: The homogeneous model of $\mathcal{D}^n$.

An oriented imaginary point of $\mathcal{D}^n$ is represented by a vector a of unit square in $\mathcal{R}^{n+1,1}$ satisfying

$$a \cdot a_0 = 0. \tag{4.27}$$

For a point $\mathbf{p}$ of $\mathcal{D}^n$, we have

$$a \cdot p = a \cdot (-a_0 + \mathbf{p}) = a \cdot \mathbf{p}. \tag{4.28}$$

Thus the inner product of a homogeneous point and an oriented imaginary point equals the negative of the signed tangent distance between them.

4.3.2 Total spheres

Below we establish the conclusion that any $(n+1)$-blade of Minkowski signature in $\mathcal{R}^{n+1,1}$ corresponds to a total sphere in $\mathcal{D}^n$.

Let s be a vector of positive signature in $\mathcal{R}^{n+1,1}$.

1. If $s \wedge a_0 = 0$, then s equals a_0 up to a nonzero scalar factor. The blade $\tilde{s}$ represents the sphere at infinity of $\mathcal{D}^n$.

2. If $s \wedge a_0$ has Minkowski signature, then $s \cdot a_0 \neq 0$. Let $(-1)^\epsilon$ be the sign of $s \cdot a_0$. Let

$$\mathbf{c} = (-1)^{1+\epsilon} \frac{P^\perp_{a_0}(s)}{|P^\perp_{a_0}(s)|}, \tag{4.29}$$

then $\mathbf{c} \in \mathcal{D}^n$. Let

$$s' = (-1)^{1+\epsilon} \frac{s}{|a_0 \wedge s|} \,, \tag{4.30}$$

then

$$s' = (-1)^{1+\epsilon} \frac{P_{a_0}^{\perp}(s)}{|a_0 \wedge s|} + (-1)^{1+\epsilon} \frac{P_{a_0}(s)}{|a_0 \wedge s|} = \mathbf{c} - (1+\rho)a_0, \tag{4.31}$$

where

$$\rho = \frac{|a_0 \cdot s|}{|a_0 \wedge s|} - 1 > 0 \tag{4.32}$$

because $|a_0 \wedge s|^2 = (a_0 \cdot s)^2 - s^2 < (a_0 \cdot s)^2$.

For any point $\mathbf{p} \in \mathcal{D}^n$,

$$s' \cdot p = (\mathbf{c} - (1+\rho)a_0) \cdot (\mathbf{p} - a_0) = \mathbf{c} \cdot \mathbf{p} + 1 + \rho. \tag{4.33}$$

So $\tilde{s}$ represents the sphere in $\mathcal{D}^n$ with center $\mathbf{c}$ and normal radius ρ; a point $\mathbf{p}$ is on the sphere if and only if $p \cdot s = 0$.

The standard form of a sphere in $\mathcal{D}^n$ is

$$c - \rho a_0. \tag{4.34}$$

3. If $s \wedge a_0$ is degenerate, then $(s \wedge a_0)^2 = (s \cdot a_0)^2 - s^2 = 0$, so $|s \cdot a_0| = |s| \neq 0$. As before, $(-1)^{\epsilon}$ is the sign of $s \cdot a_0$. Let

$$c = (-1)^{1+\epsilon} P_{a_0}^{\perp}(s), \tag{4.35}$$

then $c^2 = 0$ and $c \cdot a_0 = 0$, so c represents an oriented point at infinity of $\mathcal{D}^n$. Let

$$s' = (-1)^{1+\epsilon} s. \tag{4.36}$$

Then

$$s' = (-1)^{1+\epsilon}(P_{a_0}^{\perp}(s) + P_{a_0}(s)) = c - \rho a_0, \tag{4.37}$$

where

$$\rho = |a_0 \cdot s| = |s| > 0. \tag{4.38}$$

For any point $\mathbf{p} \in \mathcal{D}^n$,

$$s' \cdot p = (c - \rho a_0) \cdot (\mathbf{p} - a_0) = c \cdot \mathbf{p} + \rho, \tag{4.39}$$

so $\tilde{s}$ represents the horosphere in $\mathcal{D}^n$ with center c and relative radius ρ; a point $\mathbf{p}$ is on the sphere if and only if $p \cdot s = 0$.

The standard form of a horosphere in $\mathcal{D}^n$ is

$$c - \rho a_0. \tag{4.40}$$

4. The term $s \wedge a_0$ is Euclidean, but $s \cdot a_0 \neq 0$. Let

$$c = (-1)^{1+\epsilon} \frac{P_{a_0}^{\perp}(s)}{|P_{a_0}^{\perp}(s)|}, \tag{4.41}$$

then $c^2 = 1$ and $c \cdot a_0 = 0$, i.e., c represents an oriented imaginary point of $\mathcal{D}^n$. Let

$$s' = (-1)^{1+\epsilon} \frac{s}{|a_0 \wedge s|}\,, \tag{4.42}$$

then

$$s' = (-1)^{1+\epsilon} \frac{P_{a_0}^{\perp}(s)}{|a_0 \wedge s|} + (-1)^{1+\epsilon} \frac{P_{a_0}(s)}{|a_0 \wedge s|} = c - \rho a_0, \tag{4.43}$$

where

$$\rho = \frac{|a_0 \cdot s|}{|a_0 \wedge s|} > 0. \tag{4.44}$$

For any point $\mathbf{p} \in \mathcal{D}^n$,

$$s' \cdot p = (c - \rho a_0) \cdot (\mathbf{p} - a_0) = c \cdot \mathbf{p} + \rho\,, \tag{4.45}$$

so $\tilde{s}$ represents the hypersphere in $\mathcal{D}^n$ with center c and tangent radius ρ; a point $\mathbf{p}$ is on the hypersphere if and only if $p \cdot s = 0$.

The standard form of a hypersphere in $\mathcal{D}^n$ is

$$c - \rho a_0. \tag{4.46}$$

5. If $s \cdot a_0 = 0$, then $s \wedge a_0$ is Euclidean, because $(s \wedge a_0)^2 = -s^2 < 0$. For any point $\mathbf{p} \in \mathcal{D}^n$, since

$$s \cdot p = s \cdot \mathbf{p}, \tag{4.47}$$

$\tilde{s}$ represents the hyperplane of $\mathcal{D}^n$ normal to vector s; a point $\mathbf{p}$ is on the hyperplane if and only if $p \cdot s = 0$.

