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PROPER DYNAMICS OF A RIGID POINT PARTICLE

DAVID HESTENES
Department of Physics and Astronomy
Arizona State University
Tempe, Arizona 85287-1504, USA

Abstract. A spinor formulation of the classical Lorentz force is given which describes the presession of an electron's spin as well as its velocity. Solutions are worked out applicable to an electron in a uniform field, plane wave, and a Coulomb field.

Key words: Geometric algebra, Maxwell's equation, special relativity, BMT equation.

INTRODUCTION

Every evidence indicates that the Dirac theory provides an optimal description of electron motion, but for many purposes it is unnecessarily complex. The classical model of an electron as a point charge is sometimes adequate, but of course it gives no account of electron spin. The minimal generalization of the classical model is obtained simply by expressing the Lorentz force as a spinor equation. The main objective of this paper is to study the solutions of this equation in some detail.

This approach has several advantages. As will be demonstrated, it provides a new and (it seems) simpler way of integrating the classical Lorentz force and expressing the orbit as a parametrized algebraic equation. Besides providing new insights into old results, the spinor solution describes the precession of electron spin with the same accuracy as it determines the orbit. The classical spinor equations are closely related in form to the Dirac equation. This narrows the gap between classical and quantum mechanical formulations of electron motion and hopefully will help clarify the relations between them.

Since the description of electron motion given here would be impossible without the mathematical apparatus developed in Ref. [1], familiarity with the notations and results given therein is presumed.

Section 1 shows how classical electrodynamics can be used to derive a spinor equation of motion for a localized charge distribution. As a important special case, the BMT equation is derived and shown to be already in the work of Thomas in a different form. The spinor formulation of the Lorentz force is given and it's applicability to a description of the electron is discussed.

In Secs. 2, 3, and 4 the spinor Lorentz force is integrated to describe the motion of a charge in a uniform field, in a plane wave and in a Coulomb field. The problems are worked out in considerable detail to illustrate fully the efficiency of spacetime algebra in practical computations. Though the spinor solutions for uniform and plane wave fields have been found previously by other authors, the treatment here is unique in many details. I believe the spinor solution for the Coulomb field is published here for the first time.

1. PROPER DYNAMICS

In Sec. 4 of Ref. [1] the kinematics of a rigid point particle were expressed in terms of its proper angular velocity Ω. Before the equations of motion can be solved, dynamical assumptions must be made to express Ω as a definite function of the proper time τ. These depend on the nature of the particle. As an example of great importance, typical assumptions of classical electrodynamics will be put into proper form here and related to a model of the electron.

The classical force on a localized charge distribution at rest is, after a multipole expansion,

$$\mathbf{f} = e\mathbf{E} + \mathbf{p}\cdot\nabla\mathbf{E} + \nabla\mu\cdot\mathbf{B} + \cdots . \tag{1.1}$$

Assume now that the charge distribution can be regarded as a particle (of zero extent) with total electric change e, intrinsic electric dipole moment $\mathbf{p}$, intrinsic magnetic dipole moment μ, and that the higher multipole moments vanish or are negligible. Assume also that (1.1) is an expression for the relative force $\mathbf{f}$ on the particle in its instantaneous rest frame, more specifically, that

$$\mathbf{f} = f\wedge v = mc^2\dot{v}v = fv\,. \tag{1.2}$$

For vanishing $\mathbf{p}$ and μ, then, (1.1) and (1.2) reduce to the Lorentz force as already shown in (I.2.19). Therefore, it is only necessary to express the last two terms in proper form.

Define now the *proper moment bivector* M of the particle by the equations

$$M = -\mathbf{p} + i\mu\,, \tag{1.3a}$$

$$-\mathbf{p} = M\cdot v\,v\,, \tag{1.3b}$$

$$i\mu = M\wedge v\,v\,. \tag{1.3c}$$

In the instantaneous rest system $F = \mathbf{E} + i\mathbf{B}$ where $\mathbf{E} = F\cdot v\,v$ and $i\mathbf{B} = F\wedge v\,v$; so, for instance,

$$M\cdot F = -\mathbf{p}\cdot\mathbf{E} - \mu\cdot\mathbf{B}\,, \tag{1.4}$$

which is the familiar classical expression for the energy of electric and magnetic dipoles. The second term of (1.4) by itself can be written

$$\mu\cdot\mathbf{B} = -[\,(M\wedge v)\cdot v\,]\cdot F - (M\wedge v)\cdot(v\wedge F)\,. \tag{1.5}$$

The formulas used in (1.5) to rearrange the inner and outer products are established in Refs. [2] and [3]. The proper form for ∇ in (1.1) is $v\wedge\Box$, so

$$\mathbf{p}\cdot\nabla = -((M\cdot v)\wedge v)\cdot(v\wedge\Box) = -M\cdot(v\wedge\Box)\,. \tag{1.6}$$

Substituting the proper expressions in (1.2), one gets

$$\mathbf{f} = f\wedge v = eF\cdot vv - M\cdot(v\wedge\Box)F\cdot vv - v\wedge\Box(M\wedge v)\cdot(v\wedge F) \tag{1.7}$$

and substituting this in (1.2) and dividing by v one obtains finally

$$mc^2\dot{v} = f = [\,eF - M\cdot(v\wedge\Box)F - v\wedge\Box(M\wedge v)\cdot(v\wedge F)\,]\cdot v \tag{1.8}$$

The form of this equation suggests taking the term in brackets to be Ω, but as (I.4.2) shows, an expression for $\dot{v}$ determines only part of Ω, so additional dynamical assumptions are required.

In order to get equations describing the motion of an electron, assume that $\mathbf{p} = 0$, or equivalently,

$$v\cdot M = 0\,. \tag{1.9}$$

With this condition (1.5) can be replaced by the simpler relation

$$\mu\cdot\mathbf{B} = -M\cdot F\,. \tag{1.10}$$

Next assume that M has constant magnitude and is proportional to the *spin* (intrinsic angular momentum) *bivector* S, that is,

$$M = c\lambda S\,, \qquad \text{where} \qquad \lambda \equiv ge/2mc^2\,, \tag{1.11}$$

the constant g being the usual *gyromagnetic ratio*. The relation (1.11) obtains if the magnetic moment arises from a circulating charge distribution. If the distribution has a constant ratio of charge to mass density, it is easy to show that $g = 1$, in disagreement with the value $g = 2$ which obtains for an electron. However, other assumptions about the structure of the particle will give almost any desired value for g.

From (1.9) and (1.11) it follows that

$$v\cdot S = 0\,. \tag{1.12}$$

There exists a unique proper vector s called the *spin vector* such that

$$S = isv = is\wedge v\,. \tag{1.13a}$$

This can be proved simply by solving for s; thus

$$s = -iSv = iS\wedge v\,. \tag{1.13b}$$

It follows from this that $s\cdot v = 0$. The spin can now be related to the kinematical equations (I.4.3) for a rigid point particle by writing

$$s = |\,s\,|\,e_3\,. \tag{1.14}$$

But to get a definite functional form for the equations, classical dynamical considerations are helpful, at least as a guide.

For a magnetic dipole at rest in a magnetic field the classical theory gives the famous equation for the *Larmor precession* of the spin,

$$\frac{d\mathbf{s}}{dt} = \mu\times\mathbf{B}\,. \tag{1.15}$$

More generally, the classical theory adds a term proportional to $\nabla \times \mathbf{E}$ to the right side of (1.15), but, following Thomas, this can be neglected in the first approximation. To put (1.15) in proper form in accordance with the preceding assumptions, write

$$\mathbf{s} = sv = s \wedge v\,, \tag{1.16a}$$

$$\mu = c\lambda \mathbf{s} = c\lambda sv\,, \tag{1.16b}$$

$$i\mathbf{B} = B \equiv (F \wedge v)v\,. \tag{1.17}$$

Also, it is necessary to take account of the fact that (1.15) was derived for an inertial frame rather than an instantaneous rest frame. This can be done by interpreting the left side of (1.15) as a special case of $c\dot{s} \wedge v$ [just as was done for the acceleration in (I.2.19)], rather than as $cd(sv)/d\tau$, which can be shown to be inconsistent with the condition $s \cdot v = 0$. After noting that

$$\mu \times \mathbf{B} = -\tfrac{1}{2}[\,\mu, i\mathbf{B}\,] = \tfrac{1}{2}[\,i\mathbf{B}, \mu\,]\,,$$

(1.15) can be put in the form

$$c\dot{s} \wedge v = c\lambda\tfrac{1}{2}[\,B, sv\,] = c\lambda\tfrac{1}{2}[\,B, s\,]v = c\lambda B \cdot sv\,.$$

Multiplying by v and using

$$\begin{aligned}(\dot{s} \wedge v)v \;&=\; (\dot{s} \wedge v) \cdot v = \dot{s} - (\dot{s} \cdot v)v \\ &=\; \dot{s} + (\dot{v} \cdot s)v = \dot{s} - (\dot{v}v) \cdot s,\end{aligned}$$

which is a consequence of $s \cdot v = 0$, one obtains the equation of motion for s:

$$\dot{s} = \lambda B \cdot s - (\dot{v} \cdot s)s = (\dot{v}v + \lambda B) \cdot s \tag{1.18a}$$

This is the so-called Bargmann-Michel-Telegdi (BMT) equation [4]. Since derivatives of the field were neglected in the derivation of (1.18a), the same assumption must be made in the corresponding equation for v. Hence, in (1.18a)

$$\dot{v}v = \frac{e^2}{mc^2}(F \cdot v)v = \frac{e^2}{mc^2}\,(F - B)\,. \tag{1.18b}$$

While equations (1.18a, b) hold rigorously only for a homogeneous (i.e., constant in time and uniform in space) field F, they may serve as a useful approximation under other conditions. Indeed, Thomas used them in a different form to calculate the spin precession of an electron in an atom.

