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Spinor Approach to Gravitational Motion and Precession

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The translational and rotational equations of motion for a small rigid body in a gravitational field are combined in a single spinor equation. Besides its computational advantages, this unifies the description of gravitational interaction in classical and quantum theory. Explicit expressions for gravitational precession rates are derived.

1. INTRODUCTION

This article applies the *method of mobile* to describe the motion of a small rigid body in a gravitational field. Emphasis is laid on computing the gravitational precession of a gyroscope, since that is a case of major experimental interest. Aside from its perspicuity and efficiency, the method of mobiles has the great virtue of unifying the description of gravitational interactions in classical and quantum theory. This is demonstrated at the end of the paper, where gravitational interactions in the Dirac theory are determined.

A spinor form for the method of mobiles was developed in Hestenes (1974a, b), where it was used to describe the motion of charged particles, including electron spin precession in electromagnetic fields. The method was generalized to describe parallel transport along geodesics in Hestenes and Sobczyk (1984). Here I apply it specifically to the description of gravitational precession. I will freely employ the notation and results of the preceding paper (Hestenes, 1986), and begin with a brief review of concepts and results from Hestenes (1974a) and Hestenes and Sobczyk (1984).

2. THE MOBILE EQUATIONS

For present purposes, a *mobile* is a comoving orthonormal frame on a timelike curve. The motion of a rigid body of negligible dimensions is represented by a mobile $\{e_\mu = e_\mu(x(\tau)) = e_\mu(\tau)\}$. The history of the mobile is a timelike curve $x = x(\tau)$ parametrized by proper time τ. The (proper) velocity of the mobile is the vector

$$e_0 = v = dx/d\tau \tag{1}$$

The triad $\{e_k,\ k = 1, 2, 3\}$ describes the attitude of the body, with e_3 taken to be the spin axis. So the internal angular momentum or spin of the body is given by

$$s = |\,s\,| e_3 \tag{2}$$

where $|\,s\,|$ is the magnitude of the spin.

At each point of its history, the mobile $\{e_\mu\}$ is related to a given fiducial frame $\{\gamma_\mu\}$ by a Lorentz transformation

$$e_\mu = R\gamma_\mu \widetilde{R} \tag{3}$$

where the spinor R is an even multivector satisfying

$$\widetilde{R}R = 1 \tag{4}$$

Thus, $R = R(x(\tau))$ is a "spin representation" of the Lorentz transformation rotating frame γ_μ into e_μ at each point on the mobile history.

The mobile equations of motion are

$$\delta_v e_\mu = d_v e_\mu + \omega_v \cdot e_\mu = \Omega \cdot e_\mu \tag{5}$$

where the codifferential

$$\delta_v = v \cdot \Box = v^\mu \Box_\mu = \delta/\delta\tau \tag{6}$$

is equivalent to the coderivative with respect to proper time, and

$$d/d\tau \equiv d_v = v^\mu d_\mu \tag{7}$$

defines the fiducial derivative with respect to proper time as equivalent to the fiducial differential. From Hestenes (1986) we know that the fiducial differential has the property

$$d_v \gamma_\mu = 0 \tag{8}$$

Hence,

$$\frac{dv}{d\tau} = \left(\frac{dv^\mu}{d\tau}\right)\gamma_\mu \tag{9}$$

where $v^\mu = v \cdot \gamma^\mu$. The translational equation of motion for the mobile is the equation (5) for $e_0 = v$, which we can write in the form

$$dv/d\tau = (\Omega - \omega_v) \cdot v \tag{10}$$

The single rotor equation

$$dR/d\tau = \tfrac{1}{2}(\Omega - \omega_v)R \tag{11}$$

is equivalent to the four equations of motion (5), as is easily shown by differentiating (3). This single equation describes spin precession of the rigid body well as motion along its history.

The angular velocity of the mobile is a function of the physical forces acting on it. Gravitational forces are described by the bivector ω_v, while nongravitational forces are described by specifying Ω. It seems to be generally true that independent forces make independent contributions to the angular velocity of a mobile. This generalizes the *principle of superposition of forces* familiar from classical mechanics. It greatly simplifies the analysis of complex problems by allowing us to determine independently the angular velocities due to different forces and add the results.

As explained in Hestenes (1974b), for a spinning particle (such as an electron) with mass m, charge e, and gyromagnetic ratio $g = 2$ in an external electromagnetic field F, the bivector Ω in (11) is given by

$$\Omega = (e/m)F \tag{12}$$

Hestenes (1974b) finds the exact flat-space ($\omega_v = 0$) solution of equation (11) for an arbitrary combination of uniform electric and magnetic fields, for a plane wave field and for the Coulomb field.