From the above analysis we come to the following conclusion:

Theorem 3. *The intersection of any Minkowski hyperspace of $\mathcal{R}^{n+1,1}$ represented by $\tilde{s}$ with $\mathcal{N}^n_{a_0}$ is a total sphere in $\mathcal{D}^n$, and every total sphere can be obtained in this way. A point* $\mathbf{p}$ *in $\mathcal{D}^n$ is on the total sphere if and only if $p \cdot s = 0$.*

The dual of the above theorem is:

Theorem 4. *Given $n+1$ homogeneous points or points at infinity of $\mathcal{D}^n$: $a_0, \ldots, a_n$ such that*

$$\tilde{s} = a_0 \wedge \cdots \wedge a_n. \tag{4.48}$$

This $(n+1)$-blade $\tilde{s}$ represents a total sphere passing through these points or points at infinity. It is a hyperplane if

$$a_0 \wedge \tilde{s} = 0, \tag{4.49}$$

the sphere at infinity if

$$a_0 \cdot \tilde{s} = 0, \tag{4.50}$$

a sphere if

$$(a_0 \cdot \tilde{s})^\dagger (a_0 \cdot \tilde{s}) > 0, \tag{4.51}$$

a horosphere if

$$a_0 \cdot \tilde{s} \neq 0, \text{ and } (a_0 \cdot \tilde{s})^\dagger (a_0 \cdot \tilde{s}) = 0, \tag{4.52}$$

or a hypersphere if

$$(a_0 \cdot \tilde{s})^\dagger (a_0 \cdot \tilde{s}) < 0. \tag{4.53}$$

The scalar

$$s_1 * s_2 = \frac{s_1 \cdot s_2}{|s_1||s_2|} \tag{4.54}$$

is called the *inversive product* of vectors s_1 and s_2. Obviously, it is invariant under orthogonal transformations in $\mathcal{R}^{n+1,1}$. We have the following conclusion for the inversive product of two vectors of positive signature:

Theorem 5. *When total spheres $\tilde{s}_1$ and $\tilde{s}_2$ intersect, let p be a point or point at infinity of the intersection. Let m_i, $i = 1, 2$, be the respective outward unit normal vector of $\tilde{s}_i$ at p if it is a generalized sphere and p is a point, or let m_i be $s_i/|s_i|$ otherwise, then*

$$s_1 * s_2 = m_1 \cdot m_2. \tag{4.55}$$

Proof. The case when p is a point at infinity is trivial, so we only consider the case when p is a point, denoted by $\mathbf{p}$. The total sphere $\tilde{s}_i$ has the standard form $(c_i - \lambda_i a_0)^\sim$, where $c_i \cdot a_0 = 0$, $\lambda_i \geq 0$ and $(c_i - \lambda_i a_0)^2 = c_i^2 + \lambda_i^2 > 0$. Hence

$$s_1 * s_2 = \frac{c_1 \cdot c_2 + \lambda_1 \lambda_2}{|c_1 - \lambda_1 a_0||c_2 - \lambda_2 a_0|} = \frac{c_1 \cdot c_2 + \lambda_1 \lambda_2}{\sqrt{(c_1^2 + \lambda_1^2)(c_2^2 + \lambda_2^2)}}. \tag{4.56}$$

On the other hand, at point $\mathbf{p}$ the outward unit normal vector of generalized sphere $\tilde{s}_i$ is

$$m_i = \frac{\mathbf{p}(\mathbf{p} \wedge c_i)}{|\mathbf{p} \wedge c_i|}, \tag{4.57}$$

which equals $c_i = s_i/|s_i|$ when $\tilde{s}_i$ is a hyperplane. Since point $\mathbf{p}$ is on both total spheres, $\mathbf{p} \cdot c_i = -\lambda_i$, so

$$m_1 \cdot m_2 = \frac{(c_1 - \lambda_1 a_0) \cdot (c_2 - \lambda_2 a_0)}{|\mathbf{p} \wedge c_1||\mathbf{p} \wedge c_2|} = \frac{c_1 \cdot c_2 + \lambda_1 \lambda_2}{\sqrt{(c_1^2 + \lambda_1^2)(c_2^2 + \lambda_2^2)}}. \tag{4.58}$$

□

An immediate corollary is that any orthogonal transformation in $\mathcal{R}^{n+1,1}$ induces an angle-preserving transformation in $\mathcal{D}^n$.

4.3.3 Total spheres of dimensional r

Theorem 6. *For $2 \leq r \leq n+1$, every r-blade A_r of Minkowski signature in $\mathcal{R}^{n+1,1}$ represents an $(r-2)$-dimensional total sphere in $\mathcal{D}^n$.*

Proof. There are three possibilities:
Case 1. When $a_0 \wedge A_r = 0$, A_r represents an $(r-2)$-plane in $\mathcal{D}^n$. After normalization, the *standard form* of an $(r-2)$-plane is

$$a_0 \wedge \mathbf{I}_{r-2,1}, \tag{4.59}$$

where $\mathbf{I}_{r-2,1}$ is a unit Minkowski $(r-1)$-blade of $\mathcal{G}(\mathcal{R}^{n,1})$, and where $\mathcal{R}^{n,1}$ is represented by $\tilde{a}_0$.
Case 2. When $a_0 \cdot A_r = 0$, A_r represents an $(r-2)$-dimensional sphere at infinity of $\mathcal{D}^n$. It lies on the $(r-1)$-plane $a_0 \wedge A_r$. After normalization, the *standard form* of the $(r-2)$-dimensional sphere at infinity is

$$\mathbf{I}_{r-1,1}, \tag{4.60}$$

where $\mathbf{I}_{r-1,1}$ is a unit Minkowski r-blade of $\mathcal{G}(\mathcal{R}^{n,1})$.
Case 3. When both $a_0 \wedge A_r \neq 0$ and $a_0 \cdot A_r \neq 0$, A_r represents an $(r-2)$-dimensional generalized sphere. This is because

$$A_{r+1} = a_0 \wedge A_r \neq 0, \tag{4.61}$$

and the vector

$$s = A_r A_{r+1}^{-1} \tag{4.62}$$

has positive square with both $a_0 \cdot s \neq 0$ and $a_0 \wedge s \neq 0$, so $\tilde{s}$ represents an $(n-1)$-dimensional generalized sphere. According to Case 1, A_{r+1} represents an $(r-1)$-dimensional plane in $\mathcal{D}^n$. Therefore, with $\epsilon = \frac{(n+2)(n+1)}{2} + 1$,

$$A_r = sA_{r+1} = (-1)^{\epsilon} \tilde{s} \vee A_{r+1} \tag{4.63}$$

represents the intersection of $(n-1)$-dimensional generalized sphere $\tilde{s}$ with $(r-1)$-plane A_{r+1}, which is an $(r-2)$-dimensional generalized sphere.

With suitable normalization, we can write

$$s = c - \rho a_0. \tag{4.64}$$

Since $s \wedge A_{r+1} = p_0 \wedge A_{r+1} = 0$, the generalized sphere A_r is also centered at c and has normal radius ρ, and it is of the same type as the generalized sphere represented by $\tilde{s}$. Now we can represent an $(r-2)$-dimensional generalized sphere in the *standard form*

$$(c - \lambda a_0)\ (a_0 \wedge \mathbf{I}_{r-1,1}), \tag{4.65}$$

where $\mathbf{I}_{r-1,1}$ is a unit Minkowski r-blade of $\mathcal{G}(\mathcal{R}^{n,1})$.

□

Corollary: *The $(r-2)$-dimensional total sphere passing through r homogeneous points or points at infinity $p_1, \ldots, p_r$ in $\mathcal{D}^n$ is represented by $A_r = p_1 \wedge \cdots \wedge p_r$; the $(r-2)$-plane passing through $r-1$ homogeneous points or points at infinity $p_1, \ldots, p_{r-1}$ in $\mathcal{D}^n$ is represented by $a_0 \wedge p_1 \wedge \cdots \wedge p_{r-1}$.*

When the p's are all homogeneous points, we can expand the inner product $A_r^\dagger \cdot A_r$ as

$$A_r^\dagger \cdot A_r = \det(p_i \cdot p_j)_{r\times r} = (-\frac{1}{2})^r \det((\mathbf{p}_i - \mathbf{p}_j)^2)_{r\times r}. \tag{4.66}$$

When $r = n+2$, we obtain Ptolemy's Theorem for double-hyperbolic space:

Theorem 7 (Ptolemy's Theorem). *Let $\mathbf{p}_1, \cdots, \mathbf{p}_{n+2}$ be points in $\mathcal{D}^n$, then they are on a generalized sphere or hyperplane of $\mathcal{D}^n$ if and only if* $\det((\mathbf{p}_i - \mathbf{p}_j)^2)_{(n+2)\times(n+2)} = 0$.