According to (1.14), (1.18a) is an equation for the unit spacelike vector e_3. Comparison with Eqs. (I.4.2a), (I.4.3), and (I.4.6) suggests that Eqs. (1.18a,b) be interpreted as equations of motion for a rigid point particle with angular velocity

$$\begin{aligned}\Omega \;&=\; \dot{v}v + \lambda B = \frac{e}{mc^2}(F \cdot v)v + \frac{ge}{2mc^2}(F \wedge v)v \\ &=\; \frac{e}{mc^2}[\,F + (g/2 - 1)B\,]\,.\end{aligned} \tag{1.19}$$

For an electron, according to atomic theory, $g = 2$, in which case (1.19) reduces to the strikingly simple form

$$\Omega = \frac{e}{mc^2}\, F = \lambda F \tag{1.20a}$$

and the spinor equation for an electron is

$$\dot{R} = \tfrac{1}{2}\Omega R = \frac{e}{2mc^2} FR\,. \tag{1.20b}$$

Of course, the argument leading up to (1.20) can in no sense be regarded as a derivation from any consistent classical model of the electron as a spinning charge distribution. However, an equation exactly of the form (1.20b) has been derived as an approximation of the Dirac equation [Eq. (6.17) of Ref. [5]], though the significance of the approximation is not entirely clear. Therefore, it is interesting that (1.20) can be tested directly by experiments on the spin precession of electrons moving through a constant field [3], and that the anomolous magnetic moment of the electron can be evaluated by using (1.19). Equation (1.18a) describes the spin precession in the instantaneous rest frame of the particle. The equivalent equation describing spin precession in an inertial frame can be obtained directly by expressing the proper angular velocity given by (1.19) in relative form and using Eq. (I.4.45); write

$$\begin{aligned} \Omega &= \alpha + i\beta = \frac{e}{mc^2}\, F + \Big(\gamma - \frac{e}{mc^2}\Big)\, B\,, \\ F &= \mathbf{E} + i\mathbf{B}\,, \end{aligned}$$

and, with the help of (I.4.34),

$$\begin{aligned} B &= F - F\cdot vv = (1-\gamma^2)\mathbf{E} - \gamma^2(c^{-2}\mathbf{E}\cdot\mathbf{v}\,\mathbf{v} + c^{-1}\mathbf{v}\times\mathbf{E}) \\ &\qquad + i\big\{\mathbf{B} - \gamma^2(c^{-1}\mathbf{v}\times\mathbf{E} + c^{-2}(\mathbf{B}\times\mathbf{v})\times\mathbf{v})\big\}\,. \end{aligned}$$

From these equations, expressions for α and β can be read off directly, which, on substitution into (I.4.36) and some rearrangement of terms, yields

$$\begin{aligned} \omega = -\Big[\,\frac{e}{mc^2} + \gamma\Big(\lambda - \frac{e}{mc^2}\Big)\,\Big]\mathbf{B} - \Big(\frac{e}{mc^2}\frac{\gamma^2}{c(\gamma+1)} - \frac{\lambda\gamma}{c}\Big)\mathbf{v}\times\mathbf{E} \\ - \frac{\gamma^2}{c^2(1+\gamma)}\Big(\frac{e}{mc^2} - \lambda\Big)\,\mathbf{B}\cdot\mathbf{v}\,\mathbf{v}\,. \end{aligned} \tag{1.21a}$$

So the equation for the spin $\sigma \equiv |\,s\,|\,\mathbf{e}_3$ in the inertial system is, by (I.4.39),

$$\dot{\sigma} = \frac{\gamma}{c}\,\frac{d\sigma}{dt} = \omega\times\sigma \tag{1.21b}$$

This is exactly the result obtained by Thomas [Ref. [7], his Eq. (4.121)], and proves directly its equivalence to the BMT equation (1.18a).

Equations equivalent to (1.20b) have been discussed by other authors ([3], [8], [9]). However, the form (1.20b) is easier to handle than other forms because it is supported by the spacetime algebra. Equation (1.20b) describes precession of both electron spin and velocity with the same degree of accuracy that the Lorentz force describes electron motion. Even apart from equations of spin, it is sometimes easier to solve than the Lorentz equation. For these reasons, Eq. (1.20b) is important enough to be given a name and its basic solutions will be thoroughly studied in the following sections.

No attempt will be made here to generalize (1.20) to get a more precise description of the electron, since, short of the full Dirac equation, the best procedure is unclear. Equation (1.20) can be used in connection with quantum theory by taking the spin to be the quantum mechanical polarization vector. It will be referred to as "the *spinor Lorentz force*" or as "the equation of motion for a *rigid test charge*"; the adjective "test" serves to indicate that radiation of the charge is not taken into account, while the adjective "rigid" indicates that a complete comoving frame is described. Of course, the "rigid test charge" is most important as a model of the electron if the charge e is negative or a positron if e is positive.

2. RIGID TEST CHARGE IN A HOMOGENEOUS FIELD

The spinor equation of motion $\dot{R} = \frac{1}{2}\Omega R$ for a rigid point particle with constant proper angular velocity Ω integrates immediately to

$$R = \exp(\Omega\tau/2) = \sum_{n=0}^{\infty} \frac{1}{n!}(\tfrac{1}{2}\Omega\tau)^n \,, \tag{2.1}$$

where the initial condition $R(0) = 1$ has been adopted. With the dynamical assumption

$$\Omega = \lambda F = (e/mc^2)F \,, \tag{2.2}$$

the spinor (2.1) describes the motion of are rigid test charge in a homogeneous electromagnetic field F. In particular, it describes the precession of the velocity and spin of an electron. Thus, the electron velocity $v = v(\tau)$ and spin $s = s(\tau)$ are given explicitly by

$$v = Rv_0\widetilde{R} = \exp\left(F\lambda\tau/2\right) v_0 \exp\left(F\lambda\tau/2\right), \tag{2.3a}$$

$$s = Rs_0R = \exp\left(F\lambda\tau/2\right) s_0 \exp\left(F\lambda\tau/2\right). \tag{2.3b}$$

The history of the electron can be obtained by integration from (2.3a). To do this, it is convenient to assume that F is nonnull. The alternative case of a homogeneous null field has little practical significance; in any event it can be treated separately if necessary.

Since F is assumed to be nonnull, in accordance with (I.1.3), it is subject to the canonical decomposition into orthogonal blades:

$$F = \alpha f + \beta i f = fz \,, \tag{2.4a}$$

where α and β are scalars,

$$z = \alpha + i\beta \qquad \text{with} \qquad \alpha \geq 0 \,, \tag{2.4b}$$

and f is a simple unit timelike vector, that is,

$$f^2 = 1 \qquad \text{and} \qquad [\, f\,]_2 = f\,. \tag{2.4c}$$

Substituting (2.4a) into (2.1), R can be written

$$\begin{aligned} R \; &= \; \exp\left(f\alpha\lambda\tau\right)\exp\left(if\beta\lambda\tau\right) \\ &= \; (\cosh\alpha\lambda\tau + f\sinh\alpha\lambda\tau)\;(\cos\beta\lambda\tau + if\sin\beta\lambda\tau)\,. \end{aligned} \tag{2.5}$$

Now using (2.4c) and (I.1.4), the initial velocity v_0 can be decomposed into a component $v_{0\parallel}$ in the f-plane and a component $v_{0\perp}$ orthogonal to the f-plane; thus

$$v_0 = f^2 v_0 = v_{0\parallel} + v_{0\perp}\,, \tag{2.6a}$$

where

$$v_{0\parallel} = f(f\cdot v_0) = \tfrac{1}{2}(v_0 - f v_0 f) = -if(if)\wedge v_0\,, \tag{2.6b}$$
$$v_{0\perp} = f(f\wedge v_0) = \tfrac{1}{2}(v_0 + f v_0 f) = -if(if)\cdot v_0\,. \tag{2.6c}$$

