

# Mysteries and Insights of Dirac Theory

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The Dirac equation has a hidden geometric structure that is made manifest by reformulating it in terms of a real spacetime algebra. This reveals an essential connection between spin and complex numbers with profound implications for the interpretation of quantum mechanics. Among other things, it suggests that to achieve a complete interpretation of quantum mechanics, spin should be identified with an intrinsic zitterbewegung.

## I. Introduction

“I entirely agree . . . there are still many mysteries in the Dirac electron theory.” So wrote Olivier Costa de Beauregard in a letter to me dated July 8, 1968. Beginning with his doctoral thesis [1], much of his career has been devoted to exploring those mysteries. Though he himself made seminal contributions to understanding the Dirac theory, I am sure he would agree that many mysteries remain today, and their resolution is crucial to clarifying the physical content of quantum mechanics. In tribute to Olivier and acknowledging my debt to him, I cannot do better than review my own perspective on the Dirac mysteries and the insights we can gain from studying them.

A related mystery that has long puzzled me is why Dirac theory is almost universally ignored in studies on the interpretation of quantum mechanics, despite the fact that the Dirac equation is widely recognized as the most fundamental equation in quantum mechanics. That is a huge mistake, I believe, and I hope to convince you that Dirac theory provides us with insights, or hints at least, that are crucial to understanding quantum mechanics and perhaps to modifying and extending it. Specifically, I claim that an analysis of Dirac theory supports the following propositions:

(P1) *Complex numbers are inseparably related to spin in Dirac theory.* Hence spin is essential to the interpretation of quantum mechanics even in Schroedinger theory.

(P2) *Bilinear observables are geometric consequences of rotational kinematics,* so they are as natural in classical mechanics as in quantum mechanics

(P3) *Electron spin and phase are inseparable kinematic properties of electron motion (zitterbewegung).*

Though the first two propositions are not well known, they should not be controversial, because they are rigorous features of Dirac theory that are brought to light by reformulating it in a way that makes its inherent geometric structure explicit. The third proposition is debatable, but it has the virtue of providing a more complete account of the structure of Dirac theory than any alternative. I have dubbed it the *Zitterbewegung interpretation of quantum mechanics* [2].

It strongly suggests that quantum phenomena have a substructure that is not fully captured in anybody's theory.

These three propositions have been discussed at length before, so I refer to recent reviews [3, 4]. Here I supply only enough detail to make grounds for the propositions understandable and to comment on related mysteries.

## II. Spacetime Algebra

One of the most familiar mysteries of quantum mechanics is the essential role of complex amplitudes. So it may be surprising to learn that complex scalars are superfluous in the Dirac theory. This has been unequivocally proved by reformulating the Dirac theory in terms of the *real Spacetime Algebra* (STA), often called a Clifford algebra in the mathematics literature. We need a brief introduction to STA to see how it changes the form of Dirac theory.

For readers familiar with the Dirac matrix algebra, the quickest approach to the STA is by reinterpreting the Dirac matrices as an *orthonormal frame*  $\{\gamma_\mu; 0, 1, 2, 3\}$  of basis vectors for spacetime. The signature of spacetime is specified by the rules:

$$\gamma_0^2 = 1 \quad \text{and} \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1 \quad (1)$$

Note that the scalar 1 in these equations would be replaced by the identity matrix if the  $\gamma_\mu$  were Dirac matrices. Thus, (1) is no mere shorthand for matrix equations but a defining relation of vectors to scalars that encodes spacetime signature in algebraic form.

Of course, any spacetime vector  $v$  can be expressed as a linear combination  $v = v^\mu \gamma_\mu$  of the basis vectors. The entire STA is generated by defining an associative *geometric product* on the vectors. For vectors  $u, v$  the product  $uv$  can be decomposed into a symmetric *inner product*

$$u \cdot v = \frac{1}{2}(uv + vu) = v \cdot u, \quad (2)$$

and an antisymmetric *outer product*

$$u \wedge v = \frac{1}{2}(uv - vu) = -v \wedge u. \quad (3)$$

so that

$$uv = u \cdot v + u \wedge v. \quad (4)$$

It follows from (1) that the inner product  $u \cdot v$  is scalar-valued and, indeed, is the standard inner product for Minkowski spacetime. The outer product produces a new kind of geometric object  $u \wedge v$  called a *bivector*, which can be interpreted geometrically as an oriented area for the plane containing  $u$  and  $v$ .

