

## UNIVERSAL GEOMETRIC ALGEBRA

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The claim that Clifford algebra should be regarded as a universal geometric algebra is strengthened by showing that the algebra is applicable to nonmetrical as well as metrical geometry. Clifford algebra is used to develop a coordinate-free algebraic formulation of projective geometry. Major theorems of projective geometry are reduced to algebraic identities which apply as well to metrical geometry. Improvements in the formulation of linear algebra are suggested to simplify its intimate relation to projective geometry. Relations among Clifford algebras of different dimensions are interpreted geometrically as “projective and conformal splits.” The conformal split is employed to simplify and elucidate the pin and spin representations of the conformal group for arbitrary dimension and signature.

### INTRODUCTION

Alfred North Whitehead promoted the idea of UNIVERSAL ALGEBRA in his monumental treatise of 1897 [1]. He proposed two candidates for this lofty title, the algebra of Boole and Grassmann’s Algebra of Extension. Boolean algebra has since secured universal status in Set Theory and Symbolic Logic, although only the former is universally known and used by mathematicians. However, Whitehead’s work on Grassmann Algebra, which ironically is much the larger portion of his treatise, has been almost totally ignored.

Grassmann algebra has been making a comeback among mathematicians in recent decades, primarily in the guise of differential forms. But this has involved only the half of Grassmann’s algebra generated by his progressive product, while his regressive product remains unappreciated along with Whitehead’s elaborate treatment of it.

Of course, the regressive product was designed to perform an important mathematical function which must be handled by other means in modern mathematics. As Gian-Carlo Rota and coworkers ([2],[3]) have vigorously argued, the result has been a step backward in conceptual clarity and computational efficiency. I agree that the regressive product should be revived, but I wish to go further.

I claim that Grassmann and Whitehead were just one step away from a mathematical system that truly deserves to be regarded as a UNIVERSAL GEOMETRIC ALGEBRA. That system is known in mathematics as Clifford algebra. However, the true universality of Clifford algebra has remained unrecognized, even by mathematicians specializing in the subject. The main reason, I suppose, is that the demonstration of universality requires a wholesale reorganization and redesign of mathematics, integrating into a single mathematical system such superficially disparate systems as quaternion calculus, differential forms

and vector analysis. The result is much more than a set of axioms for Clifford algebra along with a few theorems revealing the structure of the algebra. It is a fullblown mathematical language for expressing and elaborating geometric ideas of every sort. I have been engaged in for many years in developing this system into a unified geometrical language for physics as well as mathematics ([4],[5],[6],[7],[8]), and demonstrating its wide applicability. The ideas and insights of hundreds of mathematicians and physicists have been incorporated into the design of the language. I have learned that most (if not all) of the important insights have occurred to many people independently. Accordingly, I regard the system as a whole as a community creation rather than the work of a few individuals. And to emphasize the geometric universality of Clifford algebra, I insist on following the lead of Clifford himself by calling it *geometric algebra* instead.

Clifford algebra is commonly regarded as “the algebra of a quadratic form.” This, it seems, has been a major barrier to recognizing the geometric universality of the algebra. For it suggests that Clifford algebra is inapplicable to nonmetrical geometry. The main objective of this article is to dispel that misconception conclusively by showing explicitly how nonmetrical geometry can be handled with Clifford algebra. We shall see how Grassmann’s progressive and regressive products can be defined within Clifford algebra and employed in a coordinate-free algebraic formulation of projective geometry. This approach has a number of advantages. First, the complementary algebraic and synthetic approaches to projective geometry are brought much closer together, because the primitives in the synthetic formulation correspond directly to algebraic entities and operations. Accordingly, it becomes easier to combine the advantages of both approaches. Second, the algebraic methods of projective and metrical geometry are unified perfectly. This should help integrate the deep ideas of projective geometry with the rest of mathematics. Third, some improvements in linear algebra are suggested to increase its geometrical perspicacity and computational power.

After formulating projective geometry and its relation to linear algebra in the language of geometric algebra, we examine its relations to affine and metrical geometry. We find that these relations are naturally expressed by certain features in the multiplicative structure of Clifford algebras which have hitherto been without geometric interpretation.

This insight helps refine the formulation of spin and representations of the conformal group to integrate the alternative representations of previous authors. This article is devoted to a compact mathematical formulation and discussion of the above ideas and results. Proofs, elaborations and further details are contained in two lengthy articles ([9], [10]).

## 1. GEOMETRIC ALGEBRA

I like to distinguish between a linear space and a vector space. A linear space is defined as usual by the operations of addition and scalar multiplication, while a vector space is a linear space on which the *geometric product* is also defined. Thus, I regard the geometric product as one of the essential properties defining the concept of “vector.”

As the geometric product has not yet received the universal recognition I think it deserves, I must repeat its simple definition here. It is defined by the usual associative and distributive rules together with the special rule that the square of any vector is some scalar.

The last rule implies that an  $n$ -dimensional *vector space*  $\mathcal{V}_n$  is not closed under the

geometric product. Rather, by addition and multiplication it generates a *geometric* (or Clifford) *algebra*  $\mathcal{G}(\mathcal{V})$ . For present purposes, the scalars in  $\mathcal{G}$  are to be identified with the real numbers. To facilitate the discussion, I need to review some definitions and properties of  $\mathcal{G}$  (fully treated in [5],[7],[4]). From geometric products  $ab$  and  $ba$  of two vectors, two new products can be defined: the scalar-valued *inner product*

$$a \cdot b = \frac{1}{2}(ab + ba), \quad (1)$$

and the bivector-valued *outer product*

$$a \wedge b = \frac{1}{2}(ab - ba). \quad (2)$$

Thus, we have three kinds of product, related by

$$ab = a \cdot b + a \wedge b. \quad (3)$$

The antisymmetrized product of vectors  $a_1, a_2, \dots, a_k$  is a blade of *step*  $k$  or *k-blade*, denoted by

$$\langle a_1 a_2 \dots a_k \rangle_k = a_1 \wedge a_2 \wedge \dots \wedge a_k. \quad (4)$$

The outer product so defined for any number of vectors can be identified with Grassmann's progressive product.