From (2.6b,c), one has

$$f v_{0\parallel} = f\cdot v_0 = -v_{0\parallel} f\,, \tag{2.7a}$$
$$f v_{0\perp} = f\wedge v_0 = -v_{0\perp} f\,. \tag{2.7b}$$

Using this and recalling $v_0 i = -iv_0$, one finds

$$v_{0\parallel}\exp\left(-f\alpha\lambda\tau/2\right)\exp\left(-if\beta\lambda\tau/2\right) = \exp\left(f\alpha\lambda\tau/2\right)\exp\left(-if\beta\lambda\tau/2\right)v_{0\parallel}\,, \tag{2.8a}$$
$$v_{0\perp}\exp\left(-f\alpha\lambda\tau/2\right)\exp\left(-if\beta\lambda\tau/2\right) = \exp\left(-f\alpha\lambda\tau/2\right)\exp\left(if\beta\lambda\tau/2\right)v_{0\perp}\,. \tag{2.8b}$$

So, substituting (2.5) in (2.3a) and using (2.8), one obtains

$$v = \frac{dx}{d\tau} = \exp\left(f\alpha\lambda\tau\right)v_{0\parallel} + \exp\left(if\beta\lambda\tau\right)v_{0\perp}\,. \tag{2.9}$$

This can be integrated immediately to get the history $x = x(\tau)$:

$$x - x_0 = \frac{(\exp\left(f\alpha\lambda\tau\right) - 1)}{\alpha\lambda}\, f\cdot v_0 + \frac{(\exp\left(if\beta\lambda\tau\right) - 1)}{\beta\lambda}\,(if)\cdot v_0\,. \tag{2.10}$$

This solution is valid even if $\alpha = 0$ and/or $\beta = 0$, as is easily established by expressing the exponential as a power series.

It is worth noting that, more generally, integration of the equations of motion can be carried out in essentially the same way as above when z is any function of τ as long as f is constant. This situation obtains when one has fields with fixed direction but spacial and/or temporal variations in magnitude.

The problem remains to reexpress the solutions (2.9) and (2.10) in terms of relative vectors such as the electric and magnetic field strengths $\mathbf{E}$ and $\mathbf{B}$, because these quantities have direct observational significance. To accomplish this, it is necessary to relate the decomposition $F = \mathbf{E} + i\mathbf{B}$ relative to a given observer u to the canonical decomposition (2.4) which is independent of any observer. The relation is a simple one in the case that

$$f \wedge u = 0\,; \tag{2.11a}$$

then,

$$\hat{f} = \hat{\mathbf{E}}, \qquad \alpha = |\,E\,|\,, \tag{2.11b}$$

$$\alpha\hat{f} = \mathbf{E}\,, \qquad \beta\hat{f} = \beta\hat{\mathbf{E}} = \mathbf{B}\,; \tag{2.11c}$$

all of which is equivalent to the condition

$$\mathbf{E} \wedge \mathbf{B} = 0\,, \tag{2.11d}$$

that is, $\mathbf{E}$ and $\mathbf{B}$ are *parallel fields*. This case is important enough in itself to work out before proceeding to the general case. Using (2.11b) in (I.2.15), on can write down immediately

$$f \cdot v_0\, u = \gamma_0 \left(\frac{\hat{\mathbf{E}} \cdot \mathbf{v}_0}{c} + \hat{\mathbf{E}} \right)\,, \tag{2.12a}$$

$$(if) \cdot v_0\, u = \frac{\gamma_0}{c}\, i\hat{\mathbf{E}} \wedge \mathbf{v}_0 = \frac{\gamma_0}{c}\, v_0 \times \hat{\mathbf{E}}\,, \tag{2.12b}$$

where

$$\gamma_0 \mathbf{v}_0 = v_0 \wedge u \qquad \text{and} \qquad \gamma_0 = v_0 \cdot u = (1 - \mathbf{v}_0^2/c^2)^{-1/2}\,. \tag{2.13}$$

Using (2.6), one obtains from (2.12)

$$v_{0\parallel} u = f(f \cdot v_0) u = \gamma_0 \left(\mathbf{E}\, \frac{(\mathbf{E} \cdot \mathbf{v}_0)}{c} + 1 \right) = \gamma_0 \left(\frac{\mathbf{v}_{0\parallel}}{c} + 1 \right)\,, \tag{2.14a}$$

$$v_{0\perp} u = -if(if) \cdot v_0\, u = \frac{\gamma_0}{c}\, \hat{\mathbf{E}} \times (\mathbf{v}_0 \times \hat{\mathbf{E}}) = \frac{\gamma_0}{c}\, \mathbf{v}_{0\perp}\,. \tag{2.14b}$$

Equation (2.9) can now be easily expressed as an equation in relative quantities by multiplying by u and using (2.12):

$$vu = \gamma \left(1 + \frac{\mathbf{v}}{c} \right) = \exp{(\mathbf{E}\lambda\tau)}\gamma_0 \left(\hat{\mathbf{E}}\, \frac{\hat{\mathbf{E}} \cdot \mathbf{v}_0}{c} + 1 \right) + \exp{(i\mathbf{B}\lambda\tau)}\gamma_0 \mathbf{v}_{0\perp}\,. \tag{2.15}$$

The scalar part of (2.15) is the equation

$$\frac{\gamma}{\gamma_0} = \cosh{(|\,\mathbf{E}\,|\lambda\tau)} + \frac{\hat{\mathbf{E}} \cdot \mathbf{v}_0}{c} \sinh{(|\,\mathbf{E}\,|\lambda\tau)}\,. \tag{2.16}$$

Writing

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \qquad \text{where} \qquad \mathbf{v}_{\parallel} = \mathbf{v} \cdot \hat{\mathbf{E}}\, \hat{\mathbf{E}}\,, \tag{2.17a}$$

one has from the vector part of (2.16)

$$\mathbf{v}_{\parallel} = \Big[\gamma_0 \sinh\left(|\,\mathbf{E}\,|\lambda\tau\right) + \frac{\hat{\mathbf{E}}\cdot\mathbf{v}_0}{c}\cosh\left(|\,\mathbf{E}\,|\lambda\tau\right)\Big]\hat{\mathbf{E}}\,, \tag{2.17b}$$

$$\gamma\mathbf{v}_{\perp} = \exp\left(i\mathbf{B}\lambda\tau\right)\gamma_0\mathbf{v}_{0\perp}\,. \tag{2.17c}$$

It will be noted that (2.17) is simplified by expressing it in terms of the relative momentum $\mathbf{p} = m\gamma\mathbf{v}$. Now multiplying (2.10) by u and using (2.12), one obtains

$$\begin{aligned}(x - x_0)u \;&=\; (t - t_0) + (\mathbf{x} - \mathbf{x}_0) \\ &=\; \frac{\gamma}{\lambda\mathbf{E}^2}\left[\exp\left(\mathbf{E}\lambda\tau\right) - 1\right]\left(\frac{\mathbf{E}\cdot\mathbf{v}_0}{c} + \mathbf{E}\right) + \frac{\gamma_0}{\lambda\mathbf{B}^2}\left[\exp\left(i\mathbf{B}\lambda\tau\right) - 1\right]\frac{\mathbf{v}_0}{c}\times\mathbf{B}\,.\end{aligned} \tag{2.18}$$

The scalar part of (2.18) gives the functional relation between the "laboratory time" t and the proper time τ:

$$(t - t_0) = \frac{\gamma_0}{\lambda|\,\mathbf{E}\,|}\left(\sinh\left(|\,\mathbf{E}\,|\lambda\tau\right) + \frac{\mathbf{E}\cdot\mathbf{v}_0}{c}\left[\,\cosh\left(|\,\mathbf{E}\,|\lambda\tau\right) - 1\,\right]\right)\,. \tag{2.19}$$

The vector part of (2.18) is a parametric equation for the orbit $\mathbf{x} = \mathbf{x}(\tau)$:

$$\begin{aligned}\mathbf{x} - \mathbf{x}_0 \;=\; &\left(\frac{\gamma_0}{\lambda\mathbf{E}^2}\left[\,\cosh\left(|\,\mathbf{E}\,|\lambda\tau\right) - 1)\,\right] + \frac{\hat{\mathbf{E}}\cdot\mathbf{v}_0}{c}\,\sinh\left(|\,\mathbf{E}\,|\lambda\tau\right)\right)\mathbf{E} \\ &+ \frac{\gamma_0}{c\lambda\mathbf{B}^2}\left(\exp\left(i\mathbf{B}\lambda\tau\right) - 1\right)\mathbf{v}_0\times\mathbf{B}\,.\end{aligned} \tag{2.20}$$

If $\mathbf{E}$, $\mathbf{B} \neq 0$, the orbit is a spiral with decreasing radius and increasing pitch as the charge loses energy to the field.