4.4 Stereographic projection

In $\mathcal{R}^{n,1}$, let $\mathbf{p}_0$ be a fixed point in $\mathcal{H}^n$. The space $\mathcal{R}^n = (a_0 \wedge \mathbf{p}_0)^\sim$, which is parallel to the tangent hyperplanes of $\mathcal{D}^n$ at points $\pm\mathbf{p}_0$, is Euclidean. By the *stereographic projection* of $\mathcal{D}^n$ from point $-\mathbf{p}_0$ to the space $\mathcal{R}^n$, every affine line of $\mathcal{R}^{n,1}$ passing through points $-\mathbf{p}_0$ and $\mathbf{p}$ intersects $\mathcal{R}^n$ at point

$$j_{\mathcal{DR}}(\mathbf{p}) = \frac{\mathbf{p}_0(\mathbf{p}_0 \wedge \mathbf{p})}{\mathbf{p}_0 \cdot \mathbf{p} - 1} = -2(\mathbf{p} + \mathbf{p}_0)^{-1} - \mathbf{p}_0. \tag{4.67}$$

Any point at infinity of $\mathcal{D}^n$ can be written in the form $\mathbf{p}_0 + a$, where a is a unit vector in $\mathcal{R}^n$ represented by $(a_0 \wedge \mathbf{p}_0)^\sim$. Every affine line passing through point $-p_0$ and point at infinity $\mathbf{p}_0 + a$ intersects $\mathcal{R}^n$ at point a. It is a classical result that the map $j_{\mathcal{DR}}$ is a conformal map from $\mathcal{D}^n$ to $\mathcal{R}^n$.

We show that in the homogeneous model we can construct the conformal map j_{SR} trivially; it is nothing but a rescaling of null vectors.

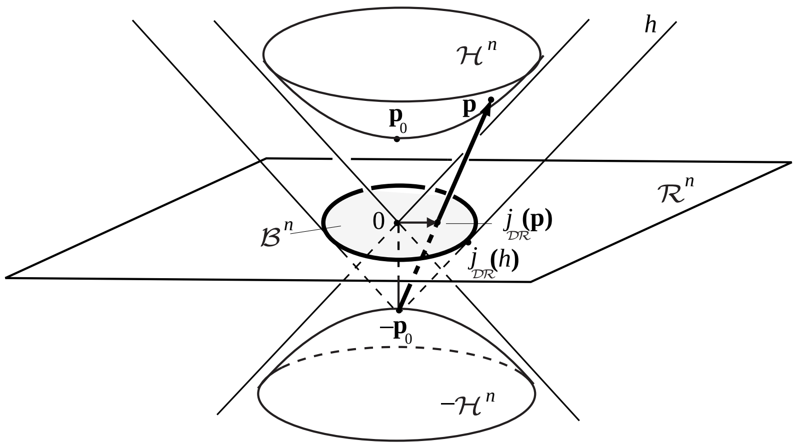

Figure 5: Stereographic projection of $\mathcal{D}^n$ from $-\mathbf{p}_0$ to $\mathcal{R}^n$.

Let

$$e = a_0 + \mathbf{p}_0, \quad e_0 = \frac{-a_0 + \mathbf{p}_0}{2}, \quad E = e \wedge e_0. \tag{4.68}$$

Then for $\mathcal{R}^n = (e \wedge e_0)^{\sim} = (a_0 \wedge \mathbf{p}_0)^{\sim}$, the map

$$i_E : x \in \mathcal{R}^n \mapsto e_0 + x + \frac{x^2}{2} e \in \mathcal{N}_e^n \tag{4.69}$$

defines a homogeneous model for Euclidean space.

Any null vector h in $\mathcal{R}^{n+1,1}$ represents a point or point at infinity in both homogeneous models of $\mathcal{D}^n$ and $\mathcal{R}^n$. The rescaling transformation $k_{\mathcal{R}} : \mathcal{N}^n \longrightarrow \mathcal{N}_e^n$ defined by

$$k_{\mathcal{R}}(h) = -\frac{h}{h \cdot e}, \text{ for } h \in \mathcal{N}^n, \tag{4.70}$$

where $\mathcal{N}^n$ represents the null cone of $\mathcal{R}^{n+1,1}$, induces the stereographic projection $j_{\mathcal{DR}}$ through the following commutative diagram:

$$\begin{tikzcd}
\mathbf{p} - a_0 \in \mathcal{N}_{a_0}^n \arrow[r, dashed, "k_{\mathcal{R}}"] & \dfrac{\mathbf{p} - a_0}{1 - \mathbf{p} \cdot \mathbf{p}_0} \in \mathcal{N}_e^n \arrow[d, "P_E^{\perp}"] \\
\mathbf{p} \in \mathcal{D}^n \arrow[u, "i_{a_0}"] \arrow[r, dashed, "j_{\mathcal{DR}}"] & \dfrac{\mathbf{p}_0(\mathbf{p}_0 \wedge \mathbf{p})}{\mathbf{p} \cdot \mathbf{p}_0 - 1} \in \mathcal{R}^n
\end{tikzcd}$$

i.e., $j_{\mathcal{DR}} = P_E^{\perp} \circ k_{\mathcal{R}} \circ i_{a_0}$. Since a point at infinity $\mathbf{p}_0 + a$ of $\mathcal{D}^n$ belongs to $\mathcal{N}_e^n$, we have

$$j_{\mathcal{DR}}(\mathbf{p}_0 + a) = P_E^{\perp}(\mathbf{p}_0 + a) = a. \tag{4.71}$$

The inverse of the map $j_{\mathcal{DR}}$, denoted by $j_{\mathcal{RD}}$, is

$$j_{\mathcal{RD}}(u) = \begin{cases} \dfrac{(1+u^2)\mathbf{p}_0 + 2u}{1-u^2}, & \text{for} \quad u^2 \neq 1, u \in \mathcal{R}^n, \\ \mathbf{p}_0 + u, & \text{for} \quad u^2 = 1, u \in \mathcal{R}^n, \end{cases} \tag{4.72}$$

When u is not on the unit sphere of $\mathcal{R}^n$, $j_{\mathcal{RD}}(u)$ can also be written as

$$j_{\mathcal{RD}}(u) = -2(u+\mathbf{p}_0)^{-1} - \mathbf{p}_0 = (u+\mathbf{p}_0)^{-1}\mathbf{p}_0(u+\mathbf{p}_0). \tag{4.73}$$

4.5 The conformal ball model

The standard definition of the conformal ball model [I92] is the unit ball $\mathcal{B}^n$ of $\mathcal{R}^n$ equipped with the following metric: for any $u, v \in \mathcal{B}^n$,

$$\cosh d(u,v) = 1 + \frac{2(u-v)^2}{(1-u^2)(1-v^2)}. \tag{4.74}$$

This model can be derived through the stereographic projection from $\mathcal{H}^n$ to $\mathcal{R}^n$. Recall that the sphere at infinity of $\mathcal{H}^n$ is mapped to the unit sphere of $\mathcal{R}^n$, and $\mathcal{H}^n$ is projected onto the unit ball $\mathcal{B}^n$ of $\mathcal{R}^n$. Using the formula (4.72) we get that for any two points u, v in the unit ball,

$$|j_{\mathcal{RD}}(u) - j_{\mathcal{RD}}(v)| = \frac{2|u-v|}{\sqrt{(1-u^2)(1-v^2)}}, \tag{4.75}$$

which is equivalent to (4.74) since

$$\cosh d(u,v) - 1 = \frac{|j_{\mathcal{RD}}(u) - j_{\mathcal{RD}}(v)|^2}{2}. \tag{4.76}$$

The following correspondences exist between the hyperboloid model and the conformal ball model:

1. A hyperplane normal to a and passing through $-\mathbf{p}_0$ in $\mathcal{D}^n$ corresponds to the hyperspace normal to a in $\mathcal{R}^n$.

