

Observables, operators, and complex numbers in the Dirac theory

David Hestenes

Abstract The geometric formulation of the Dirac theory with space-time algebra is shown to be equivalent to the usual matrix formalism. Imaginary numbers in the Dirac theory are shown to be related to the spin tensor. The relation of observables to operators and the wavefunction is analyzed in detail and compared with some purportedly general principles of quantum mechanics. An *exact formulation of Larmor and Thomas precessions in the Dirac theory is given for the first time*. Finally, some basic relations among local observables in the nonrelativistic limit are determined.

Introduction

The Dirac electron theory is widely acknowledged to be the most precise available quantum theory of a single particle. Yet the principles and interpretation of quantum theory continue to be discussed at great length in the literature without taking the Dirac theory into account. Though the mathematical formalism of the Dirac theory unfortunately does not uniquely determine its interpretation, one should expect a reasonable interpretation to provide some understanding of the mathematical structure of the theory. On this score many widely proclaimed general principles of quantum theory prove to be profoundly deficient, if not substantially wrong. For example, if the uncertainty principle is basic to the interpretation of quantum mechanics, why has it never been given a clear formulation in the full Dirac theory? And if Planck's constant provides an absolute limit on the precision of measurements, why does it also determine the magnitude of the spin?

This paper aims to clarify the geometrical and physical interpretation of the Dirac theory and determine some of its implications for the interpretation of quantum theory in general. With a formalism developed in Refs. 1, 2, 3 the Dirac theory can be given a completely geometrical formulation, involving neither matrices nor complex numbers. In the process, the following significant facts are established concerning the interpretation of the theory: (1) The four gamma matrices play a completely geometrical role in the Dirac theory. They are matrix representations of four orthonormal vectors (not four matrix components of a single vector as often suggested). The algebra of gamma matrices has the same geometrical significance as tensor algebra. The gamma matrices have no function in the Dirac theory which entitles them to be regarded as operators with any special quantum-mechanical significance. (2) The unit imaginary $i' = (-1)^{1/2}$ in the customary formulation of the Dirac has a definite geometrical and physical significance. It is a superfluous adjunct of the mathematical formalism in the sense that it can be replaced by the tensor quantity it actually represents. Specifically, the quantity $\frac{1}{2}i'\hbar$, wherever it appears in equations of the Dirac theory, is a representation of the spin tensor by one of its eigenvalues. When $\frac{1}{2}i'\hbar$ has been replaced in the formalism by the spin bivector it represents, antiparticle conjugation can be identified as a *geometrical* transformation of observables of the theory. (3) The Dirac equation and the wavefunction can be eliminated from the Dirac theory, leaving a set of conservation laws and constitutive relations for observables.

The above facts and further details have already been established in Refs. 2 and 3. However, these papers are open to misunderstanding, because they do not explain in detail exactly how they are related to the conventional matrix formulation of the Dirac theory. This paper aims to supply the missing details and make it easy to translate expressions of the matrix formulation into the geometrical language and vice-versa. The first few sections are devoted to this end, with emphasis on establishing the first two facts cited above.

Section 3 derives the geometrical form of the Dirac equation, which involves no complex numbers.

Sections 4 and 5 analyze the relation of the wave function to observables in detail. In Sec. 5 some common notions about a correspondence between observables and operators are criticized in the light of the Dirac theory. Correspondences of observables with the wavefunction are held to be fundamental. It is suggested that the significance of eigenvalues in quantum theory is to be found more in their association with constant and homogeneous local observables than with operators.

Section 6 introduces the fundamental relative observables and analyzes the physical interpretation of the energy density in considerable detail. For the first time expressions for the Larmor and Thomas precession energies are derived from the Dirac theory without any approximations.

Section 7 proves that in the nonrelativistic limit the Gordon current is equal to the local momentum density but differs from the charge current by the magnetization current associated with the spin density.

2. Geometric significance of the Dirac matrices

Recall some of the fundamental properties of the Dirac matrices (as developed, for example in Ref. 4). The Dirac matrices are defined as a set of irreducible matrices γ_μ , which satisfy the anticommutation rules

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} I, \quad (2.1a)$$

where the $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) are components of the spacetime metric tensor and I is the unit matrix. Since it can be proved that the γ_μ must be 4×4 matrices, their irreducibility can be expressed by the condition that the trace of (2.1a) gives

$$\frac{1}{4} \text{Tr} \gamma_\mu \gamma_\nu = g_{\mu\nu}. \quad (2.1b)$$

Conditions (2.1) can be satisfied only by traceless matrices, that is,

$$\frac{1}{4} \text{Tr} \gamma_\mu = 0 \quad (2.2)$$

Conditions (2.1) do not determine the Dirac matrices uniquely. However, it can be shown that any two sets of Dirac matrices $\{\gamma_\mu\}$ and $\{\gamma'_\mu\}$ are related by a similarity transformation, that is,

$$\gamma'_\mu = S \gamma_\mu S^{-1}, \quad (2.3)$$

where S is a nonsingular matrix. The γ_μ over the complex numbers generate the complete algebra of 4×4 matrices, But certain elements of the algebra are distinguished by their association (2.1) with the spacetime algebra, and this gives the entire algebra a geometric significance. The 4×4 matrix algebra with the geometric interpretation induced by the conditions (2.1) is called the *Dirac Algebra*.

Clearly, (2.1) indirectly assigns some geometric significance to the Dirac matrices themselves. But in the usual approach the full geometric significance of the γ_μ is not determined until their relation to a Dirac spinor has been specified. Equivalent results can be achieved more efficiently by a change in viewpoint, To see how, it is only necessary to realize that the key relation (2.1a) does not depend in any essential way on the assumption that the γ_μ are matrices. All that is required is that the γ_μ belong to an associative noncommutative algebra.

The appropriate change in viewpoint is achieved simply by interpreting the γ_μ as vectors of a spacetime frame instead of as matrices. By definition the scalar products $\gamma_\mu \cdot \gamma_\nu$ of these vectors are just the components $g_{\mu\nu}$ of the metric tensor. So the two equations (2.1a) and (2.1b) for matrices correspond to the single equation

$$\frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \gamma_\mu \cdot \gamma_\nu \equiv g_{\mu\nu} I, \quad (2.4)$$

for vectors. The vectors γ_μ generate an associative algebra over the reals which has been dubbed the spacetime algebra in Ref. 1, because it provides a direct and complete algebraic characterization of the geometric properties of Minkowski spacetime.

By providing a geometrical interpretation of the γ_μ which does not depend on the notion of spinor, geometric algebra releases the γ_μ from their confines in relativistic quantum theory and elevates them to a central position in the mathematical description of all physical systems in spacetime. The advantages of using spacetime algebra in place of the usual tensor methods have been adequately demonstrated in Refs. 1, 5, and 6, to which the reader is referred for details. Of course any equation in the spacetime algebra can be reexpressed as an equation in the Dirac algebra, but besides having a more direct geometric interpretation, the spacetime algebra is mathematically more efficient, as should already be evident from a comparison of (2.1) with (2.4), This becomes clearer as manipulations with the two algebras are compared in more detail.

Since the Dirac matrices in (2.1) can obviously be regarded as matrix representations of the vectors in (2.3). it is convenient to persist in using the same symbols γ_μ for both to facilitate comparison of the algebras and translation of equations from one system to the other. For the same reason, it is convenient to establish a correspondence between the nomenclatures of the two systems. The elements of the spacetime algebra are called *multivectors* and every such element can be expressed as a polynomial of the γ_μ over the reals, which has at most 16 linearly independent elements (see Refs. 1 and 5 for more details). A multivector is said to be *even* (*odd*) if it does (does not) change sign on replacement of γ_μ by $-\gamma_\mu$ in its polynomial representation. The same terminology will be applied here to matrix representations of multivectors. Of course, the Dirac algebra admits polynomials of the γ_μ over the complex numbers, but, in contrast to other elements of the algebra, the unit imaginary of the complex field has no geometric significance except, as will be shown in the next section, in connection with spinors.

It may be worth pointing out that a $(-1)^{1/2}$ is commonly introduced as a ‘scalar’ in the Dirac theory in two distinct ways, first as a root of negative terms in $g_{\mu\nu}$ and second as an essential element of the Dirac equation. Geometric algebra shows that $(-1)^{1/2}$ has a different geometrical significance in each case, and it provides the machinery to keep the distinction clear. Thus, Eq. (2.4) gives $\gamma_\mu^2 = g_{\mu\nu}$ where $\gamma_\mu = (g_{\mu\mu})^{1/2}$ (no sum on μ). More particularly, $\gamma_0^2 = 1$ has the timelike vector $\gamma_0 = 1^{1/2}$ as a “root of unity,” while $\gamma_1^2 = -1$ has the spacelike vector $\gamma_1 = (-1)^{1/2}$ as a “root of minus one.” This shows that in the first of the above cases the “scalar $(-1)^{1/2}$ ” serves to distinguish between spacelike and timelike vectors, and that such a quantity is rendered superfluous by the use of spacetime algebra. It will be shown later that in the second case $(-1)^{1/2}$ represents a spacelike bivector.

In accordance with conventions adopted in Ref. 5, the scalar term in a polynomial representation of a multivector M is denoted by $M_{(0)}$ and called the *scalar* or *(0-vector) part* of M . From (2.2) it is easy to prove that in general

$$M_{(0)} = \frac{1}{4} \text{Tr } M, \quad (2.5)$$

which shows that “scalar part” spacetime algebra corresponds to “trace” in the Dirac algebra. In particular, from (2.1b) and (2.4)

$$\gamma_\mu \cdot \gamma_\nu \equiv (\gamma_\mu \gamma_\nu)_{(0)} = \frac{1}{4} \text{Tr } (\gamma_\mu \gamma_\nu). \quad (2.6)$$

Moreover, Eq. (2.2) becomes $(\gamma_\mu)_{(0)} = 0$ in spacetime algebra and simply says that a vector is not a scalar. The factor $\frac{1}{4}$ in (2.5) and (2.6) has no geometrical import. Its appearance is another indication that matrix algebra is not ideally suited to the geometrical role it plays in the Dirac theory.

Hermitian conjugation plays an important role in matrix theory, but it has no geometrical significance in the Dirac algebra except in connection with a specific matrix representation. In spacetime algebra a similar role is played by an operation called *reversion*. The *reverse* \tilde{M} of a multivector M is obtained from M by reversing the order of γ ’s in its polynomial representation. The relation of reversion to Hermitian conjugation will be given in the next section.

Equation (2.3) was presented as a change in representation of the Dirac matrices, but the fact is the same equation appears in spacetime algebra where the notion of representation has no significance. The geometrical requirement of spacetime algebra that the γ'_μ in (2.3) must be vectors entails that they can be written as a linear combination $\gamma'_\mu = a^\nu_\mu \gamma_\nu$ of the γ_ν so (2.3) must be invariant under reversion, from which it follows that S can be chosen so that $S^{-1} = \tilde{S}$. Thus (2.3) takes on the special form

$$\gamma'_\mu = a^\nu_\mu \gamma_\nu = S \gamma_\mu \tilde{S}. \quad (2.7)$$

This equation describes a Lorentz transformation of a frame of vectors $\{\gamma_\mu\}$ into a frame $\{\gamma'_\mu\}$. Moreover, Eq. (2.7) can be solved for S as a function the γ'_μ and the γ_μ alone which proves that S is indeed a multivector and that every Lorentz transformation can be expressed in that form. Proper Lorentz transformations (i.e., transformations continuously connected to the identity) will be of special interest in the analysis of the Dirac theory. It can be shown that (2.7) is a proper Lorentz transformation if and only if S is an *even* multivector satisfying

$$S \tilde{S} = 1. \quad (2.8)$$

From this condition it is only an algebraic exercise to show that S can be put in the form

$$S = e^{B/2}, \quad (2.9)$$

where B is a bivector. Without elaborating on the important geometric notion of a bivector, it is sufficient for present purposes to remark that any bivector B can be written in the “polynomial form”

$$B = \frac{1}{2} B^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \quad (2.10a)$$

where the

$$\gamma_\mu \wedge \gamma_\nu = \frac{1}{2} [\gamma_\mu, \gamma_\nu] \equiv \frac{1}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \quad (2.10b)$$

provide a complete basis for the space of bivectors, and the

$$B^{\mu\nu} = \gamma^\mu \cdot B \cdot \gamma^\nu = B(\gamma^\mu \wedge \gamma^\nu) = (B\gamma^\mu \gamma^\nu)_{(0)} = -B^{\nu\mu} \quad (2.10c)$$

are alternative expressions for the six independent scalar coefficients. For more details and proofs of the assertions in this paragraph see Ref. 1, Chap. IV and Appendix B.

3. The Dirac Equation without Complex Numbers

At first sight the Dirac algebra appears to be more general than spacetime algebra, because its “scalar field” consists of complex numbers rather than real numbers only. But, it will be shown that the imaginary unit i' of the Dirac algebra is superfluous in the physical theory, and its use serves only to obscure the geometrical and physical interpretation. This section shows that i' can be replaced by a spacelike bivector and finds the appropriate formulations of the wavefunction and Dirac equation in terms of the spacetime algebra. The next section shows that i' is superfluous in the expression for observables of the Dirac theory.