The elements of  $\mathcal{G}_n$  are called multivectors. For any multivector  $M$ , the  $k$ -vector part  $\langle M \rangle_k$  is a multivector of step  $k$ . A  $k$ -vector is a  $k$ -blade if and only if it can be expressed as the outer product of  $k$  vectors. The main antiautomorphism of  $\mathcal{G}_n$  is called reversion. The reverse  $M^\dagger$  of  $M$  is characterized by

$$\langle M^\dagger \rangle_k = \langle M \rangle_k^\dagger = (-1)^{k(k-1)/2} \langle M \rangle_k. \quad (5)$$

For  $k = 0$ ,  $\langle M \rangle_0$  denotes the scalar part of  $M$ .

The inner and outer products defined for vectors by (1) and (2) can be extended to blades. For blades  $A$  and  $B$  of step  $r$  and  $s$  respectively, they are defined by

$$A \wedge B = \langle AB \rangle_{r+s} = (-1)^{rs} B \wedge A, \quad (6)$$

$$A \cdot B = \langle AB \rangle_{r-s} = (-1)^{s(r-s)} B \cdot A \quad \text{for } r \geq s. \quad (7)$$

Obviously  $A \wedge B$  is an  $(r+s)$ -blade. It can be proved that  $A \cdot B$  is an  $(r-s)$ -blade as well. A large number of identities involving inner and outer products are developed in [5]. They endow geometric algebra with great computational power. They include all the identities in the theory of determinants [5] and all identities in the vector calculus of Gibbs [7]. We see below that they include, as well, all algebraic identities in projective geometry. This is what we expect of a universal geometric algebra.

The outer product of  $k$  vectors (4) vanishes if and only if the vectors are linearly dependent. Therefore  $\mathcal{G}_n$  contains a non-vanishing blade  $I = \langle I \rangle_n$  of maximal step  $n$ , and all other  $n$ -blades are scalar multiples of  $I$ . Let us refer to  $I$  as the pseudoscalar of  $\mathcal{G}_n$ . A blade  $A$  is said to be *nonsingular* if  $A^2 \neq 0$ . If  $I$  is nonsingular, then  $\mathcal{G}_n$  is said to be *nondegenerate*. This is equivalent to saying that the inner product  $a \cdot b$  defines a bilinear

form. We will be concerned only with nondegenerate algebras. If  $\mathcal{V}_n$  has signature  $(r, s)$  with  $r + s = n$  and  $s = 0$  indicating Euclidean signature, I will write  $\mathcal{V}_n = \mathcal{V}(r, s)$ . It follows that

$$II^\dagger = (-1)^s |I|^2, \quad (8)$$

where  $|I|^2$  is a positive scalar. This determines a unique inverse for  $I$  and enables us to define the duality automorphism of  $\mathcal{G}_n$ .

The *dual*  $\tilde{A}$  of an  $r$ -blade  $A$  is defined by

$$\tilde{A} = AI^{-1} = A \cdot I^{-1} = (-1)^{r(n-r)} I^{-1} A. \quad (9)$$

(This corrects an important misprint on p.7 of [8].) The inner and outer products are related by duality. Specifically, for step  $A = r$  and step  $B = s$ ,

$$A \cdot (BI) = (A \wedge B)I = (-1)^{s(n-s)} (AI) \cdot B \quad (10)$$

or equivalently,

$$(A \wedge B)^\sim = A \cdot \tilde{B} = (-1)^{s(n-s)} \tilde{A} \cdot B. \quad (11)$$

## 2. THE ALGEBRA OF SUBSPACES

To every  $r$ -dimensional subspace  $\mathcal{V}_r$  in  $\mathcal{V}_n$  there corresponds an  $r$ -blade  $A = \langle A \rangle_r$  such that  $\mathcal{V}_r$  is the solution set of the equation

$$x \wedge A = 0. \quad (12)$$

Let  $A$  be called the *blade* of  $\mathcal{V}_r$  while  $\mathcal{V}_r$  is called the *support* of  $A$ . Since  $A$  has a definite orientation and magnitude  $|A|$  (defined as in (8)), it associates an oriented weight or measure with  $\mathcal{V}_r$ . Thus, every blade in  $\mathcal{G}_n$  determines a unique weighted subspace of  $\mathcal{V}_n$ .

Let  $\mathcal{B}_n$  be the set of all blades in  $\mathcal{G}_n$  including the scalars as 0-blades. This set is closed under the inner and outer products (6) and (7). Therefore, under these operations  $\mathcal{B}_n$  can be regarded as an *algebra of weighted subspaces* in  $\mathcal{V}_n$ . Note that  $\mathcal{B}_n$  is not closed under addition, though some of its subsets are. With addition all of  $\mathcal{G}_n$  can be generated from  $\mathcal{B}_n$ . The algebra  $\mathcal{B}_n$  is metrical, so its structure depends on the signature of  $\mathcal{V}_n$ . However, it contains a nonmetrical subalgebra which is independent of signature.

For blades  $A$  and  $B$ , we define a new product  $A \vee B$  by

$$A \vee B = \tilde{A} \cdot B. \quad (13)$$

This is Grassmann's *regressive product*. With this definition, Grassmann's entire algebra of extension has been perfectly imbedded in geometric algebra. All the properties of the regressive product follow immediately from the properties of the inner product and duality. Thus, it can be shown that for step  $A + \text{step } B > n$  the regressive product is associative and is related to the outer product by the "De Morgan" rule

$$(A \vee B)^\sim = \tilde{A} \wedge \tilde{B}. \quad (14)$$

Moreover, these results are *independent of signature*, because the effect of duality in  $\tilde{A} \cdot B$  is to change the inner product to an outer product in accordance with (10). This fact is so important that it deserves further discussion.