2. A hyperplane normal to a but not passing through $-\mathbf{p}_0$ in $\mathcal{D}^n$ corresponds to the sphere orthogonal to the unit sphere $\mathcal{S}^{n-1}$ in $\mathcal{R}^n$; it has center $-\mathbf{p}_0 - \dfrac{a}{a \cdot \mathbf{p}_0}$ and radius $\dfrac{1}{|a \cdot \mathbf{p}_0|}$.

3. A sphere with center c and normal radius ρ in $\mathcal{D}^n$ and passing through $-\mathbf{p}_0$ corresponds to the hyperplane in $\mathcal{R}^n$ normal to $P^{\perp}_{\mathbf{p}_0}(c)$ with signed distance from the origin $-\dfrac{1+\rho}{\sqrt{(1+\rho)^2-1}} < -1$.

4. A sphere not passing through $-\mathbf{p}_0$ in $\mathcal{D}^n$ corresponds to a sphere disjoint with $\mathcal{S}^{n-1}$.

5. A horosphere with center c and relative radius ρ in $\mathcal{D}^n$ passing through $-\mathbf{p}_0$ corresponds to the hyperplane in $\mathcal{R}^n$ normal to $P^{\perp}_{\mathbf{p}_0}(c)$ and with signed distance -1 from the origin.

6. A horosphere not passing through $-\mathbf{p}_0$ in $\mathcal{D}^n$ corresponds to a sphere tangent with $\mathcal{S}^{n-1}$.

7. A hypersphere with center c and tangent radius ρ in $\mathcal{D}^n$ passing through $-\mathbf{p}_0$ corresponds to the hyperplane in $\mathcal{R}^n$ normal to $P^{\perp}_{\mathbf{p}_0}(c)$ and with signed distance from the origin $-\dfrac{\rho}{\sqrt{1+\rho^2}} > -1$.

8. A hypersphere not passing through $-\mathbf{p}_0$ in $\mathcal{D}^n$ corresponds to a sphere intersecting but not perpendicular with $\mathcal{S}^{n-1}$.

The homogeneous model differs from the hyperboloid model only by a rescaling of null vectors.

4.6 The hemisphere model

Let a_0 be a point in $\mathcal{S}^n$. The hemisphere model [CFK97] is the hemisphere $\mathcal{S}^n_+$ centered at $-a_0$ of $\mathcal{S}^n$, equipped with the following metric: for two points a, b,

$$\cosh d(a,b) = 1 + \frac{1 - a \cdot b}{(a \cdot a_0)(b \cdot a_0)}. \tag{4.77}$$

This model is traditionally obtained as the stereographic projection $j_{\mathcal{SR}}$ of $\mathcal{S}^n$ from a_0 to $\mathcal{R}^n$, which maps the hemisphere $\mathcal{S}^n_+$ onto the unit ball of $\mathcal{R}^n$. Since the stereographic projection $j_{\mathcal{DR}}$ of $\mathcal{D}^n$ from $-\mathbf{p}_0$ to $\mathcal{R}^n$ also maps $\mathcal{H}^n$ onto the unit ball of $\mathcal{R}^n$, the composition

$$j_{\mathcal{DS}} = j^{-1}_{\mathcal{SR}} \circ j_{\mathcal{DR}} : \mathcal{D}^n \longrightarrow \mathcal{S}^n \tag{4.78}$$

maps $\mathcal{H}^n$ to $\mathcal{S}^n_+$, and maps the sphere at infinity of $\mathcal{H}^n$ to $\mathcal{S}^{n-1}$, the boundary of $\mathcal{S}^n_+$, which is the hyperplane of $\mathcal{S}^n$ normal to a_0. This map is conformal and bijective. It produces the hemisphere model of the hyperbolic space.

The following correspondences exist between the hyperboloid model and the hemisphere model:

1. A hyperplane normal to a and passing through $-\mathbf{p}_0$ in $\mathcal{D}^n$ corresponds to the hyperplane normal to a in $\mathcal{S}^n$.

2. A hyperplane normal to a but not passing through $-\mathbf{p}_0$ in $\mathcal{D}^n$ corresponds to a sphere with center on $\mathcal{S}^{n-1}$.

3. A sphere with center $\mathbf{p}_0$ (or $-\mathbf{p}_0$) in $\mathcal{D}^n$ corresponds to a sphere in $\mathcal{S}^n$ with center $-a_0$ (or a_0).

4. A sphere in $\mathcal{D}^n$ corresponds to a sphere disjoint with $\mathcal{S}^{n-1}$.

5. A horosphere corresponds to a sphere tangent with $\mathcal{S}^{n-1}$.
6. A hypersphere with center c, relative radius ρ in $\mathcal{D}^n$ and axis passing through $-\mathbf{p}_0$ corresponds to the hyperplane in $\mathcal{S}^n$ normal to $c - \rho a_0$.
7. A hypersphere whose axis does not pass through $-\mathbf{p}_0$ in $\mathcal{D}^n$ corresponds to a sphere intersecting with $\mathcal{S}^{n-1}$.

The hemisphere model can also be obtained from the homogeneous model by rescaling null vectors. The map $k_{\mathcal{S}} : \mathcal{N}^n \longrightarrow \mathcal{N}^n_{p_0}$ defined by

$$k_{\mathcal{S}}(h) = -\frac{h}{h \cdot p_0}, \text{ for } h \in \mathcal{N}^n \tag{4.79}$$

induces a conformal map $j_{\mathcal{DS}}$ through the following commutative diagram:

$$\begin{tikzcd}
\mathbf{p} - a_0 \in \mathcal{N}^n_{a_0} \arrow[r, dashed, "k_{\mathcal{S}}"] & -\dfrac{\mathbf{p} - a_0}{\mathbf{p} \cdot \mathbf{p}_0} \in \mathcal{N}^n_{\mathbf{p}_0} \arrow[d, "P^{\perp}_{\mathbf{p}_0}"] \\
\mathbf{p} \in \mathcal{D}^n \arrow[u, "i_{a_0}"] \arrow[r, dashed, "j_{\mathcal{DS}}"] & \dfrac{a_0 + \mathbf{p}_0(\mathbf{p}_0 \wedge \mathbf{p})}{\mathbf{p} \cdot \mathbf{p}_0} \in \mathcal{S}^n
\end{tikzcd}$$

i.e., $j_{\mathcal{DS}} = P^{\perp}_{\mathbf{p}_0} \circ k_{\mathcal{S}} \circ i_{a_0}$. For a point $\mathbf{p}$ in $\mathcal{D}^n$,

$$j_{\mathcal{DS}}(\mathbf{p}) = \frac{a_0 + \mathbf{p}_0(\mathbf{p}_0 \wedge \mathbf{p})}{\mathbf{p}_0 \cdot \mathbf{p}} = -\mathbf{p}_0 - \frac{\mathbf{p} - a_0}{\mathbf{p} \cdot \mathbf{p}_0}. \tag{4.80}$$