A Dirac spinor Ψ is a column matrix with four complex components, that is,

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 + i' \beta_1 \\ \alpha_2 + i' \beta_2 \\ \alpha_3 + i' \beta_3 \\ \alpha_4 + i' \beta_4 \end{pmatrix} \quad (3.1)$$

where the α 's and β 's are real numbers and i' is the $(-1)^{1/2}$ of the matrix algebra. The representation (3.1) in terms of components $\psi_1 \psi_2 \psi_3 \psi_4$ presumes a specific representation of the Dirac matrices. It is convenient to work with the so-called *standard representation*:

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad (3.2a)$$

Here I is the 2×2 unit matrix and where the σ_k are the usual 2×2 Pauli matrices, that is, are traceless Hermitian matrices satisfying

$$\sigma_1 \sigma_2 \sigma_3 = i' I. \quad (3.2b)$$

By virtue of the fact that the γ_μ are related to the spacetime metric by (2.1), Ψ and its components have a definite geometrical significance, a significance to be ascertained below by finding an expression for Ψ in terms of spacetime algebra which is independent of the matrix representation.

As explained in Refs. 1 and 5, in the spacetime algebra the quantities σ_k defined by $\sigma_k = \gamma_k \gamma_0$ ($k = 1, 2, 3$) are to be interpreted as vectors relative to the inertial system specified by the timelike vector γ_0 . The σ_k generate an algebra over the reals which is isomorphic to the Pauli algebra. To emphasize this fact it is convenient to write

$$\sigma_1 \sigma_2 \sigma_3 = i \quad (3.3)$$

in analogy to (3.2b). On the other hand, $\sigma_k = \gamma_k \gamma_0$ implies $\sigma_1 \sigma_2 \sigma_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$. This fact can be expressed only in a 4×4 matrix representation of the σ_k . The 4×4 matrix representations of the σ_k are commonly denoted by α_k in the literature (e.g., p. 69 of Ref. 4), but to help keep geometric significance to the fore the symbols σ_k are used here. From the standard representation (3.2),

$$\sigma_k = \gamma_k \gamma_0 = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad (3.4a)$$

$$i = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_5 = \begin{pmatrix} 0 & i'I \\ i'I & 0 \end{pmatrix} \quad (3.4b)$$

which imply

$$i\boldsymbol{\sigma}_k = \begin{pmatrix} i'\sigma_k & 0 \\ 0 & i'\sigma_k \end{pmatrix} \quad (3.5)$$

Now introduce a basis in spinor space

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

such that

$$\gamma_0 u_1 = u_1, \quad (3.6a)$$

$$i\boldsymbol{\sigma}_3 u_1 = \gamma_2 \gamma_1 u_1 = i' u_1, \quad (3.6b)$$

and

$$u_2 = -i\boldsymbol{\sigma}_2 u_1, \quad u_3 = \boldsymbol{\sigma}_2 u_1, \quad u_4 = \boldsymbol{\sigma}_1 u_1. \quad (3.6c)$$

Supposing (3.1) refers to this representation, by using (3.6c) to eliminate u_2, u_3, u_4 , and (3.6b) to eliminate the imaginary unit i' , Ψ can be written

$$\begin{aligned} \Psi &= \psi_1 u_1 + \psi_2 u_2 + \psi_3 u_3 + \psi_4 u_4 \\ &= (\psi_1 - i\boldsymbol{\sigma}_2 \psi_2 + \boldsymbol{\sigma}_3 \psi_3 + \boldsymbol{\sigma}_1 \psi_4) u_1 \\ &= \{ \alpha_1 + (\alpha_4 \boldsymbol{\sigma}_1 + \beta_4 \boldsymbol{\sigma}_2 + \alpha_3 \boldsymbol{\sigma}_3) + i(\beta_2 \boldsymbol{\sigma}_1 - \alpha_2 \boldsymbol{\sigma}_2 + \beta_1 \boldsymbol{\sigma}_3) + i\beta_3 \} u_1 \end{aligned}$$

Thus any Dirac spinor Ψ can be written in the form

$$\Psi = \psi u_1, \quad (3.7)$$

where ψ can be written down directly from the column matrix form (3.1) by using

$$\begin{aligned} \psi &= \alpha_1 + (\alpha_4 \boldsymbol{\sigma}_1 + \beta_4 \boldsymbol{\sigma}_2 + \alpha_3 \boldsymbol{\sigma}_3) + i(\beta_2 \boldsymbol{\sigma}_1 - \alpha_2 \boldsymbol{\sigma}_2 + \beta_1 \boldsymbol{\sigma}_3) + i\beta_3 \\ &= \alpha_1 + \alpha_4 \gamma_1 \gamma_0 + \beta_4 \gamma_2 \gamma_0 + \alpha_3 \gamma_3 \gamma_0 + \beta_2 \gamma_3 \gamma_2 + \alpha_2 \gamma_3 \gamma_1 + \beta_1 \gamma_2 \gamma_1 + \gamma_5 \beta_3. \end{aligned} \quad (3.8)$$

The significance of (3.7) and (3.8) is that the unit imaginary i' has been eliminated to express Ψ uniquely as a function of an *even* multivector ψ . But ψ can be expressed as an element of the spacetime algebra at once simply by interpreting the γ 's as vectors instead of matrices. As will be seen, this helps to make the geometrical significance of spinors explicit.

Dirac's equation for an electron with charge e and mass m in an external electromagnetic field can be written

$$\gamma_\mu (i'\hbar \partial_\mu - \frac{e}{c} A_\mu) \Psi = mc \Psi$$

where $\gamma_\mu = (\gamma_\mu)^{-1} = g^{\mu\nu} \gamma_\nu$. With

$$\square = \gamma^\mu \partial_\mu, \quad \partial_\mu = \frac{\partial}{\partial x^\mu} \quad (3.9)$$

and

$$A = A_\mu \gamma^\mu = A^\mu \gamma_\mu, \quad (3.10)$$

the Dirac equation assumes the form

$$(i'\hbar \square - \frac{e}{c} A) \Psi = mc \Psi. \quad (3.11)$$

It is crucial to note that the unit imaginary i' in the Dirac equation cannot be absorbed into the definition of the γ_μ while both ∂_μ and A_μ are kept real. Since the Dirac equation describes a physical property of the electron, the nontrivial explicit appearance of i' in the equation implies that i' has a physical significance. That significance remains to be determined.

The i' in the Dirac equation can be replaced by a multivector by using (3.7) along with (3.6b) to put (3.11) in the form

$$(\hbar \square \psi \gamma_2 \gamma_1 - \frac{e}{c} A \psi) \gamma_0 u_1 = mc \psi u_1. \quad (3.12)$$

The γ_0 was inserted in the right side of (3.12) by using (3.6a) to make the coefficients of u_1 even multivectors. Although u_1 does not have an inverse, the coefficients of u_1 in (3.12) can be equated, because, as (3.6c) shows, even multivectors operating on u_1 generate a complete basis for Dirac spinors. Therefore, (3.12) yields

$$(\hbar \square \psi \gamma_2 \gamma_1 - \frac{e}{c} A \psi) \gamma_0 = mc \psi,$$

or equivalently,

$$\hbar \square \psi \gamma_2 \gamma_1 - \frac{e}{c} A \psi = mc \psi \gamma_0. \quad (3.13)$$

Conversely, by multiplying (3.13) on the right by u_1 and using (3.6) and (3.7) the Dirac equation is recovered. So (3.13) is fully equivalent to the Dirac equation (3.11), and, by using (3.7), a solution of the one equation can be easily expressed as a solution of the other.

By interpreting the γ 's as vectors instead of matrices, Eq. (3.13) becomes an equation in spacetime algebra which may fairly be called the *Dirac equation* in that language because of its equivalence to the usual matrix equation. Likewise, it is appropriate to refer to ψ as a *spinor* in the spacetime algebra. In the spacetime algebra (3.10) simply expresses the potential A as a linear combination of basis vectors and the so-called ‘‘Dirac operator’’ introduced by (3.9) can be directly interpreted as the derivative with respect to a spacetime point $x = x^\mu \gamma_\mu$.

The most significant feature of (3.13) is that the i' which appears in (3.11) has been replaced by the bivector $\gamma_2 \gamma_1$, because bivectors have a straightforward geometrical interpretation in spacetime algebra. So translation of the Dirac theory into the language of spacetime algebra promises to reveal a hidden significance of imaginary numbers in the Dirac theory. This promise will be completely fulfilled in the next section.

The derivation of (3.13) made use of a specific matrix representation of the γ_μ . The consequence of using any other representation can be ascertained by considering an arbitrary change of basis in Dirac ‘‘spin space’’ which takes u_1 to

$$u'_1 = S' u_1. \quad (3.14)$$

Now the transformation matrix S' in (3.14) can be taken to be an even multivector; for S' has an odd part, because of (3.6a) that part can be made even without affecting (3.14) by multiplying it on the right by γ_0 ; similarly, any ‘‘complex coefficient’’ in a polynomial representation of S' can be replaced by an even multivector by using (3.6b). By an argument to be given later in connection with a more important issue, the fact that S' is even implies that it can be put in the form

$$S' = \exp(i\delta) S, \quad (3.15a)$$

where S satisfies (2.8), δ is a scalar, and it will be recalled, i satisfies (3.4b). The inverse of S' is

$$S'^{-1} = \exp(-i\delta) \tilde{S}. \quad (3.15b)$$

Hence (3.14) gives

$$u_1 = \exp(-i\delta) \tilde{S} u'_1. \quad (3.16)$$

Equations (3.6a,b) can be transformed into the new representation by multiplying them by S' and using (3.15) and (3.16) together with the fact that $i = \gamma_5$ anticommutes with the γ_μ ; one gets

$$\exp(2i\delta) \gamma'_0 u'_1 = u'_1, \quad (3.17a)$$

$$\gamma'_2 \gamma'_1 u'_1 = i' u'_1, \quad (3.17b)$$

where the γ'_μ are related to the γ_μ by (2.7). With (3.16), (3.7) can be put in the form

$$\Psi = \psi' \exp(-i\delta) u'_1, \quad (3.18a)$$

where

$$\psi' = \psi \tilde{S}. \quad (3.18b)$$

Substituting (3.18a) into the Dirac equation (3.11), and using (3.17a,b) to get an equation for ψ' in the same way that (3.6a,b) were used to get an equation for ψ , one notices that the factor $\exp(-i\delta)$ be eliminated yielding the equation

$$\hbar \square \psi' \gamma'_2 \gamma'_1 - \frac{e}{c} A \psi' = mc \psi' \gamma'_0. \quad (3.19)$$

This has the same form as (3.13). Indeed (3.19) can be obtained directly from (3.13) by multiplying it on the right by the constant factor \tilde{S} . For note that, by (2.8) and (2.7),

$$\psi \gamma_2 \gamma_1 \tilde{R} = \psi \tilde{R} \tilde{R} \gamma_2 \tilde{R} \tilde{R} \gamma_1 \tilde{R} = \psi' \gamma'_2 \gamma'_1,$$

which shows how the terms on the left of (3.13) and (3.19) are related.

Thus it has been proved that the form of (3.13) is uniquely determined, independently of the choice of a representation for the Dirac matrices. The specification of vectors γ_2 , γ_1 and γ_0 in (3.13) is determined only to within a proper Lorentz transformation; this arbitrariness corresponds to the freedom to choose a matrix representation for the Dirac theory. The form of the usual Dirac equation (3.11) is representation independent. However, no solution to that equation can be exhibited without choosing a specific representation. The significance of such a choice is hidden in the matrix formulation of the Dirac theory. It has been uncovered above. A choice of representation amounts to a specific correspondence between 4×4 matrices and vectors. Though the choice is to some extent arbitrary, a definite choice must nevertheless be made. The explicit appearance of γ 's in (3.13) and (3.19) is a consequence of such a choice. It has already been pointed out that this implies that i' must be interpreted geometrically as a spacelike bivector. A more complete interpretation will be provided in the next section.

It should be noted that in conventional treatments of the Dirac theory,⁴ it is proved that the Dirac equation retains its form under Lorentz transformations. That proof is easily translated into the present language, but it would be superfluous, for having eliminated matrices and interpreted the y , as vectors, we have put the Dirac equation in the form (3.13) which is manifestly independent of coordinates. It cannot be overemphasized that the vectors γ_0 , γ_1 and γ_2 appearing in (3.13) need not be associated with any coordinate frame; they are simply a set of arbitrarily chosen orthonormal vectors. Adoption of a coordinate frame with γ_0 as the time component is equivalent in the conventional theory to adopting a matrix representation for which γ_0 is Hermitian and the γ_μ are anti-Hermitian. Adoption of the standard representation (3.2) associates by (3.6b) the i' of the matrix representation with the bivectors $\gamma_2 \gamma_1$ in (3.13). Thus, the standard representation is distinguished by relating Hermitian conjugation and complex numbers of the matrix algebra to intrinsic features of the Dirac equation.

There is an alternative formulation of the Dirac equation in terms of spacetime algebra which should be mentioned. Define the quantity Φ by the equation

$$\Phi = \psi U \quad (3.20a)$$

where

$$U = \frac{1}{2}(1 + \gamma_0)(1 + \sigma_3). \quad (3.20b)$$

Multiplying (3.13) on the right by U and noting that $\gamma_2 \gamma_1 U = U i \sigma_3 = U i$ and $\gamma_0 U = U$, one gets

$$\hbar \square \Phi i - \frac{e}{c} A \Phi = mc \Phi. \quad (3.21)$$

Except for the choice of units and a difference in sign which is merely a matter of convention, (3.21) is identical to Eq. (13.2) of Ref. 1. It is the first formulation of the Dirac equation in terms of spacetime algebra.

Comparison of (3.21) with the matrix Dirac equation (3.11) suggests that the pseudoscalar $i = \gamma_5$ in (3.22) is the geometrical quantity which corresponds to the imaginary i' in (3.11). But this is misleading, because certain essential geometrical features of the Dirac theory are hidden in the structure of U as defined by (3.20), much as they are hidden in the properties of the base spinor u_1 in (3.7). These features were first uncovered in Ref. 2, where it was concluded that (3.13) is a much more significant equation than (3.21). To nail down the interpretation of i' in the Dirac theory, it is necessary to examine the definitions of observables. This is undertaken in the next section.