By forming all distinct products of the  $\gamma_\mu$  we obtain a complete basis for the STA consisting of the  $2^4 = 16$  linearly independent elements

$$1, \quad \gamma_\mu, \quad \gamma_\mu \wedge \gamma_\nu, \quad \gamma_\mu i, \quad i = \gamma_0 \gamma_1 \gamma_2 \gamma_3. \quad (5)$$

A generic element  $M$  of the STA, called a *multivector*, can therefore be written in the *expanded form*

$$M = \alpha + a + F + bi + \beta i, \quad (6)$$

where  $\alpha$  and  $\beta$  are scalars,  $a$  and  $b$  are vectors,  $F = \frac{1}{2}F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu$  is a bivector, and  $i$  is the unit pseudoscalar.

Computations are facilitated by introducing a few definitions and notations. Thus, the multiplicative *reverse* of  $M$  can be defined by

$$\tilde{M} = \alpha + a - F - bi + \beta i. \quad (7)$$

The scalar part of  $M$  is denoted by  $\langle M \rangle$ . The multivector  $M$  is said to be *even* if its vector part  $a$  and its pseudovector part  $ib$  vanish.

### III. Real Dirac Theory

At last we are prepared to write the Dirac equation in terms of the real STA:

$$\partial\psi\gamma_2\tilde{\gamma}_1\hbar - eA\psi = m\psi\gamma_0, \quad (8)$$

where  $\partial = \gamma^\mu\partial_\mu$ ,  $A = A^\mu\gamma_\mu$  is the electromagnetic vector potential, and the *wave function*  $\psi$  is an even multivector. This equation is called the *real Dirac equation*, because no complex scalars are involved.

Some physicists recoil at first sight of the explicit  $\gamma_2\gamma_1$  and  $\gamma_0$  in (8), claiming immediately that the equation cannot be Lorentz covariant. However, covariance is easily proved [3], and the mistaken impression that (8) is more complicated than the standard matrix form of the Dirac equation is dispelled when its geometric structure is understood. Indeed, replacement of the unit imaginary in the matrix Dirac equation by the bivector  $\gamma_2\gamma_1 = \gamma_2 \wedge \gamma_1$  in (8) points undeniably to a geometric meaning for complex numbers in quantum mechanics.

Two advantages of (8) over the matrix version can be noted at once. First, it shows that particular matrix representations are irrelevant to the physics in Dirac theory, because they have been completely eliminated. Second, it shows that complex scalars are likewise irrelevant by eliminating them in favor of real scalars. In other words, the Dirac matrix algebra over a complex scalar field has twice as many degrees of freedom as needed for the physics. Eliminating these superfluous degrees of freedom not only simplifies the theory, it opens the door to clarifying the geometric structure of the theory, and that has direct bearing on its physical interpretation.

Although I first derived the real form (8) for the Dirac equation in [5], I did not really understand its significance until [6], where I derived it in a different way and established the geometric content of the wave function  $\psi$  as follows.

One of the postulates of quantum mechanics is that observables are bilinear functions of the wave equation. Since  $\psi$  is an even multivector, the bilinear quantity  $\psi\tilde{\psi}$  can have only scalar and pseudoscalar parts, as expressed by writing

$$\psi\tilde{\psi} = \rho e^{i\beta} = \rho(\cos\beta + i\sin\beta), \quad (9)$$

where  $\rho$  and  $\beta$  are scalars. It follows that, if  $\rho \neq 0$ , we can write  $\psi$  in the invariant *canonical form*

$$\psi = (\rho e^{i\beta})^{\frac{1}{2}} R, \quad (10)$$

where

$$R\tilde{R} = \tilde{R}R = 1. \quad (11)$$

A set of bilinear vector observables is constructed from  $\psi$  by writing

$$\psi\gamma_\mu\tilde{\psi} = \rho e_\mu, \quad (12)$$

where

$$e_\mu = R\gamma_\mu\tilde{R}. \quad (13)$$

This shows that  $R$  is a *rotor* (or spin representation of a Lorentz transformation) that takes a fixed frame  $\{\gamma_\mu\}$  into a frame  $\{e_\mu\}$ . Thus, six of the eight degrees of freedom in the Dirac wave function can be identified with a Lorentz transformation.