For a point at infinity $\mathbf{p}_0 + a$, we have

$$j_{\mathcal{DS}}(\mathbf{p}_0 + a) = P^{\perp}_{\mathbf{p}_0}(\mathbf{p}_0 + a) = a. \tag{4.81}$$

We see that $\pm\mathbf{p}_0$ corresponds to $\mp a_0$. Let $\mathbf{p}$ correspond to a in $\mathcal{S}^n$. Then

$$\mathbf{p} \cdot \mathbf{p}_0 = -\frac{1}{a \cdot a_0}. \tag{4.82}$$

The inverse of the map $j_{\mathcal{DS}}$, denoted by $j_{\mathcal{SD}}$, is

$$j_{\mathcal{SD}}(a) = \begin{cases} a_0 - \dfrac{\mathbf{p}_0 + a}{a_0 \cdot a}, & \text{for} \quad a \in \mathcal{S}^n, a \cdot a_0 \neq 0, \\ \mathbf{p}_0 + a, & \text{for} \quad a \in \mathcal{S}^n, a \cdot a_0 = 0. \end{cases} \tag{4.83}$$

4.7 The half-space model

The standard definition of the half-space model [L92] is the half space $\mathcal{R}^n_+$ of $\mathcal{R}^n$ bounded by $\mathcal{R}^{n-1}$, which is the hyperspace normal to a unit vector a_0, contains point $-a_0$, and is equipped with the following metric: for any $u, v \in \mathcal{R}^n_+$,

$$\cosh d(u, v) = 1 + \frac{(u - v)^2}{2(u \cdot a_0)(v \cdot a_0)}. \tag{4.84}$$

This model is traditionally obtained from the hyperboloid model as follows: The stereographic projection $j_{\mathcal{SR}}$ of $\mathcal{S}^n$ is from a_0 to $\mathcal{R}^{n+1,1}$. As an alternative "north pole" select a point b_0, which is orthogonal to a_0. This pole determines a stereographic projection j_{b_0} with projection plane is $\mathcal{R}^n = (b_0 \wedge \mathbf{p}_0)^{\sim}$. The map $j_{\mathcal{DS}} : \mathcal{D}^n \longrightarrow \mathcal{S}^n$ maps $\mathcal{H}^n$ to the hemisphere $\mathcal{S}^n_+$ centered at $-a_0$. The map j_{b_0} maps $\mathcal{S}^n_+$ to $\mathcal{R}^n_+$. As a consequence, the map

$$j_{\mathcal{HR}} = j_{b_0} \circ j_{\mathcal{DS}} : \mathcal{D}^n \longrightarrow \mathcal{R}^n \tag{4.85}$$

maps $\mathcal{H}^n$ to $\mathcal{R}^n_+$, and maps the sphere at infinity of $\mathcal{D}^n$ to $\mathcal{R}^{n-1}$.

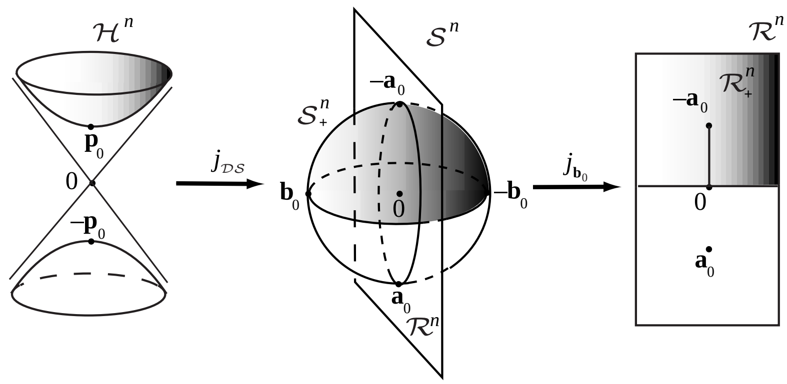

Figure 6: The hemisphere model and the half-space model.

The half-space model can also be derived from the homogeneous model by rescaling null vectors. Let $\mathbf{p}_0$ be a point in $\mathcal{H}^n$ and h be a point at infinity of $\mathcal{H}^n$, then $h \wedge \mathbf{p}_0$ is a line in $\mathcal{H}^n$, which is also a line in $\mathcal{H}^{n+1}$, the $(n+1)$-dimensional hyperbolic space in $\mathcal{R}^{n+1,1}$. The Euclidean space $\mathcal{R}^n = (h \wedge \mathbf{p}_0)^{\sim}$ is in the tangent hyperplane of $\mathcal{H}^{n+1}$ at $\mathbf{p}_0$ and normal to the tangent vector $P^{\perp}_{\mathbf{p}_0}(h)$ of line $h \wedge \mathbf{p}_0$. Let

$$e = -\frac{h}{h \cdot \mathbf{p}_0}, \quad e_0 = \mathbf{p}_0 - \frac{e}{2}. \tag{4.86}$$

Then $e^2 = e_0^2 = 0$, $e \cdot e_0 = e \cdot \mathbf{p}_0 = -1$, and $e \wedge \mathbf{p}_0 = e \wedge e_0$. The unit vector

$$b_0 = e - \mathbf{p}_0 \tag{4.87}$$

is orthogonal to both $\mathbf{p}_0$ and a_0, and can be identified with the pole b_0 of the stereographic projection j_{b_0}. Let $E = e \wedge e_0$. The rescaling map $k_R : \mathcal{N}^n \longrightarrow \mathcal{N}^n_e$

induces the map $j_{\mathcal{HR}}$ through the following commutative diagram:

$$\begin{tikzcd}
\mathbf{p} - a_0 \in \mathcal{N}^n_{a_0} \arrow[r, dashed, "k_{\mathcal{R}}"] & -\dfrac{\mathbf{p} - a_0}{\mathbf{p} \cdot e} \in \mathcal{N}^n_e \arrow[d, "P^{\perp}_E"] \\
\mathbf{p} \in \mathcal{D}^n \arrow[u, "i_{a_0}"] \arrow[r, dashed, "j_{\mathcal{HR}}"] & \dfrac{a_0 - P^{\perp}_E(\mathbf{p})}{\mathbf{p} \cdot e} \in \mathcal{R}^n
\end{tikzcd}$$

i.e., $j_{\mathcal{HR}} = P^{\perp}_E \circ k_R \circ i_{a_0}$. For a point $\mathbf{p}$ in $\mathcal{D}^n$, we have

$$j_{\mathcal{HR}}(\mathbf{p}) = \frac{a_0 - P^{\perp}_{e \wedge \mathbf{p}_0}(\mathbf{p})}{\mathbf{p} \cdot e}. \tag{4.88}$$

For a point at infinity $\mathbf{p}_0 + a$ in $\mathcal{D}^n$, we have

$$j_{\mathcal{HR}}(\mathbf{p}_0 + a) = \frac{a + e \cdot a(\mathbf{p}_0 - e)}{1 - e \cdot a}. \tag{4.89}$$

The inverse of the map $j_{\mathcal{HR}}$ is denoted by $j_{\mathcal{RH}}$:

$$j_{\mathcal{RH}}(u) = \begin{cases} a_0 - \dfrac{e_0 + u + \frac{u^2}{2} e}{a_0 \cdot u}, & \text{for} \quad u \in \mathcal{R}^n, u \cdot a_0 \neq 0, \\[2ex] e_0 + u + \dfrac{u^2}{2} e, & \text{for} \quad u \in \mathcal{R}^n, u \cdot a_0 = 0. \end{cases} \tag{4.90}$$

The following correspondences exist between the hyperboloid model and the half-space model:

1. A hyperplane normal to a and passing through e in $\mathcal{D}^n$ corresponds to the hyperplane in $\mathcal{R}^n$ normal to $a + a \cdot \mathbf{p}_0 e$ with signed distance $-a \cdot \mathbf{p}_0$ from the origin.