4. Observables and the Wavefunction

The geometrical significance of the wavefunction Ψ in Dirac's theory is determined by requiring that certain bilinear functions of Ψ be tensors. Interpretation of these tensors as observables determines the physical significance of Ψ .

This section explains how the so-called "bilinear covariants" of the Dirac theory can be expressed in terms of spacetime algebra. This makes it possible to provide a direct interpretation of the Dirac wavefunction, showing that the interpretation of the γ_μ as vectors is in complete accord with the Dirac theory and so justified by its simplicity. Moreover, the geometrical and physical interpretation of the unit imaginary i' in the Dirac theory is ascertained by proving that $\frac{1}{2}i'\hbar$ is a representation of the spin bivector by one of its eigenvalues.

Equation (3.8) explicitly shows that ψ is a sum of scalar, bivector, and pseudoscalar parts. Of course, every even multivector has this property. Note that $\tilde{i} = i$, that is, the unit pseudoscalar $i = \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ is invariant under reversion. However, every bivector changes sign under reversion as is shown by $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$ for $\mu \neq \nu$. Hence $\tilde{\psi}$ can be obtained from ψ simply by changing the sign of its bivector part. Since ψ is an even multivector, so, is $\psi\tilde{\psi}$. But $\psi\tilde{\psi}$ is clearly invariant under reversion, so its bivector part must vanish. Moreover, since $i^2 = -1$, $\psi\tilde{\psi}$ can be put in the "polar form"

$$\psi\tilde{\psi} = \rho \exp(i\beta) = \rho \cos \beta + i\rho \sin \beta, \quad (4.1)$$

where ρ and β are scalars. One can then define R by the equation $R = [\rho \exp(i\beta)]^{-\frac{1}{2}}\psi$, or just write

$$\psi = \rho^{\frac{1}{2}} \exp(i\beta) R \quad (4.2a)$$

Because of (3.9),

$$R\tilde{R} = 1. \quad (4.2b)$$

The expression (4.2a) is the "canonical form" for a spinor in the spacetime algebra first found in Ref. 2. The quantities ρ , β , and R have distinctive geometrical and physical interpretations which are independent of any matrix representation. So it is best to use them instead of the α 's and β 's in (3.8).

It is simplest to set forth the interpretation of ψ categorically and after that explain how it is related to the usual formulation of the Dirac theory. The quantity R in (4.2) determines a proper Lorentz transformation of a frame $\{\gamma_\mu\}$ into a frame $\{e_\mu\}$ according to the equation

$$e_\mu = R\gamma_\mu\tilde{R}. \quad (4.3)$$

This equation has exactly the form of the Lorentz transformation (2.7), since comparison of (4.2b) with (2.8) shows that R has the same algebraic structure as S . However $R = R(x)$ is a generally differentiable function of the spacetime point x , while S is constant. Thus, (4.3) specifies a (generally differentiable) set of four vector fields with values $c_\mu = c_\mu(x)$ at each point x determined by a proper Lorentz transformation of a fixed frame $\{\gamma_\mu\}$. This completely describes the *geometrical* significance of R . By virtue of (4.3), the spinor R may be regarded as a representation of a Lorentz transformation.

Since the pseudoscalar i anticommutes with the vector γ_μ , (4.2a) and (4.3) imply

$$\psi \gamma_\mu \tilde{\psi} = \rho e_\mu. \quad (4.4)$$

If (3.12) is regarded as a generalization of the transformation (4.3), then multiplication of the e_μ by ρ must be interpreted as a dilatation.

The geometrical interpretation of a Dirac spinor given here is more direct and detailed than the conventional one. Ordinarily, Dirac spinors are said to be representations of the Lorentz group because they *transform* in a certain way under Lorentz transformation (see Ref. 4) In contrast, we say that ψ represents a Lorentz transformation because, by Eq. (4.4), it *determines* a “rotation-dilatation” of the frame $\{\gamma_\mu\}$ into the frame $\{\rho e_\mu\}$. Actually, $\psi = \psi(x)$ determines a continuous infinity of Lorentz transformations, one at each spacetime point x . Perhaps it is worth adding for emphasis that ψ does not operate in some “abstract spin space” detached from spacetime, it transforms spacetime vectors into spacetime vectors.

The physical interpretations of ρ and R are fixed by specifying interpretations for the e_μ . The quantity

$$\rho v = \psi \gamma_0 \tilde{\psi} \quad (4.5a)$$

is to be identified as the probability current of the Dirac theory. It follows that the timelike vector

$$v \equiv e_0 = R \gamma_0 \tilde{R} \quad (4.5b)$$

can be interpreted the (local) “world velocity” of a Dirac particle, while ρ is the probability density in the local rest frame determined by v . The tensor components of the probability current relative to the frame $\{\gamma_\mu\}$ are

$$\rho v^\mu = \rho v \cdot \gamma^\mu = (\psi \gamma_0 \tilde{\psi} \gamma^\mu)_{(0)}, \quad (4.5c)$$

and the local conservation of probability is expressed by

$$\square \cdot (\rho v) = \partial_\mu (\rho \gamma^\mu) = 0. \quad (4.6)$$

The spacelike vector

$$s = \frac{1}{2} \hbar e_3 = \frac{1}{2} \hbar R \gamma_3 \tilde{R} \quad (4.7a)$$

can be identified as the (local) *spin vector* of the Dirac theory. The corresponding “current”

$$\frac{1}{2} \hbar \psi \gamma_3 \tilde{\psi} = \rho s \quad (4.7b)$$

has components

$$\rho s_\mu = \rho s \cdot \gamma_\mu = \frac{1}{2} \hbar (\psi \gamma_3 \tilde{\psi} \gamma^\mu)_{(0)}. \quad (4.7c)$$

Of course, s cannot be interpreted directly as the electron spin because angular momentum is a bivector. The “proper spin density” of the electron is ρS , where S is the (local) *spin bivector* given by

$$S = \frac{1}{2} \hbar e_2 e_1 = \frac{1}{2} \hbar R \gamma_2 \gamma_1 \tilde{R} = \frac{1}{2} \hbar R i \sigma_3 \tilde{R} = i s v. \quad (4.8a)$$

These assorted equivalent expressions are easily related by using $R \tilde{R} = 1$ and the appropriate definitions. The tensor components of S are

$$S^{\alpha\beta} = (S \gamma^\beta \gamma^\alpha)_{(0)} = i s \wedge v \wedge \gamma^\beta \wedge \gamma^\alpha = s_\mu v_\nu \epsilon^{\mu\nu\alpha\beta}, \quad (4.8b)$$

which we get by applying (2.12) to (4.8a) and introducing, the “alternating tensor” defined by

$$\epsilon^{\mu\nu\alpha\beta} = -i \gamma^\mu \wedge \gamma^\nu \wedge \gamma^\beta \wedge \gamma^\alpha = -(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\beta \gamma^\alpha)_{(0)}. \quad (4.9)$$

The right side of (4.8a) or (4.8b) shows that S is the dual of the bivector $sv = s \wedge v$. Thus, given v , s , and S are equivalent descriptions of the spin in the sense that either one determines the other by (4.8).

In (4.5), (4.7), (4.8) the e_μ defined by (4.3) have been given a physical interpretation by relating them to the electron spin and velocity. It is important to realize that the index μ in (4.3) is a “free index,” that is, it need not be related to any coordinate system. However, the physical interpretation requires that the γ_μ indexed in (4.3) be identical to the set $\gamma_0, \gamma_1, \gamma_2, \gamma_3 = i\gamma_0\gamma_1\gamma_2$ specified in the Dirac equation (3.13). It will be noted that the “change of representation” transforming (3.13) to (3.10) does not alter the e_μ since $\psi\gamma_\mu\tilde{\psi} = \psi'\gamma'_\mu\tilde{\psi}'$.

Something can now be said about the physical interpretation of the Dirac wavefunction in its canonical form (4.2a). The quantity ρ has been identified as the proper probability density. The unimodular spinor R determining the Lorentz transformation (4.3) can be specified by six scalar parameters. Five of those parameters determine the velocity and spin directions of the electron, which, of course, also determines the “spin plane” containing the vectors e_1 and e_2 orthogonal to v and s . The remaining parameter is the *phase* of the wavefunction. Geometrically, the phase determines the directions of e_1 and e_2 in the spin plane. Physically, the phase is related indirectly to the electron energy-momentum by derivatives of the wavefunction; this will be considered in the next section. To sum up, except for the phase and the parameter β in (4.2a), the Dirac wavefunction determines (or is determined by) the electron probability current and the spin direction.

To get an interpretation of β , additional physical assumptions are needed. The bivector

$$M = \frac{e\hbar}{2mc}\psi\gamma_2\gamma_1\tilde{\psi} = \frac{e}{mc}\exp(i\beta)\rho S \quad (4.10)$$

is usually interpreted as the magnetization or magnetic moment density of the electron. The right side of (4.10) was obtained by using (4.2a) with (4.8a) and shows that the ratio of magnetic moment to spin density differs from the usual Ferromagnetic ratio e/mc attributed to the electron by the factor $\exp(i\beta)$. Equation (4.10) shows that β can be interpreted geometrically as the angle of a “duality rotation” of S into M . It also lends a physical significance to β , but, as explained in Ref. 3, other features of the Dirac theory make a fully satisfactory physical interpretation difficult to come by.

Proof that the expressions (4.5), (4.7), and (4.10) are equivalent to conventional expressions for probability current, spin and magnetic moment in the Dirac theory is simply a matter of computation using the unique correspondence between ψ and the column spinor Ψ established in Sec. 3. The computations have been discussed in Appendix A of Ref. 3, so it suffices to display the results in Table I in the form of expressions for the so-called bilinear covariants using both mathematical systems. Table I uses the notation $\bar{\Psi} = \Psi^\dagger\gamma_0$ where Ψ^\dagger denotes Hermitian adjoint, in addition to conventions already explained, especially in connection with (2.5) and (2.10).

The term “observable” is used here to refer to tensor quantities such as ρv_μ , $M_{\mu\nu}$ and s_μ which (in principle, at least) are amenable to experimental observation. The relations of wavefunction to observables given by (4.1), (4.5), (4.7), (4.10) are much simpler and easier to use than those in Table I. So the table is useful only for comparison with the conventional formulation.

Table I helps reveal the role of the unit imaginary i' in the matrix formulation of the Dirac theory. Observables are always reducible to real numbers. Some of the bilinear covariants in Table I are formed from antiHermitian products of the γ_μ , so a factor i' is needed to make the quantities real. This apparently trivial mathematical fact has a physical significance. To discover that significance, it should be noted first that i' never enters any expression for observables of the Dirac theory (such as those in Table I) except as a multiple of the wavefunction Ψ , and second, that i' enters only in the combination $i\hbar$ with Planck’s constant. Also, note that

$$S\psi = \frac{1}{2}\hbar\psi\gamma_2\gamma_1 \quad (4.11)$$

is easily proved from (4.2) and (4.8). Interpreting (4.11) as a matrix equation and using (3.7) and (3.6b), one shows immediately that

$$S\psi = \frac{1}{2}i'\hbar\Psi, \quad (4.12)$$

that is, $\frac{1}{2}i'\hbar$ is an eigenvalue of the bivector S describing the spin. Equation (4.12) can be used to eliminate the explicit appearance of $i'\hbar$ in equations of the Dirac theory. For example, if Γ is some matrix operator, since $S = \frac{1}{2}S^{\mu\nu}\gamma_\mu\gamma_\nu$, (4.12) yields

$$i'\hbar\bar{\Psi}\Gamma\Psi = \bar{\Psi}\Gamma\gamma_\mu\gamma_\nu\Psi S^{\mu\nu}, \quad (4.13)$$

showing the factor $i'\hbar$ to be equivalent to a contraction of the spin tensor $S^{\mu\nu}$ with some other tensor. As Table I illustrates, the factor $i'\hbar$ appears explicitly only in those expressions for observables involving the electron spin. Equation (4.12) shows us that the factor $\frac{1}{2}i'\hbar$ in the Dirac theory is in fact a representation of the electron spin by an eigenvalue of the spin tensor. The eigenvalue is imaginary, because the spin tensor $S^{\mu\nu}$ is skew-symmetric. The i' is a representation of the direction of the spin tensor, because the conventions of Dirac theory correspond, in a devious way, to the generator of rotations in the physical “spin plane” with the generator of rotations in the abstract complex plane. Of course, the factor $\frac{1}{2}\hbar$ is just the magnitude of the spin tensor.