Blades which are the duals of vectors are called *covectors* (or pseudovectors). Each covector  $A$  determines a hyperplane in  $\mathcal{V}_n$  according to the (nonmetrical!) equation (12). However, by introducing the vector  $a = \tilde{A}$ , the hyperplane equation can be put in the alternative form

$$a \cdot x = \tilde{A} \cdot x = (x \wedge A)I^{-1} = 0. \quad (15)$$

The inner product on the left side of this equation suggests a metrical dependence, but the right side of the equation shows otherwise. In fact, the inner product here plays the nonmetrical role of a *contraction*. Note that  $\omega(x) = a \cdot x = \tilde{A} \cdot x$  is a linear form on  $\mathcal{V}_n$ , and every linear form can be defined in this way. It has become standard practice in linear algebra recently to represent covectors by linear forms. This is the backward step which Grassmann alluded to earlier. The result is a loss of clarity as well as computational power. It should be clear that the whole theory of linear forms (including differential forms) is automatically included in the algebra  $\mathcal{B}_n$  ([6], [5]). When the regressive product is used always in place of the inner product, the resulting algebra,  $\mathcal{B}'_n$  say, is Grassmann's extension algebra. Of course the inner product can be recovered if the duality is introduced as an independent operation.

With the geometric product, meet and join, products of blades can be defined which correspond uniquely to standard subspace meet and join operations on their supports. When  $\text{step } A + \text{step } B > n$  the *meet* of  $A$  and  $B$  is given by  $A \vee B$ , and its support is the subspace meet (or intersection) of the supports of  $A$  and  $B$ . Similarly, the *join* is given by  $A \wedge B$  when  $A \wedge B \neq 0$ . These definitions of meet and join can be extended to apply to all cases [9], but we need not go into that.

The important point to be made here is that the *lattice* structure of subspaces under the meet and join operations [11] is faithfully represented in the Grassmann algebra of blades  $\mathcal{B}'_n$ . Of course, the blade algebra is a more powerful system than the subspace lattice, because it can represent subspace weights and orientations. However, if desired it can easily be reduced to a lattice in the following way. Let blades  $A$  and  $B$  be regarded as *projectively equivalent* if and only if  $A = \lambda B$  where  $\lambda$  is a nonzero scalar. Express this equivalence by writing

$$A \doteq B. \quad (16)$$

Then (6) and (14) give us

$$\begin{aligned} A \wedge B &\doteq B \wedge A, \\ A \vee B &\doteq B \vee A. \end{aligned} \quad (17)$$

Thus, under projective equivalence all information about signs (hence orientations) is lost, and the meet and join operations on blades are commutative just as they are on subsets. It is an easy (but worthwhile) exercise to determine what else is needed to *reduce* the blade algebra  $\mathcal{B}'_n$  to a lattice. This establishes what I believe is a very important connection between lattice theory and geometric algebra.

### 3. PROJECTIVE GEOMETRY

The power and universality of geometric algebra derives in part from the many different geometric interpretations which can be assigned to the algebra as a whole or to some substructure within it. We obtain a representation of projective geometry within geometric algebra by adopting the standard identification of points in projective space  $\mathcal{P}_{n-1}$  with rays in the vector space  $\mathcal{V}_n$ . Thus, vectors  $a$  and  $b$  represent the *same point* if and only if  $a \wedge b = 0$ , or equivalently  $a \doteq b$ .

In projective geometry the join of distinct points  $a$  and  $b$  is *line*  $A$ , as expressed by the equation

$$A = a \wedge b. \tag{18}$$

Similarly, the join of three distinct points  $a, b, c$  is a *plane*

$$J = a \wedge b \wedge c. \tag{19}$$

Strictly speaking the 2-blade  $A$  is a representation of the line determined by  $a$  and  $b$  rather than the line itself, which is a set of points. But, since  $A$  determines the line uniquely by (12), identification of  $A$  with the line it determines is a convenient locution.

Two distinct lines  $A$  and  $B$  intersect at a point if and only if  $A \wedge B = 0$ . In the projective space  $\mathcal{P}_2$ , equation (12) applies, as the point of intersection  $d$  is given by  $d = A \vee B$ . Therefore, the condition that lines  $A, B$  are concurrent with line  $C$  is

$$d \wedge C = (A \vee B) \wedge C = A \wedge (B \vee C) = 0. \tag{20}$$

These examples show how the nonmetrical *incidence relations* of projective geometry are represented by meet and join products in geometric algebra.

The representations are so simple because the primitives of synthetic geometry, such as point, line, plane, meet, and join have exact counterparts in the algebra. One of the essentials making this possible has been the coordinate-free development of the algebra. The great simplification this entails can be seen by comparing (18) with the ‘‘classical Plücker coordinates’’ for a line, which (18) yields when decomposed in a coordinate system [5].

To illustrate the use of geometric algebra for stating and proving theorems in projective geometry, consider the famous theorem of Desargues, which concerns the configuration of lines and points shown in Fig. 1. Given the two triangles as shown in the figure, corresponding vertices of the triangles determine the three lines

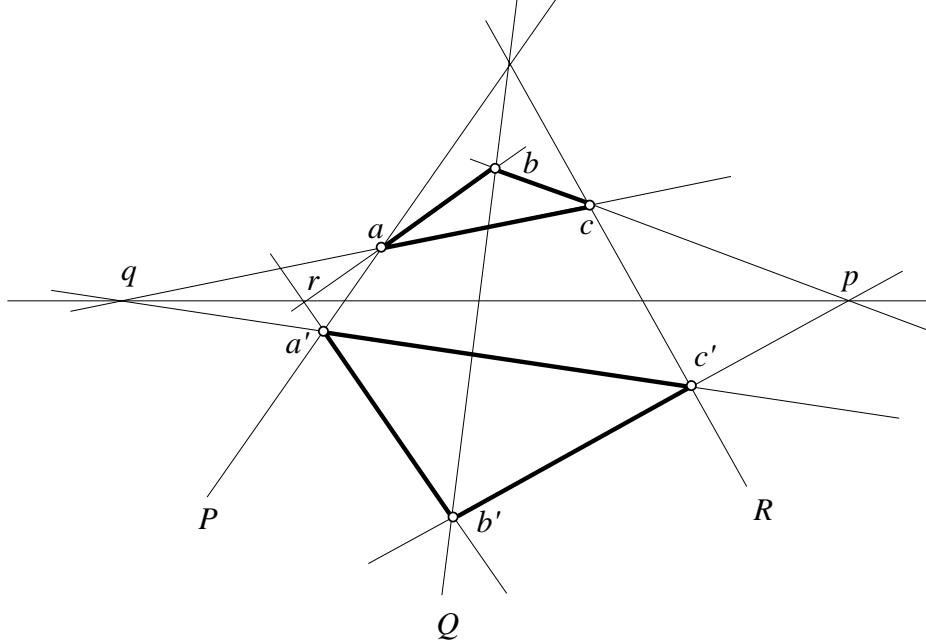
$$P = a \wedge a', \quad Q = b \wedge b', \quad R = c \wedge c', \tag{21}$$