Now returning to the general case, it is necessary to express α, β, and f in terms of $\mathbf{E}$ and $\mathbf{B}$. Squaring (2.4a), one has

$$F^2 = z^2 = \alpha^2 - \beta^2 + 2i\alpha\beta = (\mathbf{E} + i\mathbf{B})^2 = \mathbf{E}^2 - \mathbf{B}^2 + 2i\mathbf{E}\cdot\mathbf{B}\,.$$

Hence,

$$\alpha^2 - \beta^2 = \mathbf{E}^2 - \mathbf{B}^2\,, \tag{2.21a}$$

$$\alpha\beta = \mathbf{E}\cdot\mathbf{B}\,. \tag{2.21b}$$

Solving for α and β, one gets

$$\alpha = \left(\frac{|\,z\,|^2 + \mathbf{E}^2 - \mathbf{B}^2}{2}\right)^{1/2} > 0\,, \tag{2.22a}$$

$$\beta = \pm\left(\frac{|\,z\,|^{-2} - \mathbf{E}^2 + \mathbf{B}^2}{2}\right)^{1/2}\,, \tag{2.22b}$$

where

$$| z |^2 = \alpha^2 + \beta^2 = [\,(\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})\,]^{1/2}\,, \tag{2.22c}$$

and the sign of β is determined by the rule

$$\beta \neq 0 \qquad \text{if} \qquad \mathbf{E}\cdot\mathbf{B} \neq 0\,. \tag{2.22d}$$

Equation (2.4) can be solved for f by

$$f = z^{-1}F = \frac{\alpha - i\beta}{|\,z\,|^2}\,(\mathbf{E} + i\mathbf{B})\,.$$

So, expressing f in terms of relative vectors $\mathbf{e}$ and $\mathbf{b}$, one has

$$f = \mathbf{e} + i\mathbf{b}\,, \tag{2.23a}$$

where

$$\mathbf{e} = (\alpha\mathbf{E} + \beta\mathbf{B})/|\,z\,|^2\,, \tag{2.23b}$$

$$\mathbf{b} = (\alpha\mathbf{E} - \beta\mathbf{B})/|\,z\,|^2\,. \tag{2.23c}$$

It is worth noting that, from (2.4c) or from (2.23b, c),

$$f^2 = \mathbf{e}^2 - \mathbf{b}^2 = 1\,, \tag{2.24a}$$

$$\mathbf{e}\cdot\mathbf{b} = 0\,. \tag{2.24b}$$

Now, using (2.23a) in (I.2.15) one gets

$$f\cdot v_0 u = \gamma_0\left(\frac{\mathbf{e}\cdot\mathbf{v}_0}{c} + \mathbf{e} + \frac{\mathbf{v}_0\times\mathbf{b}}{c}\right)\,, \tag{2.25a}$$

$$(if)\cdot v_0 u = \gamma_0\left(-\frac{\mathbf{b}\cdot\mathbf{v}_0}{c} - \mathbf{b} + \frac{\mathbf{v}_0\times\mathbf{e}}{c}\right)\,. \tag{2.25b}$$

And using (????) with (2.25a, b), one gets

$$v_{0\parallel}u = \gamma_0\Big(\mathbf{e}^2 - \frac{\mathbf{v}_0}{c}\cdot(\mathbf{e}\times\mathbf{b}) + \mathbf{e}\frac{\mathbf{e}\cdot\mathbf{v}_0}{c} + \mathbf{b}\frac{\mathbf{b}\cdot\mathbf{v}_0}{c} - \mathbf{b}^2\frac{\mathbf{v}_0}{c} + \mathbf{e}\times\mathbf{b}\Big)\,, \tag{2.26a}$$

$$v_{0\perp}u = -\gamma_0\Big(\mathbf{b}^2 - \frac{\mathbf{v}_0}{c}\cdot(\mathbf{e}\times\mathbf{b}) + \mathbf{e}\frac{\mathbf{e}\cdot\mathbf{v}_0}{c} + \mathbf{b}\frac{\mathbf{b}\cdot\mathbf{v}_0}{c} - \mathbf{e}^2\frac{\mathbf{v}_0}{c} + \mathbf{e}\times\mathbf{b}\Big)\,. \tag{2.26b}$$

Finally, using (2.6) and (2.7) and multiplying by u, (2.9) and (2.10) can be put in the forms

$$\begin{aligned} vu \;=\; \gamma\Big(1 + \frac{\mathbf{v}}{c}\Big) &= v_{0\parallel}u\cosh(\alpha\lambda\tau) + f\cdot v_0 u\sinh(\alpha\lambda\tau) \\ &\quad + v_{0\perp}u\cos(\beta\lambda\tau) - (if)\cdot v_0\sin(\beta\lambda\tau)\,, \end{aligned} \tag{2.27}$$

$$\begin{aligned}(x-x_0)u &= (t-t_0)+(\mathbf{x}-\mathbf{x}_0)\\ &= f\cdot v_0 u\left(\frac{\cosh(\alpha\lambda\tau)-1}{\alpha\lambda}\right)+v_{0\parallel}\frac{\sinh(\alpha\lambda\tau)}{\alpha\lambda}\\ &+ (if)\cdot v_0 u\left(\frac{\cos(\beta\lambda\tau)-1}{\beta\lambda}\right)-v_{0\perp}u\sin(\beta\lambda\tau)\,. \end{aligned} \tag{2.28}$$

Substitution of (2.25) and (2.26) into (2.27) and (2.10) followed by separation into scalar and vector parts yields the complete solutions in relative form. The fact that the resulting relative formulas appear so much more complicated than the equivalent proper formulas (2.9) and (2.10) merely shows that the relative vectors $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{v}_0$ are a poor choice of parameters for the problem.

Insight which leads to a better choice of relative vectors as parameters can be gained as follows. A boost of u into

$$w = W u \widetilde{W} = W^2 u$$

can be defined by requiring that W boost $\hat{\mathbf{e}} \equiv |\,\mathbf{e}\,|^{-1}\mathbf{e}$ into f; that is

$$f = \mathbf{e} + i\mathbf{b} = W\hat{\mathbf{e}}\widetilde{W} = W^2\hat{\mathbf{e}} = \hat{\mathbf{e}}\widetilde{W}^2\,. \tag{2.29}$$

Solving for W^2, one finds

$$wu = W^2 = f\hat{\mathbf{e}} = |\,\mathbf{e}\,| + i\mathbf{b}\hat{\mathbf{e}} = |\,\mathbf{e}\,|\left(1+\frac{\mathbf{e}\times\mathbf{b}}{\mathbf{e}^2}\right) \equiv \gamma_w(1+\mathbf{w}/c)\,. \tag{2.30a}$$

Thus, with the help of (2.23)

$$\frac{\mathbf{w}}{c} = \frac{\mathbf{e}\times\mathbf{b}}{\mathbf{e}^2} = \frac{2\mathbf{E}\times\mathbf{B}}{|\,z\,|^2+\mathbf{E}^2+\mathbf{B}^2} \tag{2.30b}$$

and

$$\gamma_w = |\,\mathbf{e}\,| = (1+\mathbf{w}^2/c^2)^{-1/2}\,. \tag{2.30c}$$

Note that $fw = \hat{\mathbf{e}}u = |\,f\cdot u\,|^{-1} f\cdot u$, hence

$$f\cdot w = |\,f\cdot u\,|^{-1} f\cdot u\,, \tag{2.31a}$$

and, more important,

$$f\wedge w = 0\,. \tag{2.31b}$$

As noted earlier, the condition (2.31b) implies that the field $F = f(\alpha + i\beta)$ will consist of parallel electric and magnetic fields relative to an observer with proper velocity w. For this reason, the corresponding relative vector $\mathbf{w}$ is called the *relative drift velocity.* It is important to realize that $w = |\,f\cdot u\,|^{-1} f(f\cdot u)$ does not describe an intrinsic property of the electromagnetic field; rather, it describes a relation of the observer u to the field $F = fz$.