TABLE I. Bilinear covariants as observables.

$$\begin{aligned}
\bar{\Psi}\Psi &= (\psi\tilde{\psi})_{(0)} = \rho \cos \beta \\
\bar{\Psi}\gamma_\mu\Psi &= (\gamma_\mu\psi\tilde{\psi})_{(0)} = \gamma_\mu \cdot (\rho v) = \rho v_\mu \\
\frac{e}{mc} \frac{i'\hbar}{2} \bar{\Psi}\frac{1}{2}\gamma_\mu\wedge\gamma_\nu\Psi &= \frac{e\hbar}{2mc} (\gamma_\mu\gamma_\nu\psi\tilde{\psi})_{(0)} = (\gamma_\mu\wedge\gamma_\nu) \cdot M = M_{\mu\nu} \\
\frac{i'\hbar}{2} \bar{\Psi}\gamma_\mu\gamma_5\Psi &= \frac{\hbar}{2} (\gamma_\mu\psi\tilde{\psi})_{(0)} = \gamma_\mu \cdot (\rho s) = \rho s_\mu \\
\bar{\Psi}\gamma_5\Psi &= (i\psi\tilde{\psi})_{(0)} = -\rho \sin \beta
\end{aligned}$$

The spin tensor $S^{\alpha\beta}$, which is crucial to the understanding of $i'\hbar$, is not mentioned in standard accounts of the Dirac theory, so some explanation is in order. Standard accounts (e.g., p. 59 of Ref. 7) either implicitly or explicitly introduce the spin (density) tensor

$$\begin{aligned}
\rho S^{\nu\alpha\beta} &= \frac{1}{2}i'\hbar \bar{\Psi}\gamma^\nu \wedge \gamma^\alpha \wedge \gamma^\beta \Psi \\
&= \frac{1}{2}i'\hbar \bar{\Psi}\gamma_5\gamma_\mu\Psi \epsilon^{\mu\nu\alpha\beta} \rho s_\mu \epsilon^{\mu\nu\alpha\beta}
\end{aligned} \tag{4.14}$$

where use has been made of the identity

$$\gamma^\nu \wedge \gamma^\alpha \wedge \gamma^\beta = \gamma_5\gamma_\mu \epsilon^{\mu\nu\alpha\beta} \tag{4.15}$$

and the expression for s_μ in Table I. Contraction of (4.14) with v_ν and comparison with (4.8b) gives the desired relation

$$\rho S^{\alpha\beta} = \rho v_\nu S^{\nu\alpha\beta} = \rho s_\mu v_\nu \epsilon^{\mu\nu\alpha\beta}. \tag{4.16}$$

The $\rho S^{\alpha\beta}$ must be interpreted as the components of the electron spin angular momentum density in the local rest frame determined by the particle velocity v .

Having ascertained that the imaginary unit i' functions in the Dirac theory as a representation of the electron spin direction, it is evident that charge conjugation must have a geometrical interpretation as well as the usual physical one. Indeed, it has already been discovered in Sec. 5 of Ref. 8 that charge conjugation entails a rotation of the “observables” e_1, e_2, e_3 defined by (4.3) about an axis in the spin plane. Reference 9 pointed out that this transformation in “the space of observables” of the Dirac theory is isomorphic to the operation of charge conjugation in isospace so by regarding them as one and the same, one gets a spacetime interpretation of isospace relating spin to isospin and indicating a connection between the pseudoscalar and isospin properties of the pion. This speculative idea has not thus far been pushed by the author. Still, in view of the uncertainties of strong interaction theory. It seems sufficiently promising to merit mention once again.

5. Observables and operators

A number of statements about observables, operators, and eigenvalues are frequently put forward in one form or another as general principles of quantum theory. The difficulties these principles face when applied to the Dirac theory shows that they are not so general after all. Surely, since the Dirac theory is the most

firmly established version of quantum theory, only those principles which are required by or are at least consistent with the Dirac theory can be regarded as both fundamental and general.

TABLE II. Observables with the kinetic energy-momentum operator.

Gordon current	$k_\mu \equiv \bar{\Psi} \hat{p}_\mu \Psi \equiv i' \hbar \frac{1}{2} \{ \bar{\Psi} \partial_\mu \Psi - (\partial_\mu \bar{\Psi}) \Psi \} - \frac{e}{c} A_\mu \bar{\Psi} \Psi$ $\equiv (\tilde{\psi} \hat{p}_\mu \psi)_{(0)} \equiv \left(\tilde{\psi} \left\{ \hbar \partial_\mu \psi \gamma_2 \gamma_1 - \frac{e}{c} A_\mu \psi \right\} \right)_{(0)}$
Energy-momentum tensor	$T^{\mu\nu} = \bar{\Psi} \gamma^\nu \hat{p}_\mu \Psi = (\gamma_0 \tilde{\psi} \gamma^\nu \hat{p}_\mu \psi)_{(0)}$
Proper Energy-momentum density	$\rho p_\mu \equiv v_\nu T^{\mu\nu} = v \cdot [(\hat{p}_\mu \psi) \gamma_0 \tilde{\psi}]_{(1)}$ $= (\tilde{\psi} e^{-i\beta} \hat{p}^\mu \psi)_{(0)} = \rho (\tilde{R} \hat{p} R \psi)_{(0)}$
Kinetic Energy density	$T^\infty = \Psi^\dagger \hat{p}^0 \Psi = (\psi^\dagger \hat{p}^0 \psi)_{(0)}$
(Relative) Kinetic momentum density	$T^{0k} = \Psi^\dagger \hat{p}^k \Psi = (\psi^\dagger \hat{p}^k \psi)_{(0)}$
Total Angular momentum tensor	$J^{\mu\alpha\beta} = T^{\mu\alpha} x^\beta - T^{\mu\beta} x^\alpha - \rho s_\nu \epsilon^{\nu\mu\alpha\beta}$ $= \bar{\Psi} \{ \gamma^\mu (\hat{p}^\alpha x^\beta - \hat{p}^\beta x^\alpha) - (i' \hbar / 2) \gamma^\mu \wedge \gamma^\alpha \wedge \gamma^\beta \} \Psi$ $= (\gamma_0 \tilde{\psi} \{ \gamma^\mu (\hat{p}^\alpha x^\beta - \hat{p}^\beta x^\alpha) \psi - (\hbar / 2) \gamma^\mu \wedge \gamma^\alpha \wedge \gamma^\beta \psi \gamma_2 \gamma_1 \})_{(0)}$
Proper Total Angular momentum density	$v_\mu J^{\mu\alpha\beta} = \rho (p \wedge x + i s v) \cdot (\gamma^\beta \wedge \gamma^\alpha)$
(Relative) Total Angular momentum density ($i, j, k = (1, 2, 3)$)	$J^{0ij} = \Psi^\dagger \{ \hat{p}^i x^j - \hat{p}^j x^i - (i' \hbar / 2) \gamma^i \wedge \gamma^j \} \Psi$ $\equiv \Psi^\dagger \hat{J}_k \Psi = (\psi^\dagger \hat{J}_k \psi)_{(0)}$ $\equiv (\psi^\dagger \{ (\hat{p}^i x^j - \hat{p}^j x^i) \psi - (\hbar / 2) \gamma^i \wedge \gamma^j \psi \gamma_2 \gamma_1 \})_{(0)}$

Consider, for example, the EHO-principle that “the Eigenvalues of Hermitian Operators correspond to Observed values.” In the relation $\rho v_k = \bar{\Psi} \gamma_k \Psi = \Psi^\dagger \gamma_0 \gamma_k \Psi$ ($k = 1, 2, 3$) from Table I, the matrices $\gamma_0 \gamma_k$ are Hermitian. Since $(\gamma_0 \gamma_3)^2 = 1$, the eigenvalues of $\gamma_0 \gamma_3$ are ± 1 . So if Ψ is an eigenstate of $\gamma_0 \gamma_3$ we have $\gamma_0 \gamma_3 \Psi = \pm \Psi$ or $\gamma_3 \Psi = \pm \gamma_0 \Psi$; whence, $\rho v_3 = \Psi \gamma_3 \Psi = \pm \Psi \gamma_0 \Psi = \pm \rho v_0$. But this says that the velocity of the electron equals the speed of light, which is inconsistent with the Dirac equation, not to mention experimental evidence. Notwithstanding the absurdity of the result, application of the EHO-principle to the so-called “velocity operators” $\gamma_0 \gamma_k$ has been perpetuated in accounts of the Dirac theory for forty years, along with fruitless attempts to explain away the problem. This and many other similar difficulties are easily resolved by admitting that EHO is not a general principle of quantum mechanics and, in particular, does not apply to operators composed of the γ_μ . The role of the γ_μ in the Dirac theory is clear in the multivector formulation; the γ_μ are simply an orthonormal set of vectors. In the expression $\gamma_\mu \cdot (\rho v) = \rho v_\mu$ appearing in Table I the interpretation of the γ_μ as operators is trivial and has nothing to do with quantum mechanics; the γ_μ simply

pick out the μ th component of the vector v by inner multiplication. Nor is the noncommutative multiplication rule for the γ_μ justified by any principle concerning the impossibility of simultaneous eigenvalues in the Dirac theory; it has straightforward geometrical significance already discussed in Sec. 2 and in Refs. 1 and 5. It should be evident that the γ_μ have the same geometrical role in all the relations of Table 1.

To the extent that propositions about Hermitian operators and eigenvalues are significant in the Dirac theory, they are special properties of the “kinetic energy-momentum operators,” customarily written

$$\hat{p}_\mu = i'\hbar \partial_\mu - \frac{e}{c} A_\mu, \quad (5.1)$$

in the matrix formulation. It is often said that \hat{p}_μ ($k = 1, 2, 3$) corresponds to the kinetic momentum observable, but this is inaccurate and leaves room for misinterpretation. Rather, the operator \hat{p}_μ *determines a relation* between the wavefunction and the kinetic momentum observable. Those relations of the wavefunction to basic observables which are determined by \hat{p}_μ are given in Table II. The table gives both the multivector and matrix expressions for the observables. The relations in Table II, like those in Table I, can be proved by the method of Appendix A in Ref. 3. The operator \hat{p}_μ is defined implicitly for both the matrix and multivector formalisms by the expressions for the Gordon current. The expression for \hat{p}_μ used in the table differs from (5.1) in that it is “hermitized” to give real quantities for the observables.

Tables I and II give a complete set of relations of the wavefunction to the fundamental (local) observables in the Dirac theory. Actually, Table II adds only one relation which is mathematically independent of the seven independent relations in Table I. It relates the energy-momentum tensor to the gradient of the phase of the wavefunction. The nature of this relation is best seen by expressing the (proper) kinetic energy-momentum in the form

$$\begin{aligned} p_\mu &= (\tilde{R}\hat{p}R\psi)_{(0)} = \hbar(\gamma_1\gamma_2) \cdot (\tilde{R}\partial_\mu\psi) - \frac{e}{c} A_\mu \\ &= 2S \cdot (\partial_\mu R\tilde{R}) - \frac{e}{c} A_\mu \end{aligned} \quad (5.2)$$

The quantity $2\partial_\mu R\tilde{R} = 2(\partial_\mu R)R^{-1}$ can be regarded as the logarithmic derivative or angular velocity of the family of Lorentz transformations determined by R according to (4.3). The quantity $2S \cdot (\partial_\mu R\tilde{R})$ is the projection of this angular velocity onto the spin plane; it describes two effects: the rate at which the spin plane precesses and the rate at which the phase (angle of rotation in the spin plane) changes. Only the last effect appears in the Schrödinger’s theory, where the energy and momentum are completely determined by the phase. In spite of the fact that the “proper energy-momentum density” ρp_μ is a fundamental quantity in relativistic continuum mechanics, it is rarely considered in discussions of the Dirac theory. Indeed, the expression for ρp_μ in terms of the matrix wavefunction is so unwieldy that it has not been included in Table II, though it is not difficult to obtain from matrix expressions for $T^{\mu\nu}$ and v_μ in Tables I and II.

The significance of Eq. (5.2) lies in the insight it gives into the role of the operator $i'\hbar\partial_\mu$ in the Dirac theory. The $i'\hbar$ has been replaced in (5.2) by the physical and geometrical quantity it represents, the spin bivector S . Moreover, (5.2) shows that in relating the wavefunction to energy and momentum the $i'\hbar$ in the operator $i'\hbar\partial_\mu$ functions as a projection operator, eliminating from $\partial_\mu\psi$ the derivatives of ρ and β and retaining only the angular velocity of R in the spin plane. These specific facts about $i'\hbar\partial_\mu$ in the Dirac theory should be compared carefully with general propositions about Hermitian operators and observables in quantum mechanics. For the most part, such propositions have been developed to generalize properties of $i'\hbar\partial_\mu$. It may be concluded that in so far as they relate the phase of a wavefunction to the energy and momentum of a particle, the propositions are well grounded. Beyond this, their validity is problematic. Certainly none of the usual propositions recognize the essential relation of $i'\hbar$ to the spin. And, as shown earlier, they produce nonsense when applied to the Dirac matrices.

“Hermiticity” is often said to be an essential attribute of operators corresponding to observables. The validity of this view should be judged by examining the role of hermiticity in the Dirac theory. Several different but interrelated roles can in fact be distinguished. In the Dirac matrix algebra hermiticity is given a geometrical significance by adopting the standard representation in which the γ_k ($k = 1, 2, 3$) representing spacelike vectors are anti-Hermitian, while γ_0 representing a timelike vector is Hermitian. Thus Hermitian conjugation serves to distinguish a specific, though arbitrarily chosen, rest frame in spacetime. It should be

noted that assignment of this role to hermiticity is not entirely arbitrary; it is in large part dictated by Eq. (2.1) which relates the matrix representation to the indefinite metric of spacetime. Hermiticity also plays a role in the association of $i\hbar$ with the spin through the requirement met in Table I that only *real* bilinear covariants are to be interpreted as observables. Neither of these roles is recognized in the usual discussions of operators in quantum mechanics. Moreover, when the matrix algebra and complex numbers are replaced by spacetime algebra, the indirect representation of geometrical and physical features by hermiticity is eliminated.

The significance of Hermitian matrices is quite different from the significance of Hermitian differential operators in the Dirac theory, though these two kinds of hermiticity are continually confounded in the literature. The distinction appears in a third role of hermiticity. A study of the relations in Table II reveals that hermiticity insures that the operator $i\hbar\partial_\mu$ relates energy and momentum directly to the phase *and not* to the amplitude of the wavefunction. This is the role of Hermiticity in the Schrödinger theory and seems to be the feature responsible for the successes of abstract operator formulations of quantum mechanics.