3. SPACE-TIME SPLIT

To facilitate physical interpretation and comparison with the results of others, we need to express our results in terms of *relative variables* with respect to a specified reference frame. This is best done by following the general method laid out in Hestenes (1974a). At every spacetime point x the fiducial timelike vector $\gamma_0 = \gamma_0(x)$ determines an instantaneous rest frame, that is, a local *split* of spacetime into space and time. The proper velocity is *split* into space and time components by

$$v\gamma_0 = \beta(1 + \mathbf{v}) \tag{13}$$

where

$$\beta \equiv v \cdot \gamma_0 = (1 - \mathbf{v}^2)^{1/2} \tag{14}$$

and

$$\mathbf{v} = \frac{v \wedge \gamma_0}{v \cdot \gamma_0} \frac{d\mathbf{x}}{dt} = \left(\frac{dx^k}{dt}\right) \gamma_k\gamma_0 \tag{15}$$

is the *relative velocity* in the fiducial reference system.

The timelike bivectors

$$\boldsymbol{\sigma}_k \equiv \gamma_k\gamma_0 = \gamma_0\gamma^k \tag{16}$$

(for $k = 1, 2, 3$) compose a basis for the relative vectors in the fiducial reference system. The pseudoscalar for the fiducial frame is $i = \gamma_0\gamma_1\gamma_2\gamma_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$

A proper bivector F, such as the electromagnetic field, can be split into a part that anticommutes with γ_0 and a part that commutes. The anticommuting part is a relative vector

$$\mathbf{E} \equiv \tfrac{1}{2}(F - \gamma_0 F \gamma_0) \tag{17a}$$

while the commuting part is a relative bivector

$$i\mathbf{B} \equiv \tfrac{1}{2}(F + \gamma_0 F \gamma_0) \tag{17b}$$

Thus, the spacetime split of F is expressed by writing

$$F = \mathbf{E} + i\mathbf{B} \tag{18}$$

Similarly, the "total angular velocity" in equations (5) and (11) can be written in the split form

$$\Omega - \omega_v = \mathbf{e} + i\mathbf{b} \tag{19}$$

which defines relative vectors $\mathbf{e}$ and $\mathbf{b}$. With this definition, a split of the proper equation of motion (10) yields a relative equation of motion in the familiar "Lorentz form"

$$d\mathbf{v}/dt = \mathbf{e} + \mathbf{v} \times \mathbf{b} \tag{20}$$

In Hestenes (1974b) it can be seen that the proper equation of motion (10) is often easier to solve than the relative equation (20).

A big advantage of the space-time split is that it enables us to separate the description of spin precession from the description of translational motion. This is accomplished by a split of the spinor R into the product of spinors

$$R = LU \tag{21}$$

where

$$L\gamma_0\widetilde{L} = L^2\gamma_0 = v \tag{22}$$

and

$$U\gamma_0\widetilde{U} = \gamma_0 \tag{23}$$

The split is uniquely determined by γ_0, and it determines a unique factorization of the Lorentz transformation (3) into a spatial rotation in the fiducial frame specified by U followed by a boost specified by L.

In the instantaneous rest frame, the axes of the rigid body may be represented by the relative vectors

$$\mathbf{e}_k = U\boldsymbol{\sigma}_k\widetilde{U} \tag{24}$$

The rotational motion of the body can be described by three vector equations

$$d\mathbf{e}_k/dt = \Omega \times \mathbf{e}_k \tag{25}$$

or equivalently, by a single spinor equation

$$dU/dt = \tfrac{1}{2}i\Omega U \tag{26}$$

The quantity of greatest interest here is Ω, the angular velocity of the rigid body relative to the fiducial standard. The algebraic problem of solving for Ω in terms of $\mathbf{e}$ and $\mathbf{b}$ defined by (19) is solved in Hestenes (1974a), though the calculations given there can be simplified somewhat. The result, from equations (4.38) and (4.39) of Hestenes (1974a), is

$$\Omega = -\beta^{-1}\mathbf{b} + (1+\beta)^{-1}\mathbf{v}\times\mathbf{e} \tag{27}$$

As shown in Hestenes (1975a), this is a combination of relativistic Larmor and Thomas precessions.

It is worth noting that the classical problems of rigid body rotational dynamics are worked out in detail in Hestenes (1985) using the spinor methods employed here.