Now introduce an electromagnetic field

$$F' = \hat{\mathbf{e}}(\alpha + i\beta) = \mathbf{E}' + i\mathbf{B}' \,, \tag{2.32a}$$

where α, β and $\hat{\mathbf{e}}$ are defined as before by (2.22a,b) and (2.23b). Then, from (2.29) and (2.4) it follows that

$$F = WF'\widetilde{W} \,. \tag{2.32b}$$

Hence, from (2.1) it follows that

$$R = WR'\widetilde{W} \qquad \text{and} \qquad \widetilde{R} = W\widetilde{R}'\widetilde{W} \tag{2.33a}$$

where

$$R' = \exp{(F'\lambda\tau/2)} \,. \tag{2.33b}$$

Therefore, the equation (2.3a) for the proper velocity v of the electron can be written

$$v = R\,v_0\widetilde{R} = WR'\widetilde{W}v_0W\widetilde{R}'\widetilde{W} = Wv'\widetilde{W} \,, \tag{2.34a}$$

where

$$v' = R'v_0'\widetilde{R}' \tag{2.34b}$$

and

$$v_0' = \widetilde{W}v_0W \,. \tag{2.34c}$$

Now v' is the proper velocity of an electron with initial velocity v_0' accelerated by parallel fields $\mathbf{E}'$ and $\mathbf{B}'$ relative to the observer u, so explicit expressions for $v'u = \gamma'(1 + \mathbf{v}'/c)$ are known from the special case analyzed earlier. To get corresponding expressions for vu (2.34a) and (2.30a); thus

$$vu = \gamma(1 + \mathbf{v}/c) = Wv'\widetilde{W}u = Wv'uW = W\gamma(1 + \mathbf{v}'/c)\widetilde{W} \,,$$

the scalar part of which is

$$\gamma = \gamma'\gamma_w[\,(1 + \mathbf{v}' \cdot \mathbf{w})/c^2\,] \,, \tag{2.35a}$$

while the ratio of vector to scalar part is

$$\mathbf{v} = \frac{\mathbf{v}' + \mathbf{w} + (\gamma_w^{-1} - 1)\hat{\mathbf{w}} \times (\mathbf{v}' \times \hat{\mathbf{w}})}{1 + c^{-2}\mathbf{w} \cdot \mathbf{v}'} \,, \tag{2.35b}$$

the well-known velocity addition formula. Of course a similar formula will express $\mathbf{v}_0'$ in terms of $\mathbf{v}_0$ and $\mathbf{w}$. Also in a similar fashion, the general orbit can be found from the orbit of a particle in parallel fields by a boost in the direction of the drift velocity or by integrating (2.35). The formulas are easily worked out, and of course they will agree with (2.28), but now the general nature of the orbit is easily described; it consists of a tightening spiral in the relative direction $\hat{\mathbf{e}}$ [determined by (2.23b)] drifting with velocity $\mathbf{w}$ [given by (2.30)] in a direction orthogonal to $\hat{\mathbf{e}}$.

3. RIGID TEST CHARGE IN A PLANE WAVE FIELD

The equations of motion for a rigid test charge in an electromagnetic plane wave will now be integrated.

Any plane wave field $F = F(x)$ with proper propagation vector k can be written in the canonical form

$$F = fz\,, \tag{3.1a}$$

where f is a constant bivector and the x-dependence of F is exhibited explicitly by

$$z = a_+ \exp{(ik\cdot x)} + a_- \exp{(-ik\cdot x)}\,. \tag{3.1b}$$

As explained in Ref. [3], $a_\pm$ are the "complex" amplitudes for right and left circular polarization. Here "complex" means "having only scalar and pseudoscalar parts," i.e.,

$$\alpha_\pm = [\,\alpha_\pm\,]_0 + [\,\alpha_\pm\,]_4 = \rho_\pm \exp{(\pm i\delta_\pm)} \tag{3.1c}$$

where $\delta_\pm$, and $\rho_\pm > 0$ are scalars. In contrast to the usual use of complex numbers in electromagnetic theory, the "unit imaginary" i, being the unit pseudoscalar, has a definite geometrical significance. Maxwell's equation $\Box F = 0$ implies, since $\Box k\cdot x = k$,

$$kf = 0\,, \qquad \text{or equivalently,} \qquad kF = 0\,. \tag{3.1d}$$

Multiplying by k, one ascertains that

$$k^2 = 0\,. \tag{3.1e}$$

It can be shown further that f must have the form

$$f = ka = k\wedge a = -ak\,, \tag{3.1f}$$

where a is a unit spacelike vector orthogonal to k. When $\alpha_\pm$ have been specified, a is uniquely determined, but a rotation of a preserving $k\cdot a = 0$ can be compensated by an overall phase change of α_+ and α_- (corresponding to a gauge transformation of the electromagnetic vector potential), so to this extent factorization of F into f and z is not unique.

Before the spinor Lorentz force $\dot{R} = \frac{1}{2}\lambda FR$ can be integrated, it is necessary to express $F = F(x)$ as a function of τ. This can be done by using special properties of F to find constants of motion. Using (3.1d), one finds

$$\frac{d}{d\tau}(kR) = k\dot{R} = \tfrac{1}{2}\lambda kFR = 0\,, \tag{3.2}$$

that is, kR is a constant of motion. So, using the initial condition $R(0) = 1$, one finds

$$k = kR = Rk = k\widetilde{R}\,. \tag{3.3}$$

The second equality in (3.3) follows from the first by reversion: $Rk = R\widetilde{k} = R(\widetilde{R}\widetilde{k}) = k$. From (3.3) it follows that

$$Rk\widetilde{R} = k\,. \tag{3.4}$$

Therefore, $R = R(\tau)$ is a family of Lorentz rotations leaving the lightlike vector k invariant. Multiplying $e_\mu = R\gamma_\mu\widetilde{R}$ by k, Eq. (3.4) gives constants of motion for the e_μ:

$$k\cdot e_\mu = k\cdot\gamma_\mu\,; \tag{3.5}$$

in particular,

$$k\cdot v = k\cdot v_0\,. \tag{3.6}$$

Since $v = dx/d\tau$, this integrates to

$$k\cdot(x(\tau) - x_0) = k\cdot v_0\tau\,. \tag{3.7}$$

This is precisely the relation needed to express the electromagnetic field acting on the particle as a function of the proper time. Substituting (3.7) into (3.1b), one obtains

$$z = z(\tau) = \alpha_+ \exp{(i\omega_0\tau)} + a_- \exp{(i\omega_0\tau)}\,, \tag{3.8}$$

where $\omega_0 = k\cdot v_0$ is the frequency of the plane wave relative to an observer with proper velocity v_0, and an overall phase $\delta_0 = k\cdot x_0$ has been absorbed into the phases of α_+ and α_- [or equivalently into the definition of a in (3.1f)])

Now, by (3.1) and (3.3), the spinor Lorentz force for a plane wave has the form

$$\frac{dR}{d\tau} = \tfrac{1}{2}\lambda FR = \tfrac{1}{2}\lambda F = \tfrac{1}{2}\lambda fz\,. \tag{3.9}$$

With the initial $R(0) = 1$ and the expression (3.8) for $z = z(\tau)$, this integrates immediately to

$$R = 1 + \tfrac{1}{2}\lambda fz_1 = \exp{(\tfrac{1}{2}\lambda fz_1)}\,, \tag{3.10a}$$

where

$$z_1 \equiv \int_0^\tau z(\tau)d\tau = \frac{2}{\omega_0}\sin{(\tfrac{1}{2}\omega_0\tau)}\big[\,\alpha_+\exp{(\tfrac{1}{2}i\omega_0\tau)} - a_-\exp{(-\tfrac{1}{2}i\omega_0\tau)}\,\big]\,. \tag{3.10b}$$

Hence the expression for the comoving frame as a function of τ is

$$\begin{aligned} e_\mu &= R\gamma_\mu\widetilde{R} = (1 + \tfrac{1}{2}\lambda fz_1)\gamma_\mu(1 - \tfrac{1}{2}\lambda fz_1) \\ &= \gamma_\mu + \lambda\tfrac{1}{2}(fz_1\gamma_\mu - \gamma_\mu fz_1) - \lambda^2\tfrac{1}{4}fz_1\gamma_\mu fz_1\,, \end{aligned}$$

or, since $z_1\gamma_\mu = z_1^*\gamma_\mu$,

$$e_\mu = \gamma_\mu + \lambda(fz_1)\cdot\gamma_\mu - \lambda^2\Theta_1\tfrac{1}{2}f\gamma_\mu f\,, \tag{3.11a}$$

where, recalling (3.1c) and writing $\delta = \delta_+ + \delta_-$,

$$\Theta_1 \equiv \tfrac{1}{2}|\, z_1 \,|^2 = \frac{\sin^2 \frac{1}{2}\omega_0\tau}{2\omega_0^2}\,[\,\rho_+^2 + \rho_-^2 - 2\rho_+\rho_- \cos{(\omega_0\tau + \delta)}\,]\,. \tag{3.11b}$$

Notice that in (3.11a) $-\frac{1}{2}f\gamma_\mu f = \frac{1}{2}f(-f\gamma_\mu + f\cdot\gamma_\mu) = ff\cdot\gamma_\mu$, which is proportional to the component of γ_μ in the f plane.