A misplaced emphasis on operators in quantum mechanics has continued to cover-up the meaning of hermiticity and the relation of complex numbers to spin. The relations of observables to the wavefunction expressed in Tables I and II is fundamental to the Dirac theory. They are only indirectly and imperfectly expressible as relations of observables to Hermitian operators. By using the relations in Tables I and II along with the Dirac equation, the wavefunction and the operator $i\hbar\partial_\mu$ can be completely eliminated, resulting in a formulation of the Dirac theory as a set of conservation laws and constitutive equations for observables. This reformulation has been carried out in Ref. 3 and brings to light other features of the Dirac theory which are hidden in the conventional formulation.

Having determined that the purportedly fundamental correspondence between observables and operators is neither adequate nor necessary in the Dirac theory, the significance of eigenvalues in the theory should be examined closely. To begin with, it should be emphasized that the fundamental assumptions of the Dirac theory do not require any reference to eigenvalues, so acceptable assertions about the physical significance of eigenvalues must be derived rather than assumed. Consider the appearance of energy eigenvalues. Solution of the Dirac equation for a bound electron, together with the correspondence of energy to the wavefunction in Table II, gives a discrete spectrum of values for the energy. It is true that these numbers can be regarded as the eigenvalues of an energy operator, but this fact is not needed either to identify the numbers as observables or to explain the existence of a spectrum; the fundamental relation of the energy to the phase of the wavefunction suffices.

There is another property of energy eigenvalues which may be physically significant but goes unremarked in standard expositions of quantum mechanics, because they deal with the total energy without examining the energy density. The global (or total) energy $\langle E \rangle = \int d^3x \rho_0 E$ of an electron in a stationary state is subject to fairly direct experimental measurement. The existence of a stationary state requires only that $\langle E \rangle$ be independent of time. However, the stationary state solutions of the Dirac equation entail the additional property $\langle E \rangle = E$, that is, the energy density has the striking property of being everywhere a constant multiple E of the probability density. It may be noted that this is what one would expect if it is surmised that the stationary solutions of the Dirac equation describe an ensemble of particle motions each with the same energy E . This suggests that the physical significance of eigenvalues in quantum theory is to be found in the fact that they correspond to local observables which are homogeneous in space and constant in time, rather than in their connection to Hermitian operators.

A local observable may well be homogeneous and constant without being the eigenvalue of some Hermitian operator. For example, without specifying any Hermitian operator, one might nevertheless say that an electron is in a velocity eigenstate with eigenvalue v if the local velocity $v = R\gamma_0\tilde{R}$ is homogeneous and constant. It is easy to show that only the free particle “plane wave” states are eigenstates of the velocity in this sense. They are also eigenstates of spin and momentum in the usual sense, but they are quite unphysical because they require uniform density ρ . The physical free particle solutions of the Dirac equation are wavepackets, for which it can be shown that the local velocity cannot be constant. So it may be that some physically significant local observables cannot be associated with physically significant eigenvalues.

Angular momentum operators have an important place in the systematic analysis of quantum theory. The angular momentum operators \hat{J}_k in the Dirac theory are defined implicitly by the expressions for the (relative) angular momentum density in Table II [where the values of (i, j, k) are understood to be cyclic

permutations of (1,2,3)]. The \hat{J}_k are commonly defined by requiring first that they satisfy the well-known commutation relations of angular momentum operators and second that they commute with the Dirac Hamiltonian for a central field and so are conserved quantities (see, for example, Ref. 10). The \hat{J}_k are then found to have the form given in Table II. Such an approach gives the impression that assumptions about “angular momentum algebra” are essential to the interpretation of Dirac theory. On the contrary, it was shown in Ref. 3 that given the Dirac equation and the relations of the wavefunction to the particle velocity in Table I and the energy-momentum tensor in Table II, expressions for the spin and the total angular momentum are unambiguously determined by introducing the conventional definition of orbital angular momentum. Then the form of \hat{J}_k in Table II can be obtained simply by writing the resulting expressions in conventional operator form, and manifestly without an appeal to assumptions which could be called “quantum mechanical.” Evidently, the physical significance of the \hat{J}_k is derived from the relation they express between the wavefunction and the angular momentum tensor. But nothing in this relation implies that, as is usually supposed, the eigenvalues of $\hat{J}^2 \equiv \hat{J}_k \hat{J}_k$ should be interpreted as the square of the magnitude of the angular momentum. This is not to question the importance of \hat{J}^2 or the angular momentum algebra in calculations or the classification of states; the aim here is only to point out that some problems of interpretation exist. Further questions about the interpretation of angular momentum quantum numbers will be considered later.

Probably the most profound problem posed by the Dirac theory for conventional interpretations of quantum theory lies in the difficulty of reconciling the usual interpretations of the Heisenberg uncertainty principle with the properties of electron spin. As the nature of this problem has already been explained in Ref. 3, it will not be discussed here.

6. Relative Observables and the Interpretation of Energy

The most important observable in the Dirac theory is the total energy $\langle E \rangle$, for this quantity has been subjected to the most thorough experimental investigation especially in the hydrogen atom, and it is primarily by inferences from the experimental results that the identification of other Dirac observables, such as velocity, spin and momentum have been confirmed. To identify different physical effects contributing to the energy and study the relation to other observables, the energy must be decomposed into a sum of terms. The usual treatment associates observables with operators rather than directly with the wavefunction and, strangely, is unable to separate different physical contributions to the energy except as perturbations of the nonrelativistic limit. In contrast, the approach here is to eliminate the wavefunction along with the operators of the Dirac Hamiltonian to get an expression for the energy density of a Dirac electron in terms of the local observables identified in Tables I and II. The results are not complete in the sense that a fully satisfactory physical understanding has been achieved. However, the linear contribution of external fields is identified and exact expressions for the Thomas and Larmor precessions are found. Also, a number of peculiar features are discovered which are hidden by the operator formulation. Comparison with the usual results will be made in another paper.

Energy is a relative observable, that is, its significance is relative to some inertial frame. However, an atom binding an electron determines an inertial frame, and relative to that frame the electron’s energy has an absolute significance. This frame should be kept in mind, though the following discussion makes no assumption about bound states, and the results hold for any inertial frame designated by a specific choice of the timelike vector γ_0 .

As an aid to physical interpretation and in preparation for the “nonrelativistic limit,” the observables in Tables I and II will be expressed as “relative observables” and related by formulas derived in Ref. 3. The procedure used for introducing relative variables is explained in detail in Ref. 5, so the results and nomenclature of Ref. 5 are freely employed with only the briefest comments.

The “relative” velocity \mathbf{v} of the electron is related to the “proper” velocity v defined in (4.5) by the equation

$$\mathbf{v} \wedge \gamma_0 = \frac{v_0}{c} \mathbf{v}, \quad (6.1a)$$

where

$$v_0 \equiv v \cdot \gamma_0 = (1 - \mathbf{v}^2/c^2)^{-1/2}. \quad (6.1b)$$

Thus

$$v_0 \gamma_0 = v_0(1 + \mathbf{v}/c) = L^2, \quad (6.1c)$$

where L is the spinor in Eq. (6.15) below, which determines the “boost” of γ_0 into v . The “relative” probability density in the inertial system of γ_0 is

$$\rho_0 \equiv (\rho v) \cdot \gamma_0 = \rho v_0. \quad (6.2)$$

A spacetime point x can be designated by a time $t = c^{-1}x^0 = c^{-1}\mathbf{x} \cdot \gamma_0$ and a position $\mathbf{x} = x \wedge \gamma_0$ whence

$$\mathbf{x}\gamma_0 = ct + \mathbf{x}, \quad (6.3)$$

and the derivative $\square = \square_x$ can be expressed in terms of the derivatives

$$\partial_t = c^{-1}\partial_0 = c^{-1}\gamma_0 \cdot \square \quad \text{and} \quad \nabla = \nabla_{\mathbf{x}} = \gamma_0 \wedge \square \quad (6.4a)$$

by the equation

$$\gamma_0 \square = c^{-1}\partial_t + \nabla. \quad (6.4b)$$

Accordingly, the equation for probability conservation

$$\partial_\mu(\rho v^\mu) = \square \cdot (\rho v) = 0 \quad (6.5a)$$

can be written in the familiar relative form

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \mathbf{v}) = 0. \quad (6.5b)$$

Similarly, the proper time derivative

$$d_\tau = \frac{d}{d\tau} = v \cdot \square \quad (6.6a)$$

is related to the so-called “hydrodynamic derivative”

$$d_t \equiv \partial_t + \mathbf{v} \cdot \nabla \quad (6.6b)$$

by the equation

$$d_\tau = v \cdot \square = \frac{v_0}{c} d_t. \quad (6.6c)$$

Now consider the expression of mechanical quantities in terms of relative variables. The kinetic energy-momentum vector $p = \gamma^\mu p_\mu$, whose components are related to the wavefunction by (5.2), can be decomposed into an energy

$$\epsilon = cp \cdot \gamma_0 \quad (6.7a)$$

and a momentum

$$\mathbf{p} = p \wedge \gamma_0 \quad (6.7b)$$

by using the algebraic relation

$$p\gamma_0 = \frac{\epsilon}{c} + \mathbf{p}. \quad (6.7c)$$

(Here the energy-momentum p has been expressed in “momentum units” instead of the “energy units” used in Ref. 5.) The (kinetic) momentum density in the inertial frame of γ_0 is thus

$$\rho_0 \mathbf{p} = \rho v_0 \mathbf{p} = \rho v \cdot \gamma_0 p \wedge \gamma_0. \quad (6.8)$$

Now from (5.3) and (6.7c)

$$\begin{aligned} px &= (p\gamma_0)(\gamma_0 x) = \left(\frac{\epsilon}{c} + \mathbf{p}\right)(ct - \mathbf{x}) \\ &= \epsilon t - \mathbf{p} \cdot \mathbf{x} + ct\mathbf{p} - \frac{\epsilon}{c} \mathbf{x} - \mathbf{p} \wedge \mathbf{x}. \end{aligned}$$

The scalar part of this equation is just the relation

$$p \cdot x = \epsilon t - \mathbf{p} \cdot \mathbf{x},$$

while the proper bivector part is

$$p \wedge x = ct\mathbf{p} - \frac{\epsilon}{c}\mathbf{x} + \mathbf{x} \wedge \mathbf{p}. \quad (6.9a)$$

From this, one immediately sees that the relative vector part of $p \wedge x$ is

$$[p \wedge x]_{(1)} = ct\mathbf{p} - \frac{\epsilon}{c}\mathbf{x}. \quad (6.9b)$$

while the relative bivector part is

$$[p \wedge x]_{(2)} = \mathbf{x} \wedge \mathbf{p} = i\mathbf{x} \times \mathbf{p} \equiv i\mathbf{L}, \quad (6.9c)$$

showing that $p \wedge x$ is an appropriate generalization of the usual orbital angular momentum vector $\mathbf{L} = \mathbf{s} \times \mathbf{p}$, or better, the corresponding angular momentum bivector $\mathbf{x} \wedge \mathbf{p}$. In accordance with (6.8), the “relative orbital angular momentum density” is

$$\rho_0\mathbf{L} = \rho_0\mathbf{x} \times \mathbf{p}. \quad (6.10)$$

For the proper spin vector s , defined by (4.7),

$$s\gamma_0 = s_0 + \mathbf{s} \quad (6.11a)$$

where

$$\mathbf{s} = s \wedge \gamma_0 \quad (6.11b)$$

and, since $s \cdot v = v_0(s_0 - \mathbf{s} \cdot \mathbf{v}/c) = 0$,

$$s_0 \equiv s \cdot \gamma_0 = c^{-1}\mathbf{s} \cdot \mathbf{v}. \quad (6.11c)$$

Using (6.1) and (6.11) in (4.8), one has for the proper spin bivector

$$\begin{aligned} S &= isv = i(s\gamma_0)(\gamma_0v) \\ &= i(s_0 + \mathbf{s})v_0(1 - \mathbf{v}/c) \\ &= iv_0(\mathbf{s} - s_0\mathbf{v}/c - \mathbf{s} \wedge \mathbf{v}/c) \\ &= \frac{v_0}{a}\mathbf{s} \times \mathbf{v} + iv_0(\mathbf{s} - s_0\mathbf{v}/c). \end{aligned}$$

Hence,

$$S = \mathbf{s}_1 + S_2 = \mathbf{s}_1 + i\mathbf{s}_2, \quad (6.12a)$$

where

$$\mathbf{s}_1 \equiv [S]_{(1)} = \frac{v_0}{c}\mathbf{s} \times \mathbf{v} = \frac{v_0}{c}\mathbf{v} \cdot (i\mathbf{s}) \quad (6.12b)$$

is the relative vector part of S , and

$$S_2 = i\mathbf{s}_2 \equiv [S]_{(2)} = iv_0(\mathbf{s} - c^{-1}s_0\mathbf{v}) \quad (6.12c)$$

is the relative bivector part. From (6.9) and (6.12) one finds that the relative bivector part of the proper “total” angular momentum $J \equiv p \wedge x + S$ at a point x is

$$i\mathbf{J} \equiv [J]_{(2)} = \mathbf{x} \wedge \mathbf{p} + S_2 = i(\mathbf{x} \times \mathbf{p} + \mathbf{s}_2), \quad (6.13)$$

so

$$\rho_0\mathbf{J} = \rho_0\mathbf{x} \times \mathbf{p} + \rho_0\mathbf{s}_2, \quad (6.14)$$

is the total (relative) angular momentum density, expressed as a sum of orbital and spin parts.