Equation (27) reduces the problem of finding Ω to determining $\mathbf{e}$ and $\mathbf{b}$. Concerning the problem of solving equation (26) when Ω is known, it should be emphasized that the time derivative is a fiducial derivative, so

$$d\boldsymbol{\sigma}_k/dt = 0 \tag{28}$$

even though the $\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_k(x)$ are not constant. It follows that, for Ω sufficiently small, equation (26) is solved by

$$\begin{aligned} U(t) &\approx \left(1 + \tfrac{1}{2}\int_0^t i\Omega\, dt\right)U(0) \\ &= \left[1 + \tfrac{1}{2}i\boldsymbol{\sigma}_k(x(t))\int_0^t \Omega\cdot\boldsymbol{\sigma}_k\, dt\right]U(0) \end{aligned} \tag{29}$$

It must also be understood that in (29) the fiducial components $U(0)$ are constant, through $U(0)$ is not, because the base vectors $\boldsymbol{\sigma}_k(x(t))$ depend on t. For practical purposes, however, the $\boldsymbol{\sigma}_k$ can be regarded as constant until integration is completed.

4. SEPARATION OF GRAVITATIONAL MOTION AND PRECESSION

Considering gravitational effects alone, equation (19) reduces to

$$-\omega_v = \mathbf{e} + i\mathbf{b} \tag{30}$$

We can evaluate ω_v by the method of the preceding article (Hestenes, 1986), which gives the equations

$$\omega_v = \tfrac{1}{2}T \cdot \gamma_\mu - \Box \wedge \gamma_\mu \tag{31}$$

where the trivector T is given by

$$T = \gamma^\alpha \wedge \Box \wedge \gamma_\alpha \tag{32}$$

and

$$\Box \wedge \gamma_\mu = \eta_\mu (\Box h^\mu{}_\nu) \wedge \Box x^\nu \tag{33}$$

Here we can drop the convention in Hestenes (1986) of using carets on subscripts to distinguish fiducial components from coordinate components because we are now concerned with fiducial components only.

From (13) we have

$$v^k = v \cdot \gamma^k = \beta \mathbf{v} \cdot \boldsymbol{\sigma}_k \tag{34}$$

Therefore

$$\omega_v = v^\mu \omega_\mu = \beta(\omega_0 + \mathbf{v} \cdot \boldsymbol{\sigma}_k \boldsymbol{\omega}_k) \tag{35}$$

The most sensitive feasible satellite test of gravitation theory can be derived from a fiducial frame specified by the equations

$$\begin{aligned} \gamma_0 &= e^\Phi \Box x^0 = \gamma_0 \\ \gamma_k &= -h_k \Box x^0 - e^\lambda \Box x^k = -\gamma^k \end{aligned} \tag{36}$$

where Φ, λ, and the h_k are scalar functions describing the gravitational field. From (36) one can read off the components of the fiducial tensor:

$$h_0{}^0 = e^\Phi, \qquad h_k{}^0 = 0, \qquad h_0{}^k = h_k, \qquad h_j{}^k = e^\lambda \delta_j{}^k \tag{37}$$

for $j, k = 1, 2, 3$.

The coordinate frame $\{g_\mu = h^\nu{}_\mu \gamma_\nu\}$ is therefore given by

$$g_0 = e^\Phi \gamma_0 - h_k \gamma_k, \qquad g_k = e^\lambda \gamma_k \tag{38}$$

For purposes of comparison with the literature, using (38), we evaluate the metric tensor $g_{\mu\nu} = g_\mu \cdot g_\nu$ with the results

$$\begin{aligned} g_{00} &= e^{2\Phi} - \sum_k h_k^2 \\ g_{0k} &= -e^{\lambda} h_k \\ g_{ij} &= -e^{2\lambda}\delta_{ij} \end{aligned} \tag{39}$$

We can solve equations (38) to get the reciprocal coordinate frame $\{g^\mu = \Box x^\mu\}$ by a method given in Hestenes and Sobczyk (1984). First we compute, from (38), the pseudoscalar

$$g \equiv g_0 \wedge g_1 \wedge g_2 \wedge g_3 = e^{\Phi+3\lambda} i$$

We also compute

$$g_1 \wedge g_2 \wedge g_3 = e^{3\lambda}\gamma_1\gamma_2\gamma_3 = e^{3\lambda}\gamma_0$$

and

$$\begin{aligned} g_0 \wedge g_1 \wedge g_2 &= (e^{\Phi}\gamma_0 - h_k\gamma_k) \wedge (e^{2\lambda}\gamma_1 \wedge \gamma_2) \\ &= e^{2\lambda}(e^{\Phi}\gamma_0\gamma_1\gamma_2 - h_3\gamma_1\gamma_2) \end{aligned}$$