According to (3.11a), the equation for the proper velocity is

$$\frac{dx}{d\tau} = v_0 + \lambda(fz_1)\cdot v_0 + \lambda^2\Theta_2 ff\cdot v_0\,, \tag{3.12}$$

which integrates to a parametric equation for the particle history

$$x(\tau) - x_0 = v_0\tau + \lambda[\,fz_2\,]\cdot v_0 + \lambda^2\Theta_2 ff\cdot v_0\,, \tag{3.13a}$$

where

$$z_2 \equiv \int_0^\tau z_1(\tau)\,d\tau = -\frac{z}{\omega_0^2} + \frac{(\alpha_+ + \alpha_-)}{\omega_0^2} + \frac{(\alpha_+ - \alpha_-)}{i\omega_0} \tag{3.13b}$$

and

$$\Theta_2 \equiv \int_0^\tau \Theta_1(\tau)\,d\tau = \frac{1}{\omega_0^2}\Big((\rho_+^2 + \rho_-^2 + 2\rho_+\rho_-)\,\frac{\sin\omega_0\tau}{\omega_0} - \rho_+\rho_-\frac{\sin(2\omega_0\tau+\delta)}{2\omega_0}$$
$$+(\rho_+^2 + \rho_-^2 + 2\rho_+\rho_-\cos\delta)\tau + \frac{3}{2\omega_0}\,\sin\,\delta\Big)\,. \tag{3.13c}$$

This completes the explicit solution of a rigid charge in a plane wave. If desired, the equations can be put in relative form by the method illustrated in the last section.

4. RIGID TEST CHARGE IN A COULOMB FIELD

The spinor Lorentz force will now be integrated to describe the motion of a test charge e in the Coulomb field of a "fixed nucleus" with charge $-Ze$. Let u be the constant proper velocity of the nucleus. In terms of relative variables the Coulomb field is

$$\lambda F = \frac{e}{mc^2}\,\mathbf{E} = -k\,\frac{\mathbf{x}}{|\,x\,|^2} = \nabla\frac{k}{|\,\mathbf{x}\,|}\,, \tag{4.1a}$$

where

$$k = Ze\lambda/4\pi = Ze^2/4\pi mc^2\,. \tag{4.1b}$$

In terms of proper variables the Coulomb field is

$$\lambda F = -k\frac{x\wedge u}{|\,x\wedge u\,|^3} = \Box\left(\frac{-ku}{|\,x\wedge u\,|}\right) = u\wedge\Box\left(\frac{k}{|\,x\wedge u\,|}\right)\,, \tag{4.1c}$$

where, of course, $x = x(\tau)$ is the position of the test particle at time τ and the origin $x = 0$ has been located at some point on the history of the nucleus.

Before the spinor equation $\dot{R} = \frac{1}{2}\lambda F R$ can be integrated, it is necessary to express F as a parametric function of the particle history. This can be done by reexpressing symmetry properties of F in terms of constants of motion. The constants of motion can be found by multiplying F by the available vectors x, u, v and using the Lorentz force.

"Dotting" (4.1c) by u, one finds, with the help of (I.1.5),

$$\lambda u \cdot F = (\Box - uu \cdot \Box)\,\frac{k}{|\,x \wedge u\,|} = \Box\left(\frac{k}{|\,x \wedge u\,|}\right), \tag{4.2}$$

since $u \cdot \Box |\,x \wedge u\,|^{-1} = c^{-1}\partial_t |\,x\,|^{-1} = 0$. So, dotting the Lorentz force $\dot{v} = \lambda F \cdot v$ by u, one finds

$$\frac{d}{d\tau}\,(u \cdot v) = u \cdot \dot{v} = \lambda u \cdot F \cdot v = v \cdot \Box\left(\frac{k}{|\,x \wedge u\,|}\right) = \frac{d}{d\tau}\left(\frac{k}{|\,x \wedge u\,|}\right).$$

Hence,

$$W \equiv u \cdot v - k/|\,x \wedge u\,| = \gamma - k/|\,\mathbf{x}\,| \tag{4.3}$$

is a constant of motion. The sum of particle kinetic and potential energies is $E = mc^2(W-1) = mc^2(\gamma - 1) - Ze^2/4\pi|\,\mathbf{x}\,|$.

From (4.1c) it follows that the Coulomb field (in fact any central field) has the properties

$$F \wedge u = 0\,, \tag{4.4a}$$

$$F \wedge x = 0\,. \tag{4.4b}$$

By virtue of (I.1.8), it follows that $(F \wedge u) \cdot v = F u \cdot v - (F \cdot v) \wedge u = 0$, which, on substituting $\lambda F \cdot v = \dot{v}$, gives

$$\frac{d}{d\tau}\,(v \wedge u) = \lambda F v \cdot u\,. \tag{4.5a}$$

In terms of relative variables, this is just the usual equation for an electric force on a particle, i.e.,

$$\frac{d}{d\tau}\,(\gamma \mathbf{v}) = c^2 \lambda \mathbf{E} = \frac{e}{m}\,\mathbf{E}\,. \tag{4.5b}$$

Now applying (4.4b) to (4.5a), one finds

$$\frac{d}{d\tau}\,(v \wedge u \wedge x) = 0\,, \tag{4.6}$$

hence the dual of the trivector $v \wedge u \wedge x$,

$$l = i\,v \wedge u \wedge x\,, \tag{4.7}$$

is also a constant of motion. Using the general duality relation $(iT)\cdot x = iT\wedge x$, one obtains immediately from (4.7)

$$l\cdot x = l\cdot u = l\cdot v = 0\,. \tag{4.8}$$

In terms of the proper vector l, one can define a relative vector $\mathbf{l} = l\wedge u$ which is obviously also a constant of motion. From (4.8) and (4.7)

$$\mathbf{l} \equiv l\wedge u = lu = -i(v\wedge x\wedge u)\cdot u = -i[\,v\wedge x\,]_2\,, \tag{4.9a}$$

Since

$$\begin{aligned} vx &= (vu)(ux) = \gamma(1+\mathbf{v}/c)(ct-\mathbf{x}) \\ &= \gamma\Big[\,ct - \mathbf{x}\cdot\Big(\frac{\mathbf{v}}{c}\Big)\,\Big] + \gamma(\mathbf{v}t-\mathbf{x}) - \frac{\gamma}{c}\,\mathbf{v}\wedge\mathbf{x}\,, \end{aligned}$$

so $[\,v\wedge x\,]_2 = c^{-1}\gamma\mathbf{x}\wedge\mathbf{v} = c^{-1}\gamma i\mathbf{x}\times\mathbf{v}$, and (4.9a) yields

$$\mathbf{l} \equiv \frac{\gamma}{c}\,\mathbf{x}\times\mathbf{v} = \mathbf{x}\times\frac{d\mathbf{x}}{dt} = \frac{\mathbf{x}\times\mathbf{p}}{mc}\,. \tag{4.9b}$$

Thus $\mathbf{l}$ is the usual (relative) angular momentum per unit mc.

The constants of motion have been found; the problem now is to use them effectively. From (4.9b) one finds

$$\mathbf{l}\cdot\mathbf{v} = \mathbf{l}\cdot\mathbf{x} = 0\,, \tag{4.10}$$

which says that the relative motion is in a plane orthogonal to $\mathbf{l}$. The unit bivector $i\hat{l}$ is the generator of rotations in that plane. Hence one can write

$$\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\,\mathbf{x}\,|} = \mathbf{a}\exp{(i\hat{l}\Theta)}\,, \tag{4.11a}$$

where $\mathbf{a}\cdot\mathbf{l} = 0$ and $\Theta(\tau)$ is the angle of rotation of a fixed unit vector $\mathbf{a}$ into the direction $\hat{\mathbf{l}}$. The requirement $\dot{\Theta} = d\Theta/d\tau > 0$ entails that the rotation has the same "sense" as the particle motion. The sense of the rotation is described by the vector

$$\frac{d\hat{\mathbf{x}}}{d\Theta} = \mathbf{b}\exp{(i\hat{\mathbf{l}}\Theta)}\,, \tag{4.11b}$$

where

$$\mathbf{b} = \mathbf{a}i\hat{\mathbf{l}} = \mathbf{a}\cdot(i\hat{\mathbf{l}}) = i\,\mathbf{a}\wedge\hat{\mathbf{l}} = \hat{\mathbf{l}}\times\mathbf{a}\,. \tag{4.11c}$$