The trouble with representing spin by the relative vectors \mathbf{s} or \mathbf{s}_2 is they do not have constant magnitude, one of the principal properties of the proper spin s and S . There is an alternative definition of “relative” spin which does not suffer this defect. In the manner explained in Ref. 5, the spinor R introduced by (4.2) can be factored in the form

$$R = LU, \quad (6.15)$$

where the spinor U determines a spatial rotation and L determines a boost of γ_0 to v . Now define a relative spin bivector Σ and vector $\boldsymbol{\sigma}$ by

$$\Sigma = \frac{\hbar}{2} U i \boldsymbol{\sigma}_3 \tilde{U} = i \boldsymbol{\sigma}. \quad (6.16)$$

Using (6.15) in (4.8) one easily shows that Σ is related to S by a boost:

$$S = L \Sigma \tilde{L}, \quad \Sigma = \tilde{L} S L, \quad (6.17a)$$

$$sv = L \boldsymbol{\sigma} \tilde{L}, \quad \boldsymbol{\sigma} = \tilde{L} sv L = \tilde{L} s L \gamma_0. \quad (6.17b)$$

When expressed in terms of relative velocity, the relation of $\boldsymbol{\sigma}$ to \mathbf{s} is found to be [see Eq. (4.38) of Ref. 5],

$$\mathbf{s} = \boldsymbol{\sigma} + \frac{v_0 - 1}{\mathbf{v}^2} \boldsymbol{\sigma} \cdot \mathbf{v} \mathbf{v} = \boldsymbol{\sigma} + \frac{v_0^2}{c^2(v_0 + 1)} \boldsymbol{\sigma} \cdot \mathbf{v} \mathbf{v}, \quad (6.18a)$$

$$cs_0 = \mathbf{s} \cdot \mathbf{v} = v_0 \mathbf{v} \cdot \boldsymbol{\sigma}. \quad (6.18a)$$

Substitution of (6.18) into (6.12c) yields

$$\mathbf{s}_2 = v_0 \left(\boldsymbol{\sigma} + \frac{v_0}{c^2(v_0 + 1)} \boldsymbol{\sigma} \cdot \mathbf{v} \mathbf{v} \right), \quad (6.19)$$

so (6.14) can be written

$$\rho_0 \mathbf{J} = \rho_0 \mathbf{x} \times \mathbf{p} + \rho_0 v_0 \left(\boldsymbol{\sigma} - \frac{v_0}{c^2(v_0 + 1)} \boldsymbol{\sigma} \cdot \mathbf{v} \mathbf{v} \right). \quad (6.20)$$

This shows the correct way to combine $\boldsymbol{\sigma}$ with the orbital angular momentum to get the total relative angular momentum density. Especially in discussions of relativistic approximations, it is important to be clear about which of the several different representations of spin is employed.

From the kinetic energy-momentum vector p it is convenient to form the an energy-momentum vector P defined by

$$P = p + \frac{e}{c} A = 2\gamma^\mu S \cdot (\partial_\mu R \tilde{R}), \quad (6.21)$$

where $A = \gamma^\mu A_\mu$ is the electromagnetic vector potential and the right side of the equation has been obtained from (5.2). The quantity of chief interest in this section, the density $\rho_0 E$ of the total energy $\langle E \rangle = \int d^3x \rho_0 E$, is related to E by

$$E = c \gamma_0 \cdot P = 2c S \cdot (\partial_0 R \tilde{R}) = \epsilon + V, \quad (6.22a)$$

where (6.21) has been used, ϵ is the kinetic energy defined by (6.7a), and

$$V \equiv e A \cdot \gamma_0 \quad (6.22b)$$

is the usual electric potential energy. The corresponding total local momentum is

$$\mathbf{P} = P \wedge \gamma_0 = \mathbf{p} + \frac{e}{c} \mathbf{A} \quad (6.22c)$$

where $\mathbf{A} \equiv A \wedge \gamma_0$. Combining (6.22a) and (6.22c), one has

$$P \gamma_0 = E/c + \mathbf{P}. \quad (6.22c)$$

Equation (6.22a) explicitly exhibits the dependence of the local energy E on the time derivative of the spinor R . Instead of analyzing this expression directly, it is advantageous first to study the energy in the

local rest system determined by the velocity v . Since by (4.5), (4.7), and (4.8) the local velocity and spin are functions only of the spinor R , equations of motion for these “mechanical quantities” along a “streamline” in spacetime with tangent v are determined by an equation of motion for R , which can be put in the form

$$d_\tau R = \frac{1}{2}\Omega R \quad \text{or} \quad \Omega = 2(d_\tau R)\tilde{R}, \quad (6.23)$$

where Ω is a bivector and d_τ is the proper time derivative defined by (6.6a). Expressed in terms of the “angular velocity” Ω the equations of motion for velocity and spin are

$$d_\tau v = \frac{1}{2}[\Omega, v] = \Omega \cdot v, \quad (6.24a)$$

$$d_\tau s = \frac{1}{2}[\Omega, s] = \Omega \cdot s, \quad (6.24b)$$

$$d_\tau S = \frac{1}{2}[\Omega, S]. \quad (6.24c)$$

By using (6.23) along with (6.21), one finds that the local energy in the local rest frame is given by the projection of the angular velocity onto the local spin frame, that is,

$$v \cdot p = (\Omega S)_{(0)} = \Omega \cdot S = v \cdot p + \frac{e}{c} v \cdot A. \quad (6.25)$$

The plan now is to attain a physical interpretation of the energy density by analyzing the “proper angular velocity” Ω . To accomplish this, some physical input is needed besides the relations of observables to the wavefunction, which is all that has been used so far. That input comes from the Dirac equation, which was used in Sec. 6 of Ref. 3 to get the following expression for Ω in terms of local observables:

$$\Omega = \bar{\Omega} + v \cdot \left(mcv \cos \beta + \frac{e}{c} A \right) S^{-1}, \quad (6.26)$$

where

$$\bar{\Omega} = -\square \wedge v - iv \wedge \square \beta = \frac{e}{mc^2} F e^{i\beta} - \frac{e^{i\beta}}{mc} C, \quad (6.27a)$$

in which $F = \square \wedge A$ is the external electromagnetic field

$$C = \{ \partial_\mu W^\mu + \frac{1}{4}[\gamma^\mu \gamma^\nu, [W_\mu, W_\nu] S^{-1}] \}, \quad (6.27b)$$

$$W_\mu = (\rho e^{i\beta})^{-1} \partial_\mu (\rho e^{i\beta} S) = \partial_\mu S + S(\partial_\mu \ln \rho + i\partial_\mu \beta). \quad (6.27c)$$

Substituting (6.26) and (6.27) in (6.25) and recalling the expression (4.10) for the magnetic moment density M , one gets the following expression for the “kinetic energy density” $\rho\epsilon_\nu$ in the local rest system

$$\rho\epsilon_\nu \equiv \rho cv \cdot p = \rho mc^2 \cos \beta + M \cdot F - c^{-1} C \cdot M. \quad (6.28)$$

Lest the reader believe that physical interpretations are being arbitrarily imposed here, it should be pointed out that the identification of $\rho\epsilon_\nu$ as kinetic energy density is a consequence of adopting the conventional interpretation of the operator (5.1) as “kinetic energy-momentum operator” and of the “Dirac current” $\bar{\Psi} \gamma_\mu \Psi = \rho v_\mu$ as probability current. Unconventional as the present discussion may appear, it is based on conventional assumptions of the Dirac theory and conventional principles of relativistic continuum mechanics. The ultimate aim is to discover the full consequences of those assumptions.

According to conventional principles of relativistic continuum mechanics, the quantity $\rho\epsilon_\nu/c^2$ should be interpreted as the electron mass density. If the Dirac theory in fact describes a statistical ensemble of particle motions then v must be only a local average (not an actual) particle velocity, and a deviation of the rest mass density from ρmc^2 due to statistical effects is to be expected. Be this as it may, the term $\rho mc^2 \cos \beta$ has the appearance of a rest mass tensor, and one might guess that the unfamiliar factor $\cos \beta$ is needed to meet the constraint $v^2 = 1$ in a statistical average of particle velocities. The interpretation of the second term in (6.28) requires less speculation, for $M \cdot F = \frac{1}{2} F_{\mu\nu} M^{\mu\nu}$ will be recognized as the classical expression for the increase in mass due to the electromagnetic interaction of a dipole. Interpretation of the last term is difficult, but may be crucial to a complete understanding of the Dirac theory. Considering the

expression (6.27b,c) for C in terms of the spin and the corresponding spin dependence of the momentum flux exhibited in Eq. (3.22) of Ref. 3, it may be guessed that this last term be interpreted as the enhancement of mass due to the local spin flux.

To relate (6.28) to the energy density $\rho_0 E$ note from (6.1) and (6.7) that

$$v \cdot p = \frac{v_0}{c} (\epsilon - \mathbf{v} \cdot \mathbf{p}) : \quad (6.29)$$

hence the total energy density can be written

$$\begin{aligned} \rho_0 E &= c\rho v \cdot p + \rho_0 \mathbf{v} \cdot \mathbf{p} + \rho_0 V \\ &= mc^2 \rho \cos \beta + c\rho [\bar{\Omega} S]_{(0)} + \rho_0 \mathbf{v} \cdot \mathbf{p} + \rho_0 V \end{aligned} \quad (6.30a)$$

or

$$\rho_0 E = mc^2 \rho \cos \beta + M \cdot F - e^{-1} C \cdot M + \rho_0 \mathbf{v} \cdot \mathbf{p} + \rho_0 V. \quad (6.30b)$$

Use of the expressions (6.26), (6.27) for Ω to get the energy density in the form (6.30), which is amenable to physical interpretation, is equivalent to the usual practice of expressing the energy density in terms of the Dirac Hamiltonian, and then systematically replacing the operators and the wavefunction by local observables. To get on with the interpretation of (6.30), it is worth remarking that the magnetic moment of the Dirac electron was first identified theoretically and experimentally precisely by isolating the contribution of the term $M \cdot F$ to the total energy (by a different method, of course). But there is another contribution to the energy due to the interaction of the spin with the external field arising from the Thomas precession. Ordinarily the contribution of the Thomas precession is identified only as a correction to the nonrelativistic approximation. But the formalism used here makes it possible to discuss the role of the Thomas precession exactly.

In Sec. 4 of Ref. 5 the generalized Larmor and Thomas precession of a classical rigid point particle is discussed. All the results obtained there apply immediately to the present problem if only the proper velocity of a particle there is identified with the velocity v of a streamline in the present formulation of the Dirac theory. This paragraph recalls those results which are most pertinent to the present discussion. The angular velocity Ω defined by Eq. (6.23) determines the precession of the velocity and spin through Eq. (6.24). It will be convenient to decompose Ω into relative vector and bivector parts; thus

$$\Omega = 2(d_\tau R)\tilde{R} = \boldsymbol{\alpha} + i\boldsymbol{\beta}, \quad (6.31a)$$

$$\boldsymbol{\alpha} = [\Omega]_{(1)} = \Omega \cdot \gamma_0 \gamma_0, \quad (6.31b)$$

$$i\boldsymbol{\beta} = [\Omega]_{(2)} = \Omega \wedge \gamma_0 \gamma_0. \quad (6.31c)$$

Introducing the factorization (6.15), one finds for the spinor U the equation of motion

$$d_\tau U = \frac{v_0}{c} d_\tau U = \frac{1}{2} \omega U, \quad (6.32)$$

for which the angular velocity can be expressed in the several useful forms

$$\begin{aligned} \omega &= \tilde{L}\Omega L - 2\tilde{L}d_\tau L = \omega^L + \omega^T \\ &= i\left(\boldsymbol{\beta} - \frac{v_0}{c(1+v_0)} \boldsymbol{\alpha} \times \mathbf{v}\right), \end{aligned} \quad (6.33)$$

$$\begin{aligned} \omega^L &= [\tilde{L}\Omega L]_2 \\ &= i\left(\boldsymbol{\beta} + \frac{v_0^2}{c(1+v_0)} (\boldsymbol{\beta} \times \mathbf{v}) \times \mathbf{v} + \frac{v_0}{c} \boldsymbol{\alpha} \times \mathbf{v}\right), \end{aligned} \quad (6.34)$$

$$\begin{aligned} \omega^T &= [2(d_\tau L)\tilde{L}]_2 = [2\tilde{L}d_\tau L]_2 \\ &= \frac{((d_\tau v) \wedge v \wedge \gamma_0) \gamma_0}{1+v_0} = \frac{v_0^2}{c^2(1+v_0)} i\mathbf{v} \times d_\tau \mathbf{v} \\ &= \frac{v_0^2}{c^2(1+v_0)} i\mathbf{v} \times \left(\boldsymbol{\alpha} + \frac{\mathbf{v}}{c} \times \boldsymbol{\beta}\right). \end{aligned} \quad (6.35)$$

These equations give the decomposition of ω into a sum two terms, the generalized Larmor precession ω^L and the Thomas precession ω^T , which is due solely to the acceleration of the particle. One can solve (6.33) for in terms of ω :

$$\Omega = L(\omega + 2\tilde{L}d_\tau L)\tilde{L} = L\omega L + 2(d_\tau L)\tilde{L}, \quad (6.36)$$

which is the more useful when one has the formula

$$2(d_\tau L)\tilde{L} = \frac{(d_\tau L) \wedge (v + \gamma_0)}{1 + v_0}. \quad (6.37)$$

Applying (6.32) and (6.33) to the relative spin vector $\boldsymbol{\sigma}$ defined by (6.16), one obtains the equation of motion

$$\begin{aligned} d_\tau \boldsymbol{\sigma} &= \frac{v_0}{c} d_t \boldsymbol{\sigma} = \boldsymbol{\omega} \cdot \boldsymbol{\sigma} = \omega^L \cdot \boldsymbol{\sigma} + \omega^T \cdot \boldsymbol{\sigma} \\ &= \left(-\boldsymbol{\beta} + \frac{v_0}{c(1+v_0)} \mathbf{v} \times \boldsymbol{\alpha} \right) \times \boldsymbol{\sigma}, \end{aligned} \quad (6.38)$$

a key formula derived by Thomas. Now it must be emphasized that to speak of the Thomas precession in the Dirac theory, it is essential to introduce the relative spin $\boldsymbol{\sigma}$ defined by (6.16); this is the spin obtained, as required by Thomas, from the proper spin S by a “deboost” into an *inertial* system, a system in which the acceleration of the particle is zero. It makes no sense to speak of the Thomas precession of the spin vectors \mathbf{s} or \mathbf{s}_2 defined previously.