while corresponding sides intersect in the three points

$$\begin{aligned} p &= (b \wedge c) \vee (b' \wedge c'), \\ q &= (c \wedge a) \vee (c' \wedge a'), \\ r &= (a \wedge b) \vee (a' \wedge b'). \end{aligned} \tag{22}$$

Geometric algebra can be used to derive from this what may fairly be called *Desargues identity* [9]:

$$p \wedge q \wedge r = (JJ')^2 (P \vee Q) \wedge R \tag{23}$$



where  $J = a \wedge b \wedge c$  and  $J' = a' \wedge b' \wedge c'$ .

*Desargues theorem* simply says that the left side of (23) vanishes if and only if the right side vanishes; in synthetic terms, *the points  $p, q, r$  are collinear if and only if the lines  $P, Q, R$  are concurrent*. Of course, Desargues' theorem could have been derived from the weaker identity  $p \wedge q \wedge r \doteq (P \vee Q) \wedge R$ , since the factor  $(JJ')^2$  in (23) is just a scalar. The identity (23) transcends projective geometry and applies equally well in metrical geometry, whatever the signature.

I have employed the geometric algebra  $\mathcal{G}_3$  (with Euclidean signature) to derive Desargues identity [9], and it applies as well to the formulation and proof of all theorems about the projective plane  $\mathcal{P}_2$ . Exactly the same algebra is employed as the language for classical mechanics in [7]. Therefore the identity of Desargues must be expressible as an identity in the standard vector algebra of Gibbs. In the interest of revealing the common structure of widely separated subjects, let me show how that can be done. As shown in [7], the vector cross product  $a \times b$  is the dual of  $a \wedge b$  in  $\mathcal{G}_3$ . Therefore

$$a \times b = (a \wedge b)I^{-1} = (a \wedge b)^\sim = \tilde{a} \vee \tilde{b}. \quad (24)$$

This gives us new insight into the cross product. It shows that the cross product, which is peculiar to three dimensions, is a special case of the meet, which applies to spaces of any dimension. It also supplies a geometrical interpretation of  $a \times b$  as the intersection of planes in  $\mathcal{V}_3$  (or lines in  $\mathcal{P}_2$ ). The various factors in Desargues identity (23) have the following expressions in terms of the cross product

$$\begin{aligned} p \wedge q \wedge r &= (p \cdot q \times r)I \\ &= [(b \times c) \times (b' \times c')] \cdot [(c \times a) \times (c' \times a')] \times [(a \times b) \times (a' \times b')] I, \\ (P \vee Q) \wedge R &= (a \times a') \cdot (b \times b') \times (c \times c') I, \\ (JJ')^2 &= (a \cdot b \times c)^2 (a' \cdot b' \times c')^2. \end{aligned} \quad (25)$$

When these are substituted into (23) one gets a much more complicated identity than is ordinarily considered in vector calculus. The marvel is the simple geometry underlying this complexity. This illustrates the importance of developing diagrammatic techniques for interpreting algebraic relations, an underdeveloped subject. By the way, if the vectors in the two triangles are all taken to be unit vectors, then (23) becomes a relation between spherical triangles. Spherical trigonometry is developed in terms of  $\mathcal{G}_3$  in appendix A of [7].

Many other theorems of projective geometry are reduced to algebraic identities in [9], and the theory of poles and polars is reduced to algebraic duality.

The whole theory of projective geometry is ripe to be reworked in the language of geometric algebra to integrate it fully with the rest of mathematics, which could profit by the insights it provides into mathematical structure. This is a large task which someone should pick up and pursue vigorously. I am sure that it will produce new mathematical insights as well as recover many old insights which are unknown to most mathematicians today.

#### 4. LINEAR ALGEBRA

Linear algebra grew up with projective geometry. But conventional formulations do not do justice to the fundamental concepts of meet, join and duality in projective geometry. This defect has been corrected in [5] by introducing geometric algebra into the foundations of linear algebra. Here we note the projective interpretation of key ideas and results in [5].

The main idea is to introduce the concept of outermorphism as a fundamental concept of linear algebra. An *outermorphism* is the natural extension of a linear transformation  $f$  on  $\mathcal{V}_n$  to a linear transformation  $\underline{f}$  on  $\mathcal{G}_n$ , defined as follows for the  $k$ -blade of (4),

$$\underline{f}(a_1, a_2, \dots, a_k) = (fa_1) \wedge (fa_2) \wedge \dots \wedge (fa_k). \quad (26)$$

Consequently, the outer product of blades is preserved, as expressed by

$$\underline{f}(A \wedge B) = (\underline{f}A) \wedge (\underline{f}B). \quad (27)$$

This implies also that the step of every blade is an invariant of the outermorphism. Of course, the outermorphism merely makes explicit a concept which is inherent in linear algebra. For example, the concept of determinant derives from the outermorphism of the pseudoscalar  $I$ , as expressed by

$$\underline{f}I = (\det f) I. \quad (28)$$

The linear transformations with nonvanishing determinant can all be interpreted projectively as *collineations*. The outermorphism is precisely what is needed to prove that they map points into points, lines into lines, planes into planes, etc. The result follows from the fact that the form of equation (12) is an invariant of the mapping. Indeed, (4) maps into

$$x' \wedge A' = (\underline{f}x) \wedge (\underline{f}A) = 0 \quad (29)$$

and  $\det f \neq 0$  implies that  $A' = \underline{f}A \neq 0$  whenever  $A \neq 0$ .