2. A hyperplane not passing through e in $\mathcal{D}^n$ corresponds to a sphere with center on $\mathcal{R}^{n-1}$.

3. A sphere in $\mathcal{D}^n$ corresponds to a sphere disjoint with $\mathcal{R}^{n-1}$.

4. A horosphere with center e (or $-e$) and relative radius ρ corresponds to the hyperplane in $\mathcal{R}^n$ normal to a_0 with signed distance $-1/\rho$ (or $1/\rho$) from the origin.

5. A horosphere with center other than $\pm e$ corresponds to a sphere tangent with $\mathcal{R}^{n-1}$.

6. A hypersphere with center c, tangent radius ρ in $\mathcal{D}^n$ and axis passing through e corresponds to the hyperplane in $\mathcal{R}^n$ normal to $c - \rho a_0 + c \cdot \mathbf{p}_0 e$ with signed distance $-\dfrac{c \cdot \mathbf{p}_0}{\sqrt{1 + \rho^2}}$ from the origin.

7. A hypersphere whose axis does not pass through e in $\mathcal{D}^n$ corresponds to a sphere intersecting with $\mathcal{R}^{n-1}$.

4.8 The Klein ball model

The standard definition of the Klein ball model [I92] is the unit ball $\mathcal{B}^n$ of $\mathcal{R}^n$ equipped with the following metric: for any $u, v \in \mathcal{B}^n$,

$$\cosh d(u, v) = \frac{1 - u \cdot v}{\sqrt{(1 - u^2)(1 - v^2)}}. \tag{4.91}$$

This model is not conformal, contrary to all the previous models, and is valid only for $\mathcal{H}^n$, not for $\mathcal{D}^n$.

The standard derivation of this model is through the central projection of $\mathcal{H}^n$ to $\mathcal{R}^n$. Recall that when we construct the conformal ball model, we use the stereographic projection of $\mathcal{D}^n$ from $-\mathbf{p}_0$ to the space $\mathcal{R}^n = (a_0 \wedge \mathbf{p}_0)^\sim$. If we replace $-\mathbf{p}_0$ with the origin, replace the space $(a_0 \wedge \mathbf{p}_0)^\sim$ with the tangent hyperplane of $\mathcal{H}^n$ at point $\mathbf{p}_0$, and replace $\mathcal{D}^n$ with its branch $\mathcal{H}^n$, then every affine line passing through the origin and point $\mathbf{p}$ of $\mathcal{H}^n$ intersects the tangent hyperplane at point

$$j_K(\mathbf{p}) = \frac{\mathbf{p}_0(\mathbf{p}_0 \wedge \mathbf{p})}{\mathbf{p}_0 \cdot \mathbf{p}}. \tag{4.92}$$

Every affine line passing through the origin and a point at infinity $\mathbf{p}_0 + a$ of $\mathcal{H}^n$ intersects the tangent hyperplane at point a.

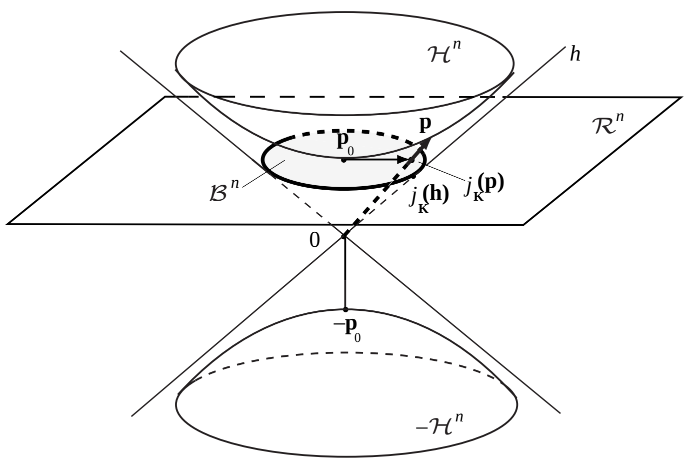

Figure 7: The Klein ball model.

The projection $j_{\mathcal{HB}}$ maps $\mathcal{H}^n$ to $\mathcal{B}^n$, and maps the sphere at infinity of $\mathcal{H}^n$ to the unit sphere of $\mathcal{R}^n$. This map is one-to-one and onto. Since it is central projection, every r-plane of $\mathcal{H}^n$ is mapped to an r-plane of $\mathcal{R}^n$ inside $\mathcal{B}^n$.

Although not conformal, the Klein ball model can still be constructed in the homogeneous model. We know that $j_{\mathcal{DS}}$ maps $\mathcal{H}^n$ to $\mathcal{S}^n_+$, the hemisphere of $\mathcal{S}^n$ centered at $-a_0$. A stereographic projection of $\mathcal{S}^n$ from a_0 to $\mathcal{R}^n$, yields a model of $\mathcal{D}^n$ in the whole of $\mathcal{R}^n$. Now instead of a stereographic projection, use a parallel projection $P_{\tilde{a}_0} = P^{\perp}_{a_0}$ from $\mathcal{S}^n_+$ to $\mathcal{R}^n = (a_0 \wedge \mathbf{p}_0)^{\sim}$ along a_0. The map

$$j_K = P^{\perp}_{a_0} \circ j_{\mathcal{DS}} : \mathcal{H}^n \longrightarrow \mathcal{B}^n \tag{4.93}$$

is the central projection and produces the Klein ball model.

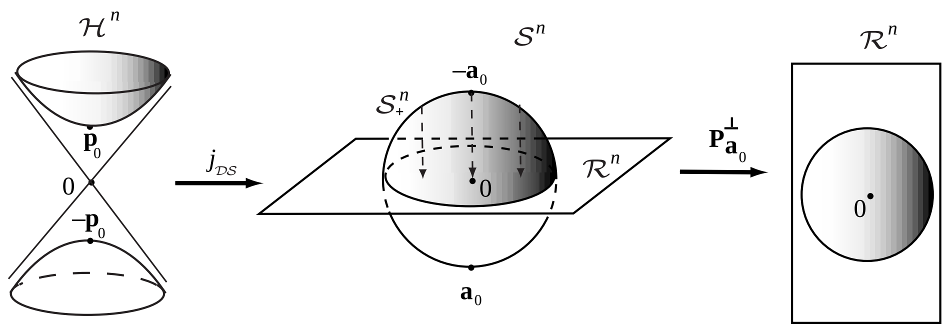

Figure 8: The Klein ball model derived from parallel projection of $\mathcal{S}^n$ to $\mathcal{R}^n$.