The formulation of the Thomas precession just given admits an immediate generalization, simply by replacing the proper time derivative d_τ by the derivatives $\partial_\mu = \gamma_\mu \cdot \square$. Then (6.31a) is replaced by

$$\Omega_\mu = 2(\partial_\mu R)\tilde{R} = \boldsymbol{\alpha}_\mu + i\boldsymbol{\beta}_\mu, \quad (6.39)$$

from which (6.31a) can be recovered since $\Omega = v^\mu \Omega_\mu$ and $d_\tau = v^\mu \partial_\mu$. Similarly, (6.32) generalizes to

$$\partial_\mu U = \frac{1}{2} \omega_\mu U, \quad (6.40a)$$

where

$$\begin{aligned} \omega_\mu &= \tilde{L}\omega_\mu L - 2\tilde{L}\partial_\mu L = \omega_\mu^L + \omega_\mu^T \\ &= i \left(\boldsymbol{\beta}_\mu + \frac{v_0}{c(1+v_0)} \boldsymbol{\alpha}_\mu \times \mathbf{v} \right), \end{aligned} \quad (6.40b)$$

$$\begin{aligned} \omega_\mu^T &= [2(\partial_\mu L)\tilde{L}]_2 = -[2\tilde{L}\partial_\mu L]_2 \\ &= \frac{v_0^2}{c^2(1+v_0)} i\mathbf{v} \times \partial_\mu \mathbf{v} = \frac{v_0^2}{c^2(1+v_0)} i\mathbf{v} \times \left(\boldsymbol{\alpha}_\mu + \frac{\mathbf{v}}{c} \times \boldsymbol{\beta}_\mu \right). \end{aligned} \quad (6.40c)$$

Also,

$$\Omega_\mu = L(\omega_\mu + 2\tilde{L}\partial_\mu R)\tilde{L} \quad (6.41)$$

and

$$\partial_\mu \boldsymbol{\sigma} = \omega_\mu \cdot \boldsymbol{\sigma} = \left(-\boldsymbol{\beta}_\mu + \frac{v_0}{c(1+v_0)} \mathbf{v} \times \boldsymbol{\alpha}_\mu \right) \times \boldsymbol{\sigma}, \quad (6.42a)$$

or, equivalently,

$$\partial_\mu \Sigma = \partial_\mu (i\boldsymbol{\sigma}) = \frac{1}{2} [\omega_\mu, \Sigma], \quad (6.42b)$$

and, differentiating (6.17a) and using (6.41),

$$\partial_\mu S = \frac{1}{2} [\Omega_\mu, S] = L(\partial_\mu \Sigma - \frac{1}{2} [\omega_\mu^T, \Sigma])\tilde{L}. \quad (6.42c)$$

We are now prepared to separate the energy-momentum due to variations in velocity from other contributions to the total energy-momentum. Using (6.39) in (6.21) followed by (6.17), (6.41), and (6.42), we find

$$\begin{aligned}\gamma_\mu \cdot P &= P_\mu = [\Omega_\mu S]_{(0)} = [(\omega_\mu + 2\tilde{L}(\partial_\mu L))\Sigma]_{(0)} \\ &= P'_\mu - \omega_\mu^T \cdot \Sigma,\end{aligned}\tag{6.43}$$

where P'_μ is defined by

$$P'_\mu \equiv \hbar[\partial_\mu U i\boldsymbol{\sigma}_3 \tilde{U}]_{(0)} = [\omega_\mu \Sigma]_{(0)} = \omega_\mu \cdot \Sigma.\tag{6.44}$$

From (6.43) one sees that

$$v \cdot P = v^\mu P_\mu = \Omega \cdot S = (\omega - \omega^T) \cdot \Sigma = \omega^L \cdot \Sigma\tag{6.45}$$

is just the generalized Larmor precession energy, whereas, from (6.44),

$$v \cdot P' = \omega \cdot \Sigma = \omega^L \cdot \Sigma + \omega^T \cdot \Sigma\tag{6.46}$$

includes also the Thomas precession energy. Removing the potential energy contribution from (6.44) by writing

$$p'_\mu = P'_\mu - \frac{e}{c} A_\mu = \hbar[\partial_\mu U i\boldsymbol{\sigma}_3 \tilde{U}]_{(0)} - \frac{e}{c} A_\mu,\tag{6.47}$$

one finds, since $\omega^T = v^\mu \omega_\mu^T = v_0(\omega_0^T - c^{-1}\mathbf{v}_k \omega_k^T)$,

$$\mathbf{v} \cdot \mathbf{p} = \mathbf{v} \cdot \mathbf{p}' + \frac{c}{v_0} \omega^T \cdot \Sigma - c\omega_0^T \cdot \Sigma.\tag{6.48}$$

It is convenient to replace Ω by $\bar{\Omega}$ in the formulation of precession energy. This is perfectly permissible since, as is easily shown by substituting (6.26) into (6.24), it does not alter the precession of the velocity and the spin. The replacement does not affect the Thomas precession; it merely changes the definition of the generalized Larmor precession from (6.34) to

$$\bar{\omega}^L = [\tilde{L}\bar{\Omega}L]_2.\tag{6.49}$$

Corresponding changes in other quantities will also be indicated by overbars.

Now switching from Ω to $\bar{\Omega}$ and substituting (6.45) and (6.48) into (6.30a), the energy density is put in the form

$$\rho_0 E = mc^2 \rho \cos \beta + c\rho \bar{\omega} \cdot \Sigma - c\rho_0 \omega_0^T \cdot \Sigma + \rho_0 \mathbf{v} \cdot \mathbf{p}' + \rho_0 V.\tag{6.50}$$

The term $c\rho \bar{\omega} \cdot \Sigma = \rho_0[(c/v_0)(\bar{\omega}^L + \omega^T)\Sigma]_{(0)}$ includes both the generalized Larmor and Thomas precession energies. It is important to note that the change in the definition of the kinetic momentum from \mathbf{p} to \mathbf{p}' is essential to make the Thomas term explicit. The term $c\rho_0 \omega_0^T \cdot \Sigma$ vanishes for stationary states, since by (6.40c) $\omega_0^T = 0$ if $c\partial_0 \mathbf{v} = \partial_t \mathbf{v} = 0$.

To study the dependence of the precession energy in (6.50) on relative observables, express F in terms of the electric and magnetic fields \mathbf{E} and \mathbf{B} relative to γ_0 ;

$$F = \mathbf{E} + i\mathbf{B}, \quad \text{hence} \quad iF = -\mathbf{B} + i\mathbf{E}.\tag{6.51}$$

Also write

$$C = \mathbf{C}_1 + C_2 = \mathbf{C}_1 + i\mathbf{C}_2\tag{6.52}$$

where $\mathbf{C}_1 \equiv [C]_1$ and $C_2 = i\mathbf{C}_2 \equiv [C]_2$. The definition (6.27b) (6.42c) shows that C depends on $\partial_\mu S$ and S ; these quantities can be expressed in terms of $\partial_\mu \Sigma$ and Σ by using (6.42a) and (6.17a), but this step will not be carried out, because it is not clear how to derive any physical insight from it. The terms which are

difficult to understand will be kept lumped together. Now using (6.51) and (6.52) in (6.27b) and then in (6.33), one gets

$$\begin{aligned}
c\rho\bar{\omega} \cdot \Sigma &= -\frac{e}{mc} \left(\boldsymbol{\sigma} \cdot \mathbf{B} + \frac{v_0}{c(1+v_0)} \boldsymbol{\sigma} \cdot (\mathbf{E} \times \mathbf{v}) \right) \rho \cos \beta \\
&+ \left(\boldsymbol{\sigma} \cdot \mathbf{C}_2 - \frac{v_0}{c(1+v_0)} \boldsymbol{\sigma} \cdot (\mathbf{C}_1 \times \mathbf{v}) \right) \frac{\rho \cos \beta}{m} \\
&- \frac{e}{mc} \left(\boldsymbol{\sigma} \cdot \mathbf{E} - \frac{v_0}{c(1+v_0)} \boldsymbol{\sigma} \cdot (\mathbf{B} \times \mathbf{v}) \right) \rho \sin \beta \\
&+ \left(\boldsymbol{\sigma} \cdot \mathbf{C}_1 + \frac{v_0}{c(1+v_0)} \boldsymbol{\sigma} \cdot (\mathbf{C}_2 \times \mathbf{v}) \right) \rho \frac{\sin \beta}{m}.
\end{aligned} \tag{6.53}$$

The terms in first line of (6.53) except for the factor $\cos \beta$ are exactly the terms derived by Thomas (see Ref. 5), who evaluated the precession energy under the assumption that $\bar{\Omega} = (e/mc^2)F$. Thomas wisely evaluated the energy only in the nonrelativistic approximation where $v_0(1+v_0)^{-1} \approx \frac{1}{2}$ (and, fortunately, $\cos \beta \approx 1$). In higher order approximations the effect of the external field through terms in the second line of (6.53) is probably important, though it is not clear how to make this explicit. It might be guessed that because of the smallness of $\sin \beta$, the unfamiliar terms in the last two rows are generally insignificant, but it will be seen that the $\boldsymbol{\sigma} \cdot \mathbf{E}$ term may be of the same order of magnitude as the $\boldsymbol{\sigma} \cdot (\mathbf{E} \times \mathbf{v})$ term in the first row.

For an electron in a central field,

$$e\mathbf{E} = -\nabla V = -V' \frac{\mathbf{x}}{|\mathbf{x}|} \quad \text{with} \quad V' = \frac{dV}{d|\mathbf{x}|},$$

in which case the second term in (6.53) can be written

$$\frac{e}{m^2 c^2} \frac{\rho}{(1+v_0)} \frac{V'}{|\mathbf{x}|} \boldsymbol{\sigma} \cdot \mathbf{L}^*, \tag{6.54a}$$

where

$$\mathbf{L}^* = \mathbf{x} \times (mv_0 \mathbf{v}) \cos \beta. \tag{6.54b}$$

The term (6.54a) has the well-known form of the spin-orbit coupling; however, L^* differs from the orbital angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$, because the relation $\mathbf{p} = mv_0 \mathbf{v}$ assumed by Thomas does not hold in the Dirac theory. To estimate the magnitude of this discrepancy the correct relation of \mathbf{p} to \mathbf{v} will now be found.

The correct relation of the kinetic energy-momentum p to the particle velocity v in the Dirac theory was found in Sec. 5 of Ref. 3 to be given by

$$\rho p = mc\rho v \cos \beta - \square \cdot (\rho S) + \square \cdot \rho (iS) \cdot \square \beta. \tag{6.55}$$

To express this as a relation among relative observables, multiply by γ_0 and recall (6.7) and (6.1) to get

$$\frac{\epsilon}{c} + \mathbf{p} m c v_0 \left(1 + \frac{v}{c}\right) - \rho \square \cdot (\rho S) \gamma_0 - (\square \beta) \cdot (iS) \gamma_0. \tag{6.56}$$

But, by virtue of (6.4) and (6.12a),

$$\begin{aligned}
\gamma_0 \square \cdot (\rho S) &= [\gamma_0 \square (\rho S)]_{(0+1)} = [(\partial_0 + \nabla) \rho (\mathbf{s}_1 + i\mathbf{s}_2)]_{(0+1)} \\
&= \nabla \cdot (\rho \mathbf{s}_1) + \partial_0 (\rho \mathbf{s}_1) + \nabla \cdot (\rho i\mathbf{s}_2)
\end{aligned}$$

and

$$\begin{aligned}
\gamma_0 (\square \beta) \cdot (iS) &= [\gamma_0 (\square \beta) iS]_{(0+1)} = [(\partial_0 \beta + \nabla \beta) (i\mathbf{s}_1 - \mathbf{s}_2)]_{(0+1)} \\
&= -\mathbf{s}_2 \cdot \nabla \beta - \mathbf{s}_2 \partial_0 \beta + (\nabla \beta) \cdot (i\mathbf{s}_1).
\end{aligned}$$

Hence the scalar part of (6.56) can be written

$$\epsilon = mc^2 v_0 \cos \beta - c\rho^{-1} \nabla \cdot (\rho \mathbf{s}_1) + c\mathbf{s}_2 \cdot \nabla \beta, \quad (6.57a)$$

while the vector part can be written

$$\mathbf{p} = m\mathbf{v}v_0 \cos \beta - \rho^{-1} \nabla \times (\rho \mathbf{s}_2) + \mathbf{s}_1 \times \nabla \beta + \rho^{-1} \partial_0 (\rho \mathbf{s}_1) - \mathbf{s}_2 \partial_0 \beta. \quad (6.57b)$$

Using (6.57b) to eliminate $m\mathbf{v}v_0 \cos \beta$ in (6.54), one indeed gets the desired spin-orbit term with \mathbf{L} instead of \mathbf{L}^* , but there are several additional terms as well. Some understanding of the additional terms can be achieved by comparing with results known in the literature, but to do so it is necessary to take the nonrelativistic limit, since it is only in connection with that approximation that the Thomas precession has been discussed previously. This will be done later.