Every linear transformation  $f$  has an *adjoint* (or transpose) transformation  $\bar{f}$ , and this can be extended to any outermorphism denoted by  $\underline{f}$  also. The adjoint transformation can be defined directly in terms of  $\underline{f}$  by

$$\langle M \bar{f} N \rangle_0 = \langle (\bar{f} N) M \rangle_0 = \langle N \underline{f} M \rangle_0, \quad (30)$$

assumed to hold for all  $M, N$  in  $\mathcal{G}_n$ .

In, contrast to the outer product, the inner product is not generally preserved by an outermorphism. Rather, it obeys the fundamental transformation law

$$A \cdot (\bar{f} B) = \bar{f}[(\underline{f} A) \cdot B], \quad (31)$$

which holds for (step  $A$ )  $\leq$  step  $B$  and for interchange of  $\underline{f}$  and  $\bar{f}$ . When I first encountered this law in [5] I was greatly puzzled as to its geometric meaning and import. Its importance is evident in applying it to the special case  $B = I$ , which produces immediately an explicit expression for the inverse outermorphism

$$\underline{f}^{-1} A = \frac{\bar{f}(AI)I^{-1}}{\det f}. \quad (32)$$

Note how clearly this shows the essential roles of the adjoint and double duality in computing the inverse of a linear transformation.

The full meaning of (31) did not dawn on me until I derived in [10] the following transformation law for the meet

$$(\underline{f} A) \vee (\underline{f} B) = (\det f) \underline{f}(A \vee B). \quad (33a)$$

The factor  $(\det f)$  can be removed by defining duality on the left with respect to the transformed pseudoscalar  $I' = \underline{f} I = (\det f) I$ . Then (32) implies that the transformation  $A' = \underline{f} A$  entails the induced transformation  $\tilde{A} = \bar{f} \tilde{A}'$  on the dual  $\tilde{A}' = A'(I')^{-1}$ , and (33a) can be put in the form

$$\underline{f}[\tilde{A} \cdot B] = \tilde{A}' \cdot B'. \quad (33b)$$

This transformation law is mathematically equivalent to (31) for  $\det f \neq 0$ , but, in contrast, its geometric meaning is transparent. It tells us that the “incidence properties” in projective geometry are invariant under collineations, or alternatively, that the “subspace intersection property is preserved by nonsingular linear transformations. I think (33a,b) should be counted as one of the fundamental results of linear algebra, but, as far as I know, this is the first time it has been published, though I suppose an equivalent result must lie buried in the works of the old masters of projective geometry.

## 5. PROJECTIVE SPLIT AND CROSS RATIO

The points of projective  $n$ -space  $\mathcal{P}_n$  can be represented as rays in  $\mathcal{V}_{n+1}$ , as we have already done, or as vectors in  $\mathcal{V}_n$ . The change in representation from  $\mathcal{V}_n$  to  $\mathcal{V}_{n+1}$  is accomplished by introducing *homogeneous coordinates*. I aim to show now that geometric algebra enables

us to express this relation between  $\mathcal{V}_n$  and  $\mathcal{V}_{n+1}$  in an algebraic coordinate-free form which is intimately related to the formal structure of geometric algebra.

Let  $x$  and  $\mathbf{e}_0$  be vectors in  $\mathcal{V}_{n+1}$  with  $\mathbf{e}_0^2 \neq 0$ . For fixed  $\mathbf{e}_0$  and variable  $x$ , define  $\mathcal{V}_n$  as the set

$$\mathcal{V}_n = \{x \wedge \mathbf{e}_0\}. \quad (34)$$

It is readily verified that  $\mathcal{V}_n$  is a vector space under the geometric product, and the algebra  $\mathcal{G}_n = \mathcal{G}(\mathcal{V}_n)$  that it generates is just the *even subalgebra* of  $\mathcal{G}_{n+1} = \mathcal{G}(\mathcal{V}_{n+1})$ , as expressed by

$$\mathcal{G}_n = \mathcal{G}_{n+1}^+. \quad (35)$$

From (34) we see that  $\mathcal{V}_n$  can be interpreted projectively as *the pencil of all lines through the point  $\mathbf{e}_0$* . This provides projective interpretation for the relation of a geometric algebra to its even subalgebra.

The function  $x \wedge \mathbf{e}_0$  is a linear mapping of  $\mathcal{V}_{n+1}$  into  $\mathcal{V}_n$ . The projective mapping relating each ray  $\{\lambda x\}$  in  $\mathcal{V}_{n+1}$  to a unique vector  $\mathbf{x}$  in  $\mathcal{V}_n$  is defined by the following relation

$$x\mathbf{e}_0 = x \cdot \mathbf{e}_0 + x \wedge \mathbf{e}_0 = x_0(1 + \mathbf{x}), \quad (36)$$

where  $x_0 = x \cdot \mathbf{e}_0$  so  $\mathbf{x} = x \wedge \mathbf{e}_0 / x \cdot \mathbf{e}_0$ .

Vectors in  $\mathcal{V}_n$  are denoted in boldface to distinguish them from vectors in  $\mathcal{V}_{n+1}$ . I call the relation of  $\mathcal{V}_{n+1}$  to  $\mathcal{V}_n$  determined by (36) a *projective split* of  $\mathcal{V}_n$ . The same kind of relation is inherent in physics when spacetime is split into space and time components, but it was first formulated explicitly in [4]. I call it the *spacetime split*. Another important application of the projective split is to ‘‘Clifford Analysis’’ [12]. If  $\mathcal{V}_{n+1}$  has Euclidean signature, then the projective split implies that  $\mathcal{V}_n$  has anti-Euclidean signature. This is precisely the relation among variables in the seemingly different formulation of Clifford analysis in [5] and [12].