So far everything said about  $\psi$  is simply mathematics. Its importance is that it establishes the purely geometric properties of the wave function. Now the problem is to use this geometric insight in establishing a physical interpretation of the wave function. First, we can identify

$$\psi\gamma_0\tilde{\psi} = \rho e_0 = \rho v \quad (14)$$

as the *Dirac probability current* by deriving its conservation law

$$\langle \partial(\psi\gamma_0\tilde{\psi}) \rangle = \partial \cdot (\rho v) = 0 \quad (15)$$

from the Dirac equation (8). This supports the standard physical interpretation of  $\rho$  as a *probability density* and the unit timelike vector  $v = e_0$  as *electron velocity* along streamlines of the continuity equation (15).

The notation

$$s = \frac{1}{2}\hbar e_3 \quad (16)$$

suggests identification as a *spin vector*. Analysis of angular momentum conservation shows that spin is properly represented by the bivector quantity

$$S = \frac{1}{2}\hbar R\gamma_2\gamma_1\tilde{R} = \frac{1}{2}\hbar e_2e_1 = \frac{1}{2}\hbar i e_3e_0 = i s v. \quad (17)$$

The last term here justifies the representation of spin by a vector.

These considerations show that the physical interpretation given to the frame field  $\{e_\mu\}$  is a key to interpretation of the entire Dirac theory. Their identification with electron velocity and spin shows that the  $e_\mu$  can be interpreted directly as descriptors of the local kinematics of electron motion. It follows from (13),

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**TABLE I: BILINEAR COVARIANTS**

Scalar	$\tilde{\Psi}\Psi = \Psi^\dagger\gamma_0\Psi = \langle\psi\tilde{\psi}\rangle = \rho\cos\beta$
Vector	$\begin{aligned}\tilde{\Psi}\gamma_\mu\Psi &= \Psi^\dagger\gamma_0\gamma_\mu\Psi = \langle\psi\gamma_0\tilde{\psi}\gamma_\mu\rangle = \langle\psi^\dagger\gamma_0\gamma_\mu\psi\rangle \\ &= (\psi\gamma_0\tilde{\psi})\cdot\gamma_\mu = (\rho v)\cdot\gamma_\mu = \rho v_\mu\end{aligned}$
Bivector	$\begin{aligned}\frac{e}{m}\frac{i'\hbar}{2}\tilde{\Psi}\frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\Psi &= \frac{e\hbar}{2m}\langle\gamma_\mu\gamma_\nu\psi\gamma_2\gamma_1\tilde{\psi}\rangle \\ &= (\gamma_\mu\wedge\gamma_\nu)\cdot M = M_{\nu\mu} = \frac{e}{m}\rho(ie^{i\beta}sv)\cdot(\gamma_\mu\wedge\gamma_\nu)\end{aligned}$
Pseudovector	$\frac{1}{2}i'\hbar\tilde{\Psi}\gamma_\mu\gamma_5\Psi = \frac{1}{2}\hbar\langle\gamma_\mu\psi\gamma_3\tilde{\psi}\rangle = \gamma_\mu\cdot(\rho s) = \rho s_\mu$
Pseudoscalar	$\tilde{\Psi}\gamma_5\Psi = \langle i\psi\tilde{\psi}\rangle = -\rho\sin\beta$

In this table the column matrix representation of the wave function is denoted by  $\Psi$ , the scalar imaginary unit is denoted by  $i'$ , and the more conventional symbol  $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$  is used for the matrix representation of the unit pseudoscalar  $i$ . The electron charge has been inserted in the bivector expression to show its identification as the magnetization tensor in the Dirac theory.

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therefore, that the rotor  $R$  component of the electron wave function (10) is a descriptor of local electron kinematics.