Thus, the vectors $\mathbf{a}$, $\mathbf{b}$, $\hat{\mathbf{l}}$ form a right-handed orthonormal frame. So do the vectors $\mathbf{x}$, $d\hat{\mathbf{x}}/d\Theta$, $\hat{\mathbf{l}}$, since

$$\hat{\mathbf{x}}\,\frac{d\hat{\mathbf{x}}}{d\Theta} = \mathbf{a}\exp{(i\hat{\mathbf{l}}\Theta)}\,\mathbf{b}\exp{(i\hat{\mathbf{l}}\Theta)} = \mathbf{ab} = i\hat{\mathbf{l}} = i\hat{\mathbf{x}}\times\frac{d\hat{\mathbf{x}}}{d\Theta}\,. \tag{4.11d}$$

Conservation of $\mathbf{l}$ implies that the direction and the magnitude of $\mathbf{l}$ are conserved separately. The implication of the directional conservation has been expressed by

(4.11). The implication of the magnitude conservation can be obtained by substituting

$$\frac{\gamma\mathbf{v}}{c} = \frac{d\mathbf{x}}{d\tau} = \frac{\Theta d\mathbf{x}}{d\Theta} = \Theta\left(\mathbf{x}\frac{d\,|\,\mathbf{x}\,|}{d\Theta} + |\,\mathbf{x}\,|\,\frac{d\hat{\mathbf{x}}}{d\Theta}\right) \tag{4.12}$$

into (4.9b) and using (4.11d); thus

$$\mathbf{l} = |\,\mathbf{x}\,|^2\dot{\Theta}\,\hat{\mathbf{x}} \times \frac{d\hat{\mathbf{x}}}{d\Theta}$$

or

$$|\,\mathbf{l}\,| = |\,\mathbf{x}\,|^2\dot{\Theta}\,. \tag{4.13}$$

Now, with (4.13) and (4.11a), the Coulomb field (4.1a) can be put in the simple parametric form

$$\lambda F = -\kappa\mathbf{x}\dot{\Theta} = -\kappa\mathbf{a}\exp{(i\hat{\mathbf{l}}\Theta)}\dot{\Theta} \tag{4.14a}$$

where

$$\kappa = k/|\,\mathbf{l}\,| = Ze^2/4\pi mc\,|\,\mathbf{l}\,|\,. \tag{4.14b}$$

Hence on changing variables from τ to Θ, the spinor equation $\dot{R} = \frac{1}{2}\lambda FR$ assumes the simple form

$$\frac{dR}{d\Theta} = -\frac{\kappa}{2}\,\hat{\mathbf{x}}R = -\frac{\kappa}{2}\,\mathbf{a}\exp{(i\hat{\mathbf{l}}\Theta)}R\,. \tag{4.15}$$

Of course (4.14) and (4.15) assume $|\,\mathbf{l}\,| \neq 0$; the case $\mathbf{l} = 0$ is easily integrated separately, since then the direction of the field is a constant of motion.

To solve (4.15), guess that the solution has the general form

$$R = \exp{(-B\Theta/2)}\exp{(-A\Theta/2)}R_0 \tag{4.16a}$$

where B, A, and R_0 are independent of Θ, and, to satisfy the conditions that R be even and $R\widetilde{R} = 1$, B and A must be proper bivectors and $R_0\widetilde{R}_0 = 1$. It may be noted that no generality is gained by adding "phases" to the angles in (4.16a) since they can be "absorbed" in the definitions of the constants A, B, R_0. Substituting (4.16a) into (4.15) one obtains conditions on A and B; thus

$$\begin{aligned} -\kappa\mathbf{a}\exp{(i\hat{\mathbf{l}}\Theta)} \;&=\; 2\,\frac{dR}{d\Theta}\widetilde{R} = -B - \exp{(-B\Theta/2)}A\exp{(B\Theta/2)} \\ &=\; -B - A_+ - A_-\exp{(B\Theta)}\,, \end{aligned}$$

where to carry out the last step, A has been expressed as the sum of a part A_+ which commutes with B and a part A_- which anticommutes with B. Equating independent parts of the equation, one finds

$$B = i\hat{\mathbf{l}}\,, \tag{4.16b}$$

$$A = \kappa\mathbf{a} - i\hat{\mathbf{l}}\,. \tag{4.16c}$$

Hence (4.16a) subject to (4.16b,c) is a general solution of (4.15). The form of this solution is peculiar to the Coulomb field and does not apply to any other central field.

The "initial value" R_0 of the *Coulomb spinor* (4.16a) can be written $R_0 = L_0 U_0$ where L_0 determines a boost and U_0 a spatial rotation. By an appropriate choice of the initial conditions for the comoving frame, the spinor U_0 can be set equal to unity. The spinor L_0 is determined from the velocity v_0 at $\Theta = 0$ by the equation

$$v_0 = R_0 u \widetilde{R}_0 = L_0 u \widetilde{L}_0 = L_0^2 u$$

or

$$v_0 u = L_0^2 = \gamma_0 \left(1 + \frac{\mathbf{v}_0}{c}\right) \qquad \text{where} \qquad \gamma_0 = (1 - v_0^2/c^2)^{-1/2}\,. \tag{4.17}$$

[The use of the symbol γ_0 in (4.17) should not be confused with the use of the same symbol to represent a vector elsewhere in this paper.] According to (4.10) $\mathbf{l} \cdot \mathbf{v}_0 = 0$, although it is not necessary, it is convenient to require also $\mathbf{a} \cdot \mathbf{v}_0 = \hat{\mathbf{x}}_0 \cdot \mathbf{v}_0 = 0$, so by (4.11)

$$\mathbf{v}_0 = |\,\mathbf{v}_0\,|\,\mathbf{b}\,. \tag{4.18}$$

This eliminates previous arbitrariness in the choice of the direction $\mathbf{a}$ and the zero for Θ. The constants of motion $|\,\mathbf{l}\,|$ and W can be expressed in terms of the initial values $|\,\mathbf{v}_0\,|$ and $|\,\mathbf{x}_0\,|$ and vice versa. Because of (4.18), (4.9), and (4.11) imply

$$|\,\mathbf{l}\,| = |\,\mathbf{x}_0\,|\,\frac{|\,d\mathbf{x}_0\,|}{d\tau} = |\,\mathbf{x}_0\,|\,\gamma_0\,\frac{|\,\mathbf{v}_0\,|}{c}\,. \tag{4.19a}$$

Using this in (4.3) one obtains

$$W = \gamma_0 \left(1 - \kappa\,\frac{\mathbf{v}_0}{c}\right) = \gamma_0^{-1} - \kappa(\gamma_0^2 - 1)^{1/2}\,. \tag{4.19b}$$

Solving (4.19b) for $|\,\mathbf{v}_0\,|$ and γ_0, one obtains

$$\frac{|\,\mathbf{v}_0\,|}{c} = \frac{\kappa \pm W\sqrt{W^2 + \kappa^2 - 1}}{W^2 + \kappa^2} \tag{4.20a}$$

$$\gamma_0 = \frac{W \pm \kappa\sqrt{W^2 + \kappa^2 - 1}}{1 - \kappa^2} \tag{4.20b}$$

The physical roots must, of course, satisfy the condition $0 < |\,\mathbf{v}_0\,| < c$.

The Coulomb spinor (4.16a) gives immediately the explicit expression for the particle proper velocity

$$v = R u \widetilde{R} = \exp(-B\Theta/2)\exp(-A\Theta/2) v_0 \exp(A\Theta/2)\exp(B\Theta/2)\,. \tag{4.21}$$

Three classes of motion can be distinguished: when A^2 is zero, positive or negative. If $A^2 = 0$ then $\exp \frac{1}{2}A\Theta = 1 + \frac{1}{2}A\Theta$ and the motion is most easily analyzed by substituting this in (4.21). If $A^2 \neq 0$ the motion is most easily analyzed by decomposing v_0 into a component

$$v_{0\parallel} = \frac{v_0 \cdot A\, A}{A^2} \tag{4.22a}$$

in the A plane and a component

$$v_{0\perp} = \frac{v_0 \wedge A\, A}{A^2} = -\frac{v_0 \cdot (iA) iA}{A^2} \tag{4.22b}$$

orthogonal to the A plane, so (4.21) becomes

$$v = \exp(-B\Theta/2)\big(v_{0\perp} + v_{0\parallel} \exp(-A\Theta)\big) \exp(B\Theta/2)\,. \tag{4.23}$$

From (4.16c) one finds $A^2 = \kappa^2 - 1$. If $\kappa^2 > 1$, then $\exp A\Theta = \cosh|\,A\,|\Theta + |\,A\,|^{-1} A \sinh|\,A\,|\,\Theta$ where $|\,A\,| = (\kappa^2 - 1)^{1/2}$; this is characteristic of the "scattering states" of the particle. If $\kappa^2 < 1$, then

$$\exp(A\Theta) = \cos\,|\,A\,|\,\Theta + \widehat{A} \sin\,|\,A\,|\,\Theta\,, \tag{4.24}$$

where

$$|\,A\,| = (1 - \kappa^2)^{1/2} \qquad \text{where} \qquad \widehat{A} = |\,A\,|^{-1} A\,. \tag{4.25}$$

This is a necessary (but not sufficient) condition for "bound states" of the particle. In the following, this case will be studied in more detail.