Whence,

$$g^0 = g_1 \wedge g_2 \wedge g_3\, g^{-1} = e^{-\Phi}\gamma_0$$

and

$$\begin{aligned} g^3 &= -g_0 \wedge g_1 \wedge g_2\, g^{-1} = e^{-\Phi-\lambda}\,(e^{\Phi}\gamma_0\gamma_1\gamma_2 - h_3\gamma_3\gamma_1\gamma_2)\, i \\ &= e^{-\lambda}\,(\gamma^3 + e^{-\Phi} h_3\gamma_0) \end{aligned}$$

Therefore,

$$\begin{aligned} \Box x_0 &= e^{-\Phi}\gamma_0 \\ \Box x^k &= e^{-\lambda}(\gamma^k + e^{-\Phi} h_k\gamma_0) \end{aligned} \tag{40}$$

This result can be checked by using $g^\mu \cdot g_\nu = \delta^\mu{}_\nu$.

Now, taking the curl of (40) and using (33), we get

$$\begin{aligned} \Box \wedge \gamma_0 &= (\Box\Phi) \wedge \gamma_0 \\ \Box \wedge \gamma_k &= \gamma^k \wedge \Box\Phi + e^{\Phi}\gamma_0 \wedge (\Box h_k + h_k \Box\lambda) \end{aligned} \tag{41}$$

We substitute this in (32) to get

$$T = \gamma^k \wedge \Box \wedge \gamma_k = e^{-\Phi}\sum_k \gamma_k \wedge \gamma_0 \wedge (\Box h_k + h_k \Box\lambda) \tag{42}$$

We can substitute (41) and (42) into (31) immediately to get expressions for the ω_μ. But we want our result in terms of relative variables, so let us first introduce the necessary notation for that end.

We represent the time coordinate by $t = x^0$ and intrduce the notaions

$$\partial_t = \gamma_0 \cdot \Box, \qquad \nabla = \gamma_0 \wedge \Box \tag{43}$$

as well as

$$\mathbf{h} = h_k \boldsymbol{\sigma}_k, \qquad \mathbf{D} = \boldsymbol{\sigma}_k d_k \tag{44,45}$$

Whence

$$\mathbf{D} \wedge \mathbf{h} = (\nabla h_k) \wedge \boldsymbol{\sigma}_k$$

Now (41) assumes the form

$$\begin{aligned} \Box \wedge \gamma_0 &= -\nabla\Phi \\ \Box \wedge \gamma_k &= -\boldsymbol{\sigma}_k \wedge \nabla\Phi - \boldsymbol{\sigma}_k \partial_t \Phi + e^{-\Phi}(\nabla h_k + h_k \nabla\lambda) \end{aligned} \tag{46}$$

Also, from (42) we get

$$\begin{aligned} T \cdot \gamma_0 &= -e^{-\Phi}\{d_t\,\mathbf{h} + \mathbf{h}\,\partial_t\lambda + \nabla \wedge \mathbf{h} - \mathbf{h} \wedge \nabla\lambda\} \\ T \cdot \gamma_k &= -e^{-\Phi}\{d_k\,\mathbf{h} + \mathbf{h}\,\partial_k\lambda - \nabla h_k - h_k \nabla\lambda\} \end{aligned} \tag{47}$$

Putting (46) and (47) into (31), we obtain

$$\omega_0 = \nabla\Phi - \tfrac{1}{2}e^{-\Phi}\{d_t\,\mathbf{h} + \mathbf{h}\,\partial_t\lambda + \mathbf{D} \wedge \mathbf{h} - \mathbf{h} \wedge \nabla\lambda\} \tag{48a}$$

$$\omega_k = \boldsymbol{\sigma}_k \wedge \nabla\lambda + \boldsymbol{\sigma}_k \partial_t \lambda - \tfrac{1}{2}e^{-\Phi}\,\{d_t\,\mathbf{h} + \mathbf{h}\,\partial_t\lambda + \nabla h_k + h_k \nabla\lambda\} \tag{48b}$$

For a weak static gravitational field, equations (48a) and (48b) assume the approximate form

$$\omega_0 = \nabla\Phi - \tfrac{1}{2}\mathbf{D} \wedge \mathbf{h}\,, \qquad \omega_k = -\tfrac{1}{2}\nabla h_k + \boldsymbol{\sigma}_k \wedge \nabla\lambda \tag{49}$$