To see how the representation for a line in  $\mathcal{P}_n$  by a 2-blade  $a \wedge b$  in  $\mathcal{G}_{n+1}$ , is related to the representation in  $\mathcal{G}_n$ , we use the splits  $a\mathbf{e}_0 = a_0(1 + \mathbf{a})$  and  $b\mathbf{e}_0 = b_0(1 + \mathbf{b})$  to derive (for  $\mathbf{e}_0^2 = 1$ )

$$a \wedge b = a_0 b_0 (\mathbf{a} - \mathbf{b} + \mathbf{b} \wedge \mathbf{a}) = a_0 b_0 (\mathbf{u} + \mathbf{a} \wedge \mathbf{u}). \quad (37)$$

This represents a line with direction  $\mathbf{u} = \mathbf{a} - \mathbf{b}$  and moment  $\mathbf{M} = \mathbf{a} \wedge \mathbf{u} = \mathbf{b} \wedge \mathbf{a}$  passing through the point  $\mathbf{a}$  in  $\mathcal{V}_n$ . This interpretation can be verified by performing a similar split on the equation  $x \wedge (a \wedge b)$  for the line, with the result

$$x \wedge a \wedge b = x_0 a_0 b_0 [(\mathbf{a} - \mathbf{x}) \wedge \mathbf{u} + \mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u}] \mathbf{e}_0 = 0. \quad (38)$$

Thus in  $\mathcal{G}_n$  the equation for the line is  $(\mathbf{x} - \mathbf{a}) \wedge \mathbf{u} = 0$ , and  $\mathbf{x} \wedge \mathbf{a} \wedge \mathbf{u} = 0$  is automatically satisfied. The ‘‘Plücker line coordinates’’  $a \wedge b = \mathbf{u} + \mathbf{M}$  are applied to mechanics in Sec. 7–1 of [7].

For three distinct points  $a, b, c$  on the same line, the split (37) yields the invariant ratio

$$\frac{a \wedge c}{b \wedge c} = \frac{a_0}{b_0} \left[ \frac{\mathbf{a} - \mathbf{c}}{\mathbf{b} - \mathbf{c}} \right]. \quad (39)$$

This is a projective invariant in two senses: it is independent of the chosen split vector  $\mathbf{e}_0$ , and it is invariant under collineations. The internal ratio  $(\mathbf{a} - \mathbf{b})/(\mathbf{b} - \mathbf{c})$  is not a projective

invariant. However, the invariant *cross ratio* for four distinct points  $a, b, c, d$  on the same line is given by

$$\left[ \frac{a \wedge c}{b \wedge c} \right] \left[ \frac{b \wedge d}{a \wedge d} \right] = \left[ \frac{\mathbf{a} - \mathbf{c}}{\mathbf{b} - \mathbf{c}} \right] \left[ \frac{\mathbf{b} - \mathbf{d}}{\mathbf{a} - \mathbf{d}} \right]. \quad (40)$$

All the well-known properties of the *cross ratio* are easily derived from this. The considerable advantage of using geometric algebra here should be obvious. It insures, for example, that division in (40) is well-defined without abuse of notation.

## 6. CONFORMAL SPLIT AND CONFORMAL GROUP

Geometric algebras can be factored multiplicatively according to the fundamental theorem

$$\mathcal{G}_{n+2} = \mathcal{G}_n \otimes \mathcal{G}_2. \quad (41)$$

The Kronecker product  $\otimes$  is used here to emphasize that all elements of  $\mathcal{G}_2$  commute with all elements  $\mathcal{G}_n$ , but the product is actually the geometric product defined for  $\mathcal{G}_{n+2}$ . I believe that this theorem has not been exploited to its fullest in applications of Clifford algebra because it has been lacking a geometric interpretation. The fact that the projective split by a vector relates  $\mathcal{G}_{n+1}$  to  $\mathcal{G}_n$  suggests that (41) might be obtained from a similar split by a 2-blade and so given a similar geometric interpretation. Accordingly, by analogy with (36), we select a nonsingular 2-blade  $\mathbf{e}_0$  in  $\mathcal{G}_{n+2}$  and for every  $x$  in  $\mathcal{V}_{n+2}$  we perform the split

$$x\mathbf{e}_0 = x \cdot \mathbf{e}_0 + x \wedge \mathbf{e}_0 = x_0 + \rho \mathbf{x} \quad (42)$$

and we define the vector spaces

$$\mathcal{V}_2 = \{x_0 = x \cdot \mathbf{e}_0 = -\mathbf{e}_0 \cdot x\}, \quad (43)$$

$$\mathcal{V}_n = \{\rho \mathbf{x} = x \wedge \mathbf{e}_0 = \mathbf{e}_0 \wedge x\}, \quad (44)$$

where the boldface  $\mathbf{x}$  distinguishes a vector in  $\mathcal{V}_n$  from the corresponding vector  $x$  in  $\mathcal{V}_{n+2}$  and  $\rho$  is a scale factor to be determined.

Note that elements of  $\mathcal{V}_2$  do indeed commute with the elements of  $\mathcal{V}_n$  so the geometric algebras they generate satisfy (41) as desired. Also,  $\mathcal{V}_2$  is a 2-dimensional subspace of  $\mathcal{V}_{n+2}$  determined by  $\mathbf{e}_0$ , whereas  $\mathcal{V}_n$  is the subspace of 3-blades in  $\mathcal{G}_{n+2}$  with  $\mathbf{e}_0$  as a common factor. Projectively, the rays of  $\mathcal{V}_2$  are points on the line  $\mathbf{e}_0$ , while  $\mathcal{V}_n$  is the pencil of all planes passing through the line  $\mathbf{e}_0$ . Thus, we have a geometric interpretation for the split.