To find an expression for the relative velocity of the particle, substitute (4.24) into (4.23) and note that by (4.16b) that commutes with B, so

$$\begin{aligned} vu \;=\; \gamma(1 + \mathbf{v}/c) = \exp(-B\Theta/2)\Big[\, & v_0 + v_0 \cos\,|\,A\,|\,\Theta \\ & + \frac{\mathbf{v}_0 \cdot A}{|\,A\,|} \sin\,|\,A\,|\,\Theta\Big] u \exp(-B\Theta/2)\,. \end{aligned} \tag{4.26}$$

The terms involving v_0 can be put in relative form with the help of the general formula (1.2.15); thus, since $A = \kappa\mathbf{a} - i\hat{\mathbf{l}}$ and $v_0 u = \gamma_0(1 + \mathbf{b}|\,\mathbf{v}_0\,|/c)$ where $\mathbf{ab} = i\hat{\mathbf{l}}$, one obtains

$$\begin{aligned} v_0 \cdot A\, u \;&=\; -A \cdot v_0 u = -\gamma_0 \left(\kappa \frac{\mathbf{a} \cdot \mathbf{v}_0}{c} + \kappa\mathbf{a} - i\hat{\mathbf{l}}\, \frac{\mathbf{v}_0}{c}\right) \\ &=\; \gamma_0 \left(-\kappa + \kappa \frac{\mathbf{v}_0}{c}\right) \mathbf{a} = \pm\big(W^2 - |\,A\,|^2\big)^{1/2} \mathbf{a}\,, \end{aligned} \tag{4.27}$$

where in the last step (4.19b) and (4.20b) were used convert initial values to constants of motion. The two signs in (4.27) correspond to an arbitrariness in the choice of orientation of $\mathbf{a}$ and $\mathbf{b}$. It is convenient to choose the *positive* sign. Repeating the procedure which lead to (4.27) with $iA = \hat{\mathbf{l}} + i\kappa\mathbf{a}$ instead of A, one obtains

$$(iA) \cdot v_0 u = \gamma_0 \left(\mathbf{l} + i\kappa\mathbf{ab}\, \frac{|\,\mathbf{v}_0\,|}{c}\right) = \gamma_0 \left(1 - \kappa\, \frac{|\,\mathbf{v}_0\,|}{c}\right) \hat{\mathbf{l}} = W\hat{\mathbf{l}}\,. \tag{4.28}$$

From (4.27) one gets

$$\begin{aligned} v_{0\|}u &= \frac{A(A\cdot v_0)u}{-|\,A\,|^2} = \frac{(W^2-|\,A\,|^2)^{1/2}}{|\,A\,|^2}\,(\kappa\mathbf{a}-i\hat{\mathbf{l}})\mathbf{a} \\ &= \frac{(W^2-|\,A\,|^2)^{1/2}}{|\,A\,|^2}\,(\kappa+\mathbf{b}) \end{aligned} \tag{4.29}$$

and from (4.28)

$$v_{0\perp}u = \frac{-iA(iA)\cdot v_0}{-|\,A\,|^2} = \frac{W}{|\,A\,|^2}\,(\hat{\mathbf{l}}+i\kappa\mathbf{a})\hat{\mathbf{l}} = \frac{W}{|\,A\,|^2}\,(1+\kappa\mathbf{b})\,. \tag{4.30}$$

Substituting (4.27, 4.29, 4.30) into (4.26), one has

$$\begin{aligned} \gamma\left(1+\frac{\mathbf{v}}{c}\right) = \exp\left(-B\Theta/2\right)\Bigg(&\frac{W}{|\,A\,|^2}\,(1+\kappa\mathbf{b}) + \frac{(W^2-|\,A\,|^2)^{1/2}}{|\,A\,|^2}\,(\kappa+\mathbf{b})\cos|\,A\,|\Theta \\ &+\frac{(W^2-|\,A\,|^2)^{1/2}}{|\,A\,|}\,\mathbf{a}\sin|\,A\,|\,\Theta\Bigg)\exp\left(B\Theta/2\right). \end{aligned} \tag{4.31}$$

The scalar part of (4.31) is

$$\gamma = \frac{1}{|\,A\,|^2}\left[\,W+\kappa(W^2-|\,A\,|^2)^{1/2}\cos|\,A\,|\,\Theta)\,\right], \tag{4.32}$$

while the vector part of (4.31) is

$$\begin{aligned} \frac{d\mathbf{x}}{d\tau} = \frac{\gamma\mathbf{v}}{c} = \Bigg[&\frac{\mathbf{b}}{|\,A\,|^2}\Big(\kappa W + (W^2-|\,A\,|^2)^{1/2}\cos|\,A\,|\,\Theta\Big) \\ &+\mathbf{a}\,\frac{(W^2-|\,A\,|^2)^{1/2}}{|\,A\,|}\sin|\,A\,|\,\Theta\Bigg]\exp\left(B\Theta\right). \end{aligned} \tag{4.33}$$

This is the desired equation for the relative velocity.

An equation for the orbit of the particle can be obtained immediately from the radial component of (4.33) without integration. Using (4.11) and (4.13) in (4.12), one gets

$$\frac{d\mathbf{x}}{d\tau} = |\,\mathbf{l}\,|\left[\,\frac{1}{|\,\mathbf{x}\,|}\,\mathbf{b}-\mathbf{a}\,\frac{d}{d\Theta}\left(\frac{1}{|\,\mathbf{x}\,|}\right)\right]\exp\left(B\Theta\right). \tag{4.34}$$

Equating (4.34) to (4.33), one obtains from the radial component

$$\frac{|\,\mathbf{l}\,|\,|\,\mathbf{A}\,|^2}{|\,\mathbf{x}\,|} = \kappa W + (W^2-|\,A\,|^2)^{1/2}\cos|\,A\,|\,\Theta\,. \tag{4.35}$$

This is the well-known equation for a precessing ellipse derived long ago by Sommerfeld.

When $\kappa < 1$, the bivector $-A = -\kappa\mathbf{a} + i\hat{\mathbf{l}}$ can be obtained from $B = i\hat{\mathbf{l}}$ by a boost. Thus, as in (2.29) and (2.30),

$$-A = -\kappa A + i\hat{\mathbf{l}} = |\,A\,|Ki\hat{\mathbf{l}}\widetilde{K} = |\,A\,|K^{-2}i\hat{\mathbf{l}}\,. \tag{4.36}$$

So

$$K^2 = |\,A\,|^{-1}Ai\hat{\mathbf{l}} = |\,A\,|^{-1}(1+\kappa\mathbf{a}i\hat{\mathbf{l}})$$

or, by (4.11c),

$$K^2 = |\,A\,|^{-1}(1+\kappa\mathbf{b})\,. \tag{4.37}$$

Notice that K produces a boost in the direction of the initial velocity $\mathbf{v}_0 = |\,\mathbf{v}_0\,|\mathbf{b}$. Indeed, from (4.34) it is obvious that the orbit is circular if $W^2 = |\,A\,|^2$, and according to (4.27) this is equivalent to the condition $\kappa = |\,\mathbf{v}_0\,|/c$, which implies that K is equal to the initial boost L_0 in (4.17). Using (4.36) the Coulomb spinor, (4.16a) can be put in the form

$$R = \exp{(-B\Theta/2)}K\exp{(B|\,A\,|\Theta/2)}\,\widetilde{K}L_0\,, \tag{4.38}$$

which, for circular motion, reduces to

$$R = \exp{(-B\Theta/2)}\,K\exp{(B\,|\,A\,|\,\Theta/2)} = K'\exp{[\,-(1-|\,A\,|)B\Theta/2\,]}\,, \tag{4.39}$$

where

$$K' \equiv \exp{(-B\Theta/2)}K\exp{(B\Theta/2)}\,.$$

The right side of (4.34) displays R factored into a boost by K' preceded by a spatial rotation through an angle $(1-|\,A\,|)$, which evaluated for a period gives the Thomas precession immediately. The Thomas precession for arbitrary angular momentum can be obtained algebraically by factoring $\exp{(-B\Theta/2)}\exp{(-A\Theta/2)}$ into a boost preceded by a spatial rotation.

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