The following are some properties of the map j_K. There is no correspondence between spheres in $\mathcal{H}^n$ and $\mathcal{B}^n$ because the map is not conformal.

1. A hyperplane of $\mathcal{H}^n$ normal to a is mapped to the hyperplane of $\mathcal{B}^n$ normal to $P^{\perp}_{\mathbf{p}_0}(a)$ and with signed distance $-\dfrac{a \cdot \mathbf{p}_0}{\sqrt{1 + (a \cdot \mathbf{p}_0)^2}}$ from the origin.

2. An r-plane of $\mathcal{H}^n$ passing through $\mathbf{p}_0$ and normal to the space of $\mathbf{I}_{n-r}$, where $\mathbf{I}_{n-r}$ is a unit $(n-r)$-blade of Euclidean signature in $\mathcal{G}(\mathcal{R}^n)$, is mapped to the r-space of $\mathcal{B}^n$ normal to the space $\mathbf{I}_{n-r}$.

3. An r-plane of $\mathcal{H}^n$ normal to the space of $\mathbf{I}_{n-r}$ but not passing through $\mathbf{p}_0$, where $\mathbf{I}_{n-r}$ is a unit $(n-r)$-blade of Euclidean signature in $\mathcal{G}(\mathcal{R}^n)$, is mapped to an r-plane $\mathcal{L}$ of $\mathcal{B}^n$. The plane $\mathcal{L}$ is in the $(r+1)$-space, which is normal to the space of $\mathbf{p}_0 \cdot \mathbf{I}_{n-r}$ of $\mathcal{R}^n$, and is normal to the vector $\mathbf{p}_0 + (P_{\mathbf{I}_{n-r}}(\mathbf{p}_0))^{-1}$ in the $(r+1)$-space, with signed distance $-\dfrac{1}{\sqrt{1 + (P_{\mathbf{I}_{n-r}}(\mathbf{p}_0))^{-2}}}$ from the origin.

The inverse of the map $j_{\mathcal{HB}}$ is

$$j_K^{-1}(u) = \begin{cases} \dfrac{u + \mathbf{p}_0}{|u + \mathbf{p}_0|}, & \text{for} \quad u \in \mathcal{R}^n,\ u^2 < 1, \\ u + \mathbf{p}_0, & \text{for} \quad u \in \mathcal{R}^n,\ u^2 = 1. \end{cases} \tag{4.94}$$

The following are some properties of this map:

1. A hyperplane of $\mathcal{B}^n$ normal to n with signed distance δ from the origin is mapped to the hyperplane of $\mathcal{H}^n$ normal to $n - \delta \mathbf{p}_0$.

2. An r-space $\mathbf{I}_r$ of $\mathcal{B}^n$, where $\mathbf{I}_r$ is a unit r-blade in $\mathcal{G}(\mathcal{R}^n)$, is mapped to the r-plane $a_0 \wedge \mathbf{p}_0 \wedge \mathbf{I}_r$ of $\mathcal{H}^n$.

3. An r-plane in the $(r+1)$-space $\mathbf{I}_{r+1}$ of $\mathcal{B}^n$, normal to vector n in the $(r+1)$-space with signed distance δ from the origin, where $\mathbf{I}_{r+1}$ is a unit $(r+1)$-blade in $\mathcal{G}(\mathcal{R}^n)$, is mapped to the r-plane $(n - \delta \mathbf{p}_0)\,(a_0 \wedge \mathbf{p}_0 \wedge \mathbf{I}_{r+1})$ of $\mathcal{H}^n$.

4.9 A universal model for Euclidean, spherical and hyperbolic spaces

We have seen that spherical and Euclidean spaces and the five well-known analytic models of the hyperbolic space, all derive from the null cone of a Minkowski space, and are all included in the homogeneous model. Except for the Klein ball model, these geometric spaces are conformal to each other. No matter how viewpoints are chosen for projective splits, the correspondences among spaces projectively determined by common null vectors and Minkowski blades are always conformal. This is because for any nonzero vectors c, c' and any null vectors $h_1, h_2 \in \mathcal{N}^n_{c'}$, where

$$\mathcal{N}^n_{c'} = \{x \in \mathcal{N}^n | x \cdot c' = -1\}, \tag{4.95}$$

we have

$$\left| -\frac{h_1}{h_1 \cdot c} + \frac{h_2}{h_2 \cdot c} \right| = \frac{|h_1 - h_2|}{\sqrt{|(h_1 \cdot c)(h_2 \cdot c)|}}, \tag{4.96}$$

i.e., the rescaling is conformal with conformal coefficient $1/\sqrt{|(h_1 \cdot c)(h_2 \cdot c)|}$.

Recall that in previous constructions of the geometric spaces and models in the homogeneous model, we selected special viewpoints: $\mathbf{p}_0$, a_0, b_0, $e = \mathbf{p}_0 + a_0$, $e_0 = \frac{\mathbf{p}_0 - a_0}{2}$, etc. We can select any other nonzero vector c in $\mathcal{R}^{n+1,1}$ as the viewpoint for projective split, thereby obtaining a different realization for one of these spaces and models. For the Euclidean case, we can select any null vector in $\mathcal{N}^n_e$ as the origin e_0. This freedom in choosing viewpoints for projective and conformal splits establishes an equivalence among geometric theorems in conformal geometries of these spaces and models. From a single theorem, many "new" theorems can be generated in this way. We illustrate this with a simple example.

The original Simson's Theorem in plane geometry is as follows:

Theorem 8 (Simson's Theorem). *Let ABC be a triangle, D be a point on the circumscribed circle of the triangle. Draw perpendicular lines from D to the three sides AB, BC, CA of triangle ABC. Let C_1, A_1, B_1 be the three feet respectively. Then A_1, B_1, C_1 are collinear.*

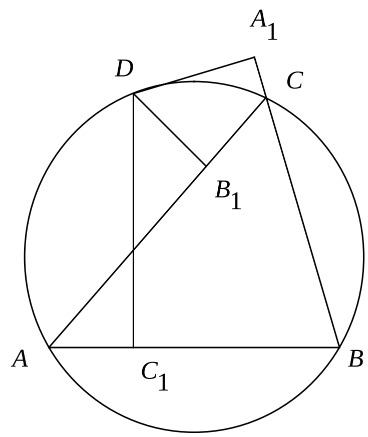

Figure 9: Original Simson's Theorem.

When $A, B, C, D, A_1, B_1, C_1$ are understood to be null vectors representing the corresponding points in the plane, the hypothesis can be expressed bt the following constraints:

$$
\begin{array}{ll}
A \wedge B \wedge C \wedge D = 0 & A, B, C, D \text{ are on the same circle} \\
e \wedge A \wedge B \wedge C \neq 0 & ABC \text{ is a triangle} \\
e \wedge A_1 \wedge B \wedge C = 0 & A_1 \text{ is on line } BC \\
(e \wedge D \wedge A_1) \cdot (e \wedge B \wedge C) = 0 & \text{Lines } DA_1 \text{ and } BC \text{ are perpendicular} \\
e \wedge A \wedge B_1 \wedge C = 0 & B_1 \text{ is on line } CA \\
(e \wedge D \wedge B_1) \cdot (e \wedge C \wedge A) = 0 & \text{Lines } DB_1 \text{ and } CA \text{ are perpendicular} \\
e \wedge A \wedge B \wedge C_1 = 0 & C_1 \text{ is on line } AB \\
(e \wedge D \wedge C_1) \cdot (e \wedge A \wedge B) = 0 & \text{Lines } DC_1 \text{ and } AB \text{ are perpendicular}
\end{array}
\tag{4.97}
$$

The conclusion can be expressed as

$$e \wedge A_1 \wedge B_1 \wedge C_1 = 0. \tag{4.98}$$

Both the hypothesis and the conclusion are invariant under rescaling of null vectors, so this theorem is valid for all three geometric spaces, and is free of the requirement that $A, B, C, D, A_1, B_1, C_1$ represent points and e represents the point at infinity of $\mathcal{R}^n$. Various "new" theorems can be produced by interpreting the algebraic equalities and inequalities in the hypothesis and conclusion of Simson's theorem differently.