The discussion of the Dirac energy density in this section has concentrated on a detailed interpretation of a few terms. A satisfactory interpretation of all the terms has not been found, but let us review the general approach. The effective mass density given by (6.28) differs from what appears to be a rest energy term $\rho mc^2 \cos \beta$ by a term commonly called the internal energy density. Accordingly, it is natural to call the “generalized Larmor” term $c\bar{\Omega} \cdot S = c\bar{\omega}^T \cdot \Sigma$ the *internal energy* of the system. The magnitude of the internal energy depends, of course, on the interaction with external fields, which (6.30b) expresses to terms linear in the external field F by $c\rho\bar{\omega}^L \cdot \Sigma = M \cdot F + e^{-1}C \cdot M$. The kinetic energy density $\rho_0 \mathbf{v} \cdot \mathbf{p}$ the system is also influenced by external fields, which was found to terms linear in the field by separating the Thomas precession energy from other contributions to the kinetic energy with $\rho_0 \mathbf{v} \cdot \mathbf{p} = \rho_0 \mathbf{v} \cdot \mathbf{p}' + \rho c\omega^T \cdot \Sigma - c\rho_0 \omega_0^T \cdot \Sigma$. The Larmor and Thomas precessions were combined in (6.50) to get the total influence of external fields (aside from the potential energy V of course). Nevertheless, it is very important to be able to separate contributions to the mass density from contributions to the kinetic energy.

7. Observables in the Nonrelativistic Limit

The exact constitutive relations found for relative observables in the last section are rather complicated and difficult to interpret. The relations simplify greatly in the nonrelativistic limit to be determined here; still they remain nontrivial. It will be shown that the local momentum and the Gordon current are equal in that limit, but they differ from the local velocity by a “magnetization current.” In a subsequent paper this result will be shown to have important implications for the interpretation of spin-orbit coupling and the Pauli and Schrödinger theories. Also, for future use, the nonrelativistic form of the spin-electric energy density is found.

The adjective “relativistic” was criticized in Ref. 5, but it will nevertheless be employed here, because it is almost universally used in connection with the topic under discussion. It may be well to recall, therefore, that sometimes the word “relativistic” means that the relative speed $|\mathbf{v}|$ is of the order of the velocity of light c ; sometimes it means that accelerations are small, and sometimes it means that an expression or quantity is completely independent of the relative velocity. The term “nonrelativistic” will be used here especially to mean $|\mathbf{v}| \ll c$. Furthermore, it should be emphasized that the so-called “nonrelativistic limit” of the Dirac theory involves a number of other assumptions—reasonable assumptions about the magnitude of external fields, and about which quantities are slowly varying functions of position, in particular, about the curious quantity β .

It will not be necessary to spell out such assumptions, because they will be implicit in the approximate equations written down. From (6.16) one gets in the nonrelativistic limit

$$v_0 = (1 - \mathbf{v}^2/c^2)^{-1/2} = 1 + \frac{1}{2} \frac{\mathbf{v}^2}{c^2} + \dots \approx 1. \quad (7.1)$$

Using this in the several equations defining the relative spins one finds

$$\mathbf{s} \approx \mathbf{s}_2 \approx \boldsymbol{\sigma}, \quad (7.2a)$$

$$S \approx S_2 \approx \Sigma \approx i\mathbf{s}, \quad (7.2b)$$

$$s_0 \ll |\mathbf{s}|, \quad (7.2c)$$

$$\mathbf{s}_1^2 \ll \mathbf{s}^2 \approx |S|^2 = -S^2 = \frac{1}{4}\hbar^2. \quad (7.2d)$$

Also, of course,

$$\rho_0 = \rho v_0 \approx \rho,$$

and (6.20) reduces to the familiar expression

$$\rho_0 \mathbf{J} \approx \rho \mathbf{J} \approx \rho(\mathbf{x} \times \mathbf{p} + \mathbf{s}).$$

From Eq. (2.18) of Ref. 3, one gets

$$-mc\rho_0 \sin \beta = c^{-1}\partial_t(\rho s_0) + \nabla \cdot (\rho \mathbf{s}).$$

Hence for stationary or slowly varying states one has

$$\sin \beta = \frac{-1}{mc} \frac{\nabla \cdot (\rho \mathbf{s})}{\rho}. \quad (7.3a)$$

To this may be added the condition

$$|\sin \beta| \approx |\beta| \leq |\mathbf{v}|/c, \quad (7.3b)$$

which seems reasonable in view of the numerical factor $|\mathbf{s}|/mc \approx \hbar/2mc$ on the right side of (7.3a). Better justification will be given in a subsequent paper.

Employing the above approximations, one finds that (6.57b) reduces to the important equation

$$\mathbf{p} = m\mathbf{v} - \rho^{-1}\nabla \times (\rho \mathbf{s}) = m\mathbf{v} - \rho^{-1}\nabla \cdot (\rho S). \quad (7.4)$$

Higher order terms must be carried to get the appropriate approximation to (6.57a), since the corrections to the large “rest energy” term mc^2 are of interest. Accordingly, recalling especially (6.12b), one find:

$$\begin{aligned} \epsilon - mc^2 &\approx \frac{1}{2}m\mathbf{v}^2 - \frac{1}{2}mc^2\beta^2 + \rho^{-1}\nabla \cdot (\rho \mathbf{s} \times \mathbf{v}) + c\mathbf{s} \cdot \nabla \beta \\ &= \frac{1}{2}m\mathbf{v}^2 + \frac{1}{2}mc^2\beta^2 + \rho^{-1}\nabla \cdot [\rho(c\beta \mathbf{s}) + \mathbf{v} \times \mathbf{s}] \\ &\approx \frac{1}{2}m\mathbf{v}^2 + \frac{1}{2m} \left(\frac{\nabla \cdot (\rho \mathbf{s})}{\rho} \right)^2 + \rho^{-1}\nabla \cdot \left(-\frac{1}{2m} \mathbf{s} \nabla \cdot (\rho \mathbf{s}) + \rho \mathbf{v} \times \mathbf{s} \right). \end{aligned} \quad (7.5)$$

This is an appropriate place to discuss the physical interpretation and examine the N. R. limit of the Gordon current k , whose components are defined in Table II. In Sec. 5 of Ref. 3 the Gordon current was found to be related to the velocity and magnetization or spin by the exact equation

$$j \equiv e\rho v = \frac{e}{mc} k + \square \cdot M = \frac{e}{mc} k + e\square \cdot (\rho e^{i\beta} S), \quad (7.6)$$

and to the proper energy-momentum density by

$$k = \rho p \cos \beta - \rho \sin \beta, \quad (7.7a)$$

where

$$q = \gamma^\mu v \cdot \partial_\mu s = -\gamma^\mu s \cdot \partial_\mu v. \quad (7.7b)$$

Given the conservation law (6.5) for the Dirac current $j = e\rho v$ and the identity $\square \cdot (\square \cdot M) = 0$, one finds from (7.6), the conservation law

$$\square \cdot k = \partial_\mu k^\mu = 0. \quad (7.8)$$

Accepting the conventional interpretation of j as the total charge current and identifying $\square \cdot M$ as a magnetization current, one is lead by (7.6) to interpret $(e/mc)k$ as a convection current. In this way the

Gordon current is given a physical interpretation, however, the significance of its close relation to the energy-momentum exhibited by (7.7) remains obscure, though it is clearly tied up with the significance of β . Since, as has already been mentioned, $\sin \beta$ should be regarded as a small quantity, (7.7) shows that the Gordon current is nearly proportional to the energy-momentum density. However, it is not possible in the exact Dirac theory to identify $(e/mc)\rho\mathbf{p}$ as a charge current density, because it does not have vanishing divergence. One can express $\square \cdot (\rho\mathbf{p})$ in terms of other observables by taking the divergence of (6.55), obtaining immediately

$$\square \cdot (\rho\mathbf{p}) = -mc\rho v \cdot (\square\beta) \sin \beta + \{\square \cdot (\rho iS)\} \cdot \square\beta.$$

But, as shown in Sec. 5 of Ref. 3 the Dirac equation also implies

$$\square \cdot (i\rho S) = mc\rho v \sin \beta + (\square\beta) \cdot (\rho S) + \rho q.$$

Hence, one gets the exact relations

$$\square \cdot (\rho\mathbf{p}) = \rho q \cdot \square\beta = \rho v \cdot \{(\square\beta) \cdot \square\mathbf{s}\} = -\rho\mathbf{s} \cdot \{(\square\beta) \cdot \square\mathbf{v}\}. \quad (7.9)$$

Regarding β as small, one gets immediately from (7.7a)

$$k \approx \rho\mathbf{p}. \quad (7.10)$$

By (6.7) and (7.5), the “time component” of the Gordon current is, in the N.R. limit,

$$\gamma_0 \cdot k \approx \rho \frac{e}{c} \approx mc\rho, \quad (7.11a)$$

while the “space component” is

$$\mathbf{k} \equiv k \wedge \gamma_0 = \rho\mathbf{p}. \quad (7.11b)$$

And, if β is slowly varying, one gets from (7.9)

$$\square \cdot (\rho\mathbf{p}) \approx m\partial_t \rho + \nabla \cdot (\rho\mathbf{p}) = 0. \quad (7.12)$$

Substituting (7.11b) into (7.4), one gets

$$e\rho\mathbf{v} = \frac{e}{m}\mathbf{k} + \nabla \times \left(\rho \frac{e}{m}\mathbf{s}\right), \quad (7.13)$$

the N. R. expression for a charge current expressed as a conduction current plus a magnetization current.

Returning now to the expression (6.53) for the interaction energy density, we are particularly interested in the N. R. limit of the two terms explicitly involving the electric field. Let us refer to these terms collectively as the spin-electric energy density and denote them by $\rho_0 E_{SE}$. Recalling (7.3b), we find that (6.53) gives us

$$\rho_0 E_{SE} \approx -\frac{e}{2mc^2} \rho\mathbf{s} \cdot (\mathbf{E} \times \mathbf{v}) - \frac{e}{mc} \mathbf{s} \cdot \mathbf{E} \rho\beta. \quad (7.14)$$

Using (7.4) we can determine how the first term in (7.14) couples to the momentum instead of the velocity. Thus, using the fact that $\mathbf{s}^2 = \frac{1}{4}\hbar^2$ implies $\mathbf{s} \cdot (\partial_k \mathbf{s}) = 0$, we find that

$$\begin{aligned} m\rho\mathbf{s} \cdot (\mathbf{E} \times \mathbf{v}) - \rho\mathbf{s} \cdot (\mathbf{E} \times \mathbf{p}) &= \mathbf{s} \cdot [\mathbf{E} \times (\nabla \times \rho\mathbf{s})] \\ &= -\mathbf{s}^2 \mathbf{E} \cdot \nabla \rho + \mathbf{E} \cdot \mathbf{s}\mathbf{s} \cdot \nabla \rho + \rho \mathbf{E} \cdot (\mathbf{s} \cdot \nabla \mathbf{s}). \end{aligned} \quad (7.15)$$

By virtue of (7.3a) the last term in (7.14) can be written

$$-\frac{e}{mc} \mathbf{s} \cdot \mathbf{E} \rho\beta = \frac{e}{m^2 c^2} \mathbf{s} \cdot \mathbf{E} \nabla \cdot (\rho\mathbf{s}), \quad (7.16)$$

which reveals that it gives a contribution of the same order of magnitude as the next to last term in (7.14). With the help of (7.15) and (7.16) we can write (7.14) in the form

$$\begin{aligned} \rho_0 E_{SE} = & -\frac{e}{2m^2c^2} \{ \rho \mathbf{s} \cdot (\mathbf{E} \times \mathbf{p}) + \mathbf{s}^2 (\rho \nabla \cdot \mathbf{E} - \nabla \cdot (\rho \mathbf{E})) \} \\ & - \frac{e}{2m^2c^2} \mathbf{E} \cdot \{ \mathbf{s} (\mathbf{s} \cdot \nabla \rho + 2 \nabla \cdot (\rho \mathbf{s})) + \mathbf{s} \cdot \nabla \mathbf{s} \}. \end{aligned} \quad (7.17)$$

Discussion of this result will be deferred until a subsequent paper, when we will be in a position to complete it with results obtained by conventional methods.

¹D. Hestenes, *Space-Time Algebra* (Gordon and Breach, New York, 1966).

²D. Hestenes, *J. Math. Phys.* **8**, 798 (1967).

³D. Hestenes, *J. Math. Phys.* **14**, 893 (1973). [By an error in proof, the physical interpretations of the Dirac and Gordon currents in (5.6) of this reference got interchanged in the subsequent sentence.]

⁴S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row Peterson, New York, 1961), p.70–80.

⁵D. Hestenes, *J. Math. Phys.* **15**, 1768 (1974).

⁶D. Hestenes, *J. Math. Phys.* **15**, 1778 (1974).

⁷J. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading Mass., 1955).

⁸D. Hestenes, *J. Math. Phys.* **8**, 809 (1967). Equation (2.9b) of this paper should be corrected by adding the term $-2m\mathbf{s} \cdot \mathbf{J}_\mu \sin \beta$ to its right-hand side. (The calculation follows Appendix B of Ref. 3.) This correction does not affect the main ideas of the paper.

⁹D. Hestenes, *J. Math. Phys.* **8**, 809 (1967).

¹⁰H. Bethe and E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer-Verlag, Berlin, 1957), p.54.