To have a definite relation between points in  $\mathcal{V}_n$  and  $\mathcal{V}_{n+2}$  we need to impose restrictions on the two extra degrees of freedom in  $\mathcal{V}_2$ . There is a natural way to do this if the signature of  $\mathcal{V}_2$  is taken to be (1,1), as expressed by  $\mathcal{V}_2 = \mathcal{V}(1,1)$ . Our considerations will hold for any signature  $(r,s)$  of  $\mathcal{V}_n = \mathcal{V}(r,s)$ , in which case  $\mathcal{V}_{n+2} = \mathcal{V}(r+1, s+1)$ . With this proviso, I call the 2-blade split (42) a *conformal split*, because of its relation to the conformal group on  $\mathcal{V}_n$  given below.

Because of its indefinite signature,  $\mathcal{V}_2$  has a unique basis  $\mathbf{e}_+, \mathbf{e}_-$  of singular vectors:  $\mathbf{e}_+^2 = \mathbf{e}_-^2 = 0$ . It is convenient to normalize them by

$$\frac{1}{2} \mathbf{e}_+ \mathbf{e}_- = 1 + \mathbf{e}_0, \quad \text{with} \quad \mathbf{e}_0^2 = 1. \quad (45)$$

Now the value of  $x_0 = x \cdot \mathbf{e}_0$  can be related to  $\mathbf{x}$  in (42) by imposing the null condition  $x^2 = 0$ . As a consequence, (42) can be put in the form

$$2x\mathbf{e}_0 = x \cdot \mathbf{e}_+(\mathbf{e}_- + \mathbf{x}^2\mathbf{e}_+ + 2\mathbf{x}). \quad (46)$$

Note that the functional form of the scale factor  $\rho = x \cdot \mathbf{e}_+$  in (42) is now determined and can be seen to play the role of a homogeneous coordinate. Thus, by (46) every vector  $x$  in  $\mathcal{V}_n$  corresponds to a unique ray in the null cone  $x^2 = 0$  of  $\mathcal{V}_{n+2}$ . This might be the ideal representation for a systematic treatment of the projective theory of quadratic forms, but that remains to be seen.

The quadratic equation  $x^2 = 0$  is an invariant of the orthogonal group  $O(r+1, s+1)$  on  $\mathcal{V}_{n+2}$ . Each orthogonal transformation can be put in the canonical form

$$x' = Gx(G^{-1})^*, \quad (47)$$

where  $G$  is a versor in  $\mathcal{G}_{n+2}$  and  $G^*$  is the involute of  $G$ . A versor  $G$  is a multivector which can be expressed as the geometric product

$$G = \pm a_1 a_2 \dots a_k \quad (48)$$

of nonsingular vectors. The versor  $G$  is said to have *odd parity* if  $G^* = (-1)^k G = -G$  or *even parity* if  $G^* = G$ . The versors form a multiplicative group  $\text{Pin}(r+1, s+1)$  called the *pin group* of  $\mathcal{V}_{n+2}$ . The versors of even parity form a subgroup  $\text{Spin}(r+1, s+1)$  called the *spin group* of  $\mathcal{V}_{n+2}$ . The transformation (47) is called a *rotation* if  $G$  has even parity. The group of rotations is the *special orthogonal group*  $\text{SO}(r+1, s+1)$ .

By virtue of the constraint (46), the orthogonal transformation  $x \rightarrow x'$  defined by (47) induces a transformation  $\mathbf{x} \rightarrow \mathbf{x}' = g(\mathbf{x})$  of  $\mathcal{V}_n$ . An explicit relation between  $G$  and  $g$  is obtained by combining (46) and (47):

$$G(\mathbf{e}_- + \mathbf{x}^2\mathbf{e}_+ + 2\mathbf{x})\widehat{G} = \sigma_g[\mathbf{e}_- + [g(\mathbf{x})]^2\mathbf{e}_+ + 2g(\mathbf{x})], \quad (49)$$

where  $\widehat{G} = \mathbf{e}_0(G^{-1})^*\mathbf{e}_0$  and  $\sigma_g = \sigma_g(\mathbf{x}) = x' \cdot \mathbf{e}_+ / x \cdot \mathbf{e}_+$ . This equation can be solved for  $g$  as an explicit function of  $G$  if  $G$  is expressed in a form which explicitly distinguishes its parts in  $\mathcal{G}_2$  and  $\mathcal{G}_n$ . The most suitable generic form seems to be

$$G = \frac{1}{2}[A(1 + \mathbf{e}_0) + B\mathbf{e}_+ + C\mathbf{e}_- + D(1 - \mathbf{e}_0)], \quad (50)$$

where  $A, B, C, D$  are versors in  $\mathcal{G}_n$  and so commute with  $\mathbf{e}_+, \mathbf{e}_-, \mathbf{e}_0$ . From (50) it is evident that  $A$  and  $D$  must have the same parity while  $C$  and  $B$  have the opposite parity. Moreover, group properties  $GG^{-1} = 1$  and (47) imply that

$$A\widetilde{D} - B\widetilde{C} = \pm 1 \quad (51)$$

and  $\widetilde{A}B, \widetilde{A}C, D\widetilde{B}, D\widetilde{C}$  are in  $\mathcal{V}_n$ , where  $\widetilde{A} = (A^*)^\dagger$  is the reverse in  $\mathcal{G}_n$ . Aside from these restrictions the versors are arbitrary.

The solution of (49) for  $g$  is most simply and elegantly expressed as a *homographic* transformation:

$$g(\mathbf{x}) = (A\mathbf{x} + B)(C\mathbf{x} + D)^{-1}. \quad (52)$$

It can be verified that  $g$  is a conformal transformation or  $\mathcal{V}_n = \mathcal{V}(r, s)$  and, for the range of coefficients allowed by (51), gives the entire *conformal group*  $\mathcal{C}(r, s)$ . Thus we have established the 2 to 1 homomorphism between  $\text{Pin}(r+1, s+1)$  and  $\mathcal{C}(r, s)$ . The correspondence between elements in the two groups is given by (50) and (52). A representation for  $G$  which is less cumbersome than (50) can be obtained by adopting a  $2 \times 2$  matrix representation for  $\mathbf{e}_\mu$  which puts (50) in the matrix form used by Ahlfors [13]:

$$[g] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (53)$$

This has the advantage that group composition can be carried out by matrix multiplication. It has the disadvantage of hiding the dependence on  $\mathbf{e}_+$ ,  $\mathbf{e}_-$  and  $\mathbf{e}_0$ , along with the geometric structure that entails. That is evident in Table 1, which displays the elementary group elements from which all other group elements can be generated.