This is the case of experimental interest. Using (49) in (35), we get

$$\omega_v = \beta\{\nabla(\Phi - \tfrac{1}{2}\mathbf{h} \cdot \mathbf{v}) + i(\mathbf{v} \times \nabla\lambda - \tfrac{1}{2}\mathbf{D} \times \mathbf{h})\,\} \tag{50}$$

Comparing this with (30), we can read off the values of **e** and **b**. Substituting these results into (20), we get the equation of motion for a particle in a gravitational field

$$\frac{d\mathbf{v}}{dt} = \beta\{\nabla(\Phi - \tfrac{1}{2}\mathbf{h} \cdot \mathbf{v}) + \mathbf{v} \times (\mathbf{v} \times \nabla\lambda - \tfrac{1}{2}\mathbf{D} \times \mathbf{h})\,\} \tag{51}$$

For small velocities, equation (51) reduces to

$$\Omega = \mathbf{v} \times \nabla(\lambda - \tfrac{1}{2}\Phi) - \tfrac{1}{2}\mathbf{D} \times \mathbf{h} \tag{52}$$

This is in agreement with equation (40.33) of Misner *et al.* (1973, p. 1118), where $\lambda = \gamma U$ and $\Phi = -U$. The reader is referred to this reference for a discussion of physical applications.

5. GRAVITATIONAL INTERACTION IN THE DIRAC THEORY

In Hestenes (1967) the Dirac equation without gravitational interaction was first put in the form

$$\gamma^\mu \partial_\mu \psi \gamma_2 \gamma_1 = eA\psi + m\psi\gamma_0 \tag{53}$$

where the spinor ψ has the canonical form

$$\psi = (\rho e^{i\beta})^{\frac{1}{2}} R \tag{54}$$

and determines a mobile field

$$e_\mu = R\gamma_\mu \widetilde{R} \tag{55}$$

The vector field

$$J = \psi\gamma_0\widetilde{\psi} = \rho e_0 = \rho v \tag{56}$$

is the Dirac current, and

$$e_3 = R\gamma_3\widetilde{R} \tag{57}$$

is equivalent to the so-called "Pauli-Lubanski" spin vector.

The main advantage of this formulation of the Dirac theory is that it employs the spacetime algebra only, without employing complex numbers which do not have a physical interpretation. The simplest proof that it is equivalent to the conventional matrix form of the Dirac theory is given in Appendix A of Hestenes (1973).

We can generalize equation (53) to gravitationally curved spacetime by identifying the γ_μ with a fiducial frame and noting that equation (55) differs mathematically from (3) only in being defined over an extended region of spacetime instead of on a timelike curve alone. Therefore, the equations of motion (5) must be generalized to give coderivatives in all directions, as described by

$$\gamma_\mu \cdot \Box\, e_\nu = d_\mu e_\nu + \omega_\mu \cdot e_\nu = \Omega_\mu \cdot e_\nu \tag{58}$$

Hence (10) must be generalized to

$$d_\mu R = \tfrac{1}{2}(\Omega_\mu - \omega_\mu)R \tag{59}$$

An explicit form for Ω_μ in the presence of electromagnetic interactions can be found from Hestenes (1973), but that is not of interest here. The important point is that these considerations show that the desired generalization of the Dirac equation (53) is achieved simply by replacing the operator ∂_μ on the left side of (53) by the operator $d_\mu + \omega_\mu$.

Thus, defining a fiducial derivative D by

$$D = \gamma^\mu d_\mu \tag{60}$$

and introducing the abbreviation

$$\Gamma = \tfrac{1}{2}\gamma^\mu \omega_\mu \tag{60}$$

we generalize the Dirac equation (53) to

$$(D + \Gamma)\psi\gamma_2\gamma_1 = eA\psi + m\psi\gamma_0 \tag{61}$$

This is equivalent to the derivation of the gravitational interaction from a gauge invariance argument given in Section 24 of Hestenes (1966).

Along the streamlines of the Dirac equation, equations (56) and (59) give the spinor equation of motion

$$dR/d\tau = \tfrac{1}{2}(\Omega_v - \omega_v)R \tag{62}$$

where

$$d/d\tau = v \cdot D = v^\mu d_\mu \tag{63}$$

Comparison of (62) with (11) tells us immediately that the *gravitational effects on electron motion are exactly the same as on the classical rigid body motion*, which we have discussed already, including spin precession.

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