For example, let us exchange the roles that D, e play in Euclidean geometry. The constraints become

$$\begin{array}{l} e \wedge A \wedge B \wedge C = 0 \\ A \wedge B \wedge C \wedge D \neq 0 \\ A_1 \wedge B \wedge C \wedge D = 0 \\ (e \wedge D \wedge A_1) \cdot (e \wedge B \wedge C) = 0 \\ A \wedge B_1 \wedge C \wedge D = 0 \\ (e \wedge D \wedge B_1) \cdot (e \wedge C \wedge A) = 0 \\ A \wedge B \wedge C_1 \wedge D = 0 \\ (e \wedge D \wedge C_1) \cdot (e \wedge A \wedge B) = 0\,, \end{array} \tag{4.99}$$

and the conclusion becomes

$$A_1 \wedge B_1 \wedge C_1 \wedge D = 0. \tag{4.100}$$

This "new" theorem can be stated as follows:

Theorem 9. *Let DAB be a triangle, C be a point on line AB. Let A_1, B_1, C_1 be the symmetric points of D with respect to the centers of circles DBC, DCA, DAB respectively. Then D, A_1, B_1, C_1 are on the same circle.*

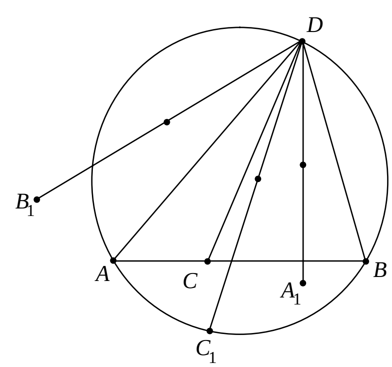

Figure 10: Theorem 9.

We can get another theorem by interchanging the roles of A, e. The constraints become

$$\begin{array}{l} e \wedge B \wedge C \wedge D = 0 \\ e \wedge A \wedge B \wedge C \neq 0 \\ A \wedge A_1 \wedge B \wedge C = 0 \\ (A \wedge D \wedge A_1) \cdot (A \wedge B \wedge C) = 0 \\ e \wedge A \wedge B_1 \wedge C = 0 \\ (A \wedge D \wedge B_1) \cdot (e \wedge C \wedge A) = 0 \\ e \wedge A \wedge B \wedge C_1 = 0 \\ (A \wedge D \wedge C_1) \cdot (e \wedge A \wedge B) = 0\,, \end{array} \tag{4.101}$$

and the conclusion becomes

$$A \wedge A_1 \wedge B_1 \wedge C_1 = 0. \tag{4.102}$$

This "new" theorem can be stated as follows:

Theorem 10. *Let ABC be a triangle, D be a point on line AB. Let EF be the perpendicular bisector of line segment AD, which intersects AB, AC at E, F respectively. Let C_1, B_1 be the symmetric points of A with respect to points E, F respectively. Let AG be the tangent line of circle ABC at A, which intersects EF at G. Let A_1 be the intersection, other than A, of circle ABC with the circle centered at G and passing through A. Then A, A_1, B_1, C_1 are on the same circle.*

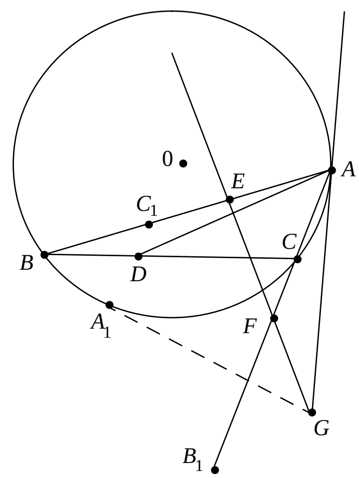

Figure 11: Theorem 10.

There are equivalent theorems in spherical geometry. We consider only one case. Let $e = -D$. A "new" theorem as follows:

Theorem 11. *Within the sphere there are four points A, B, C, D on the same circle. Let A_1, B_1, C_1 be the symmetric points of $-D$ with respect to the centers of circles $(-D)BC$, $(-D)CA$, $(-D)AB$ respectively. Then $-D, A_1, B_1, C_1$ are on the same circle.*

There are various theorems in hyperbolic geometry that are also equivalent to Simson's theorem because of the versatility of geometric entities. We present one case here. Let A, B, C, D be points on the same branch of $\mathcal{D}^2$, $e = -D$.

Theorem 12. *Let A, B, C, D be points in the Lobachevski plane $\mathcal{H}^2$ and be on the same generalized circle. Let L_A, L_B, L_C be the axes of hypercycles (1-dimensional hyperspheres) $(-D)BC$, $(-D)CA$, $(-D)AB$ respectively. Let A_1, B_1, C_1 be the symmetric points of D with respect to L_A, L_B, L_C respectively. Then $-D, A_1, B_1, C_1$ are on the same hypercycle.*

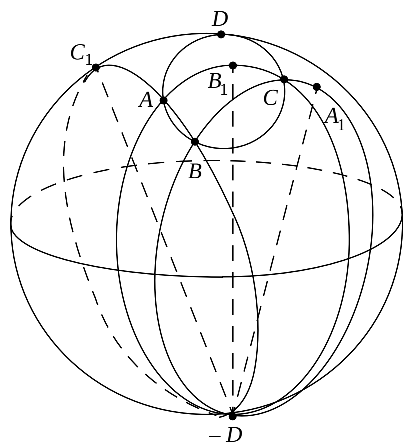

Figure 12: Theorem 11.

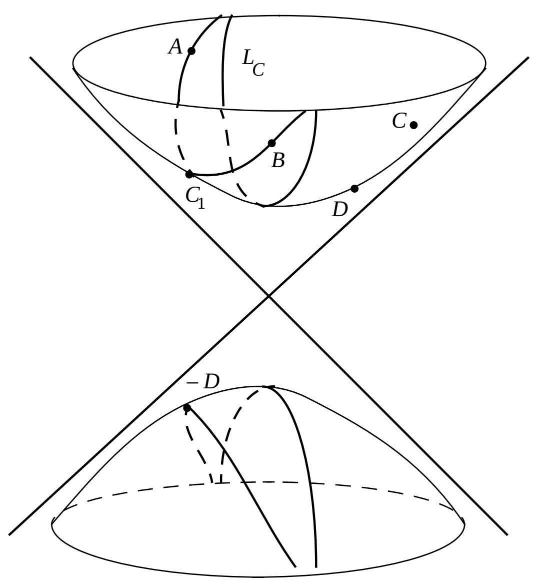

Figure 13: Construction of C_1 from A, B, D in Theorem 12.

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