The versors with even parity in Table 1 generate the group  $\text{Spin}(r+1, s+1)$ . Note that  $\mathbf{e}_0$  is the only one of those elements which is not continuously connected to the identity. With  $\mathbf{e}_0$  excluded  $\text{Spin}(r+1, s+1)$  is a spin representation of the *special conformal group*  $\text{SC}(r, s)$ . Every element in the group can be represented in the canonical form  $G = \pm K_{\mathbf{a}} T_{\mathbf{b}} D_{\lambda} R$  where  $R = R^*$ . This has the matrix representation

$$[K_{\mathbf{a}} T_{\mathbf{b}} D_{\lambda} R] = \begin{bmatrix} \lambda R & \mathbf{b} \lambda^{-1} R \\ \mathbf{a} \lambda R & (1 + \mathbf{a} \mathbf{b}) \lambda^{-1} R \end{bmatrix}, \quad (54)$$

which relates it directly to  $g$  by (52). In the pin group, the transversion can be reduced to a composite of two inversions by  $K_{\mathbf{b}} = \mathbf{e}_1 T_{\mathbf{b}} \mathbf{e}_1$ .

Mathematicians have considered the spin and pin representations of the conformal group before by several different approaches ([13], [14], [15]). I believe that the present approach has the advantage of greater geometrical perspicuity, bringing together the advantages of the previous approaches by showing how they are related [10]. Spin representations of the conformal group for spacetime have been considered by physicists in many articles (for example, [16]). I hope the present formulation will clarify and facilitate applications of the conformal group in the future.

**Table 1.** Correspondence between generating elements of  $\text{Pin}(r + l, s + 1)$  and the full conformal group  $\mathcal{C}(r, s)$ .

<u>Type</u>			<u>Group element</u>
conformal	$g(\mathbf{x})$	$\iff$	$\pm G$
orthogonal	$R\mathbf{x}R^{-1}$		$R = \mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_k$
translation	$\mathbf{x} + \mathbf{a}$		$T_{\mathbf{a}} = 1 + \frac{1}{2}\mathbf{a}\mathbf{e}_+$
dilation	$\lambda^2\mathbf{x}$		$D_\lambda = e^{\varphi\mathbf{e}_0}, \quad \lambda = e^\varphi$
transversion	$\frac{\mathbf{x} + \mathbf{b}\mathbf{x}^2}{1 + \mathbf{b} \cdot \mathbf{x} + \mathbf{b}^2\mathbf{x}^2}$		$K_{\mathbf{b}} = 1 + \frac{1}{2}\mathbf{b}\mathbf{e}_-$
inversion	$\mathbf{x}^{-1}$		$\mathbf{e}_1 = \frac{1}{2}(\mathbf{e}_+ + \mathbf{e}_-)$
involution	$-\mathbf{x}$		$\mathbf{e}_0$

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## REFERENCES

- [1] A. N. Whitehead, *A Treatise on Universal Algebra with Applications*, Cambridge Press, Cambridge (1898). [Reprint: Hafner, New York (1960)].
- [2] P. Doubilet, G.-C. Rota and J. Stein, *On the Foundations of Combinatorial Theory: IX. Combinatorial Methods of Invariant Theory*. Stud. Appl. Math. **53** (1974) 185–216.
- [3] M. Barnabei, A. Brini and G.-C. Rota, *On the Exterior Calculus of Invariant Theory*, J. Algebra **96** (1985) 120-160.
- [4] D. Hestenes, *Spacetime Algebra*, Gordon and Breach, New York (1966).
- [5] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, D. Reidel Publ. Co., Dordrecht/Boston (1984).
- [6] D. Hestenes, *A Unified Language for Mathematics and Physics*. In: J. S. R. Chisholm and A. K. Common (eds.), *Clifford Algebras and their Applications in Mathematical Physics*, D. Reidel Publ. Co., Dordrecht/Boston (1986), pp. 1-23.
- [7] D. Hestenes, *New Foundations for Classical Mechanics*, D. Reidel Publ. Co., Dordrecht/Boston (1986).
- [8] D. Hestenes, *New Foundations for Mathematical Physics*, Kluwer, Dordrecht/Boston (in preparation).
- [9] D. Hestenes and R. Ziegler, *Projective Geometry with Clifford Algebra*, (submitted).

- [10] D. Hestenes, *The Design of Linear Algebra and Geometry* (submitted).
- [11] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloquium Publ. **25**, revised edition, New York (1948).
- [12] F. Brackx, R. Delange, F. Sommen, *Clifford Analysis*, Pitman, London (1982).
- [13] L. V. Ahlfors, *Clifford Numbers and Möbius Transformations in  $R^n$* . In *Clifford Algebras and their Applications in Mathematical Physics*, J. S. R. Chisholm and A. K. Common (eds.), D. Reidel Publ. Co., Dordrecht/Boston (1986).
- [14] P. Angles, *Construction de revêtements du groupe conforme d'un espace vectoriel muni d'une métrique de type  $(\Gamma, q)$* , Ann. Inst. Henri Poincaré, **33** (1980) 33.
- [15] P. Lounesto and E. Latvamaa, *Conformal Transformations and Clifford Algebras*, Proc. Am. Math. Soc. **79** (1980) 533.
- [16] W. A. Hepner, *The Inhomogeneous Lorentz Group and the Conformal Group*, Nuovo Cimento **26** (1962) 351–367.