But spin and velocity determine only five of the six parameters in the local Lorentz transformation (13) specified by  $R$ . Through (13) they determine the plane containing the vectors  $e_2$  and  $e_1$ , but one more parameter is needed to determine the orientation of these vectors in that plane. That parameter is the phase of the wave function. Thus, *STA reveals that the quantum mechanical phase has a geometrical interpretation relating it to local kinematics of electron motion*. This striking fact cries out for a physical interpretation! We return to that problem at the end of the paper.

For comparison, the equivalence of STA expressions to standard matrix expressions for bilinear covariants is shown in Table I.

At this point real Dirac theory is sufficiently well developed to assess some of its novel implications. The creation of the Dirac equation was shrouded in mystery that persists to this day. Dirac created a first order relativistic wave equation, and, miraculously, spin appeared with no further assumptions. Where did it come from? The conventional answer is that it came from the Dirac matrices, a relativistic generalization of the Pauli matrices. However, STA tells

us that the  $\gamma_\mu$  have nothing to do with spin; they are merely vectors that provide an algebraic encoding of spacetime properties. Moreover, STA removes the mystery from the so-called “Dirac operator”  $\gamma^\mu\partial_\mu$  by identifying it as the “vector derivative with respect to a spacetime point” [3]; as such, it combines divergence and curl in a single differential operator that simplifies Maxwell’s equations and is recognized as the fundamental tool for a general *Geometric Calculus* [7].

If the  $\gamma_\mu$  have nothing to do with spin, how did spin get into the Dirac equation in the first place? The answer comes from the association of the bivector  $\gamma_2\gamma_1$  with electron spin in (19). Referring to the real Dirac equation (8), we see that spin was inadvertently incorporated into the Dirac equation by assigning the imaginary unit to the derivatives. Here is another striking feature of the Dirac theory that cries out for analysis. More about it below.

Thus we see that STA dispells much of the original mystique of the Dirac theory. But it raises new questions that hopefully can lead us to a more coherent interpretation of quantum mechanics where the full kinematical significance of spin is evident.

This is where real Dirac theory stood after my first published paper on the subject [6], although I was not so proficient at articulating its novel implications at that time. I still had much more to learn about the Dirac theory. Soon afterward I heard from Costa de Beauregard for the first time.

## IV. Observables & Operators in real Dirac theory

In a most diplomatic manner, Costa de Beauregard called attention to a mistake in my first paper [6] by simply asking a question about one of my results. From the Dirac equation one can derive an equation for the divergence of the vector spin density in much the same way that divergence of the probability current is established. The curious result is

$$\partial \cdot (\rho s) = -m\rho \sin \beta. \tag{18}$$

In my initial derivation the right side arises as the sum of two terms, which in my naiveté, I had cancelled to get a new conservation law. Olivier’s searching question directed my attention immediately to my careless treatment of signs. In a subsequent letter he informed me about the original derivation of the equation by Uhlenbeck and Laporte [8].

This was a good lesson for a fledgling physicist. Besides enabling me to correct my mistake in a subsequent paper [9], it raised problems with physical interpretation of the *mysterious parameter*  $\beta$ . In my first paper I noted that  $\beta$  correctly distinguishes between electron and positron plane wave states, and I suggested that it describes an admixture of particle and antiparticle states in general. However, I could not square that general interpretation with equation (18). This stimulated me to study, with my student R. Gurtler, the strange behavior of  $\beta$  in Dirac solutions of the hydrogen atom, which only exacerbated the problem of interpretation. To this day the physical significance of  $\beta$  remains

an abiding mystery of the Dirac theory. Of course, no one has made any sense of equation (18) either.

Why should one care about the physical interpretation of  $\beta$  when the textbooks don't even mention it? The reason is that (9) and (10) show that  $\beta$  is a Lorentz invariant property of the wave function, and all the other parameters of the wave function can be given clear geometrical and physical interpretations.

The new insights brought by the STA formulation and the problems of physical interpretation exacerbated by input from Costa de Beauregard stimulated me to undertake a systematic study of observables, identities and conservation laws in the real Dirac theory. The project turned out to be surprisingly complex. I am still quite proud of the result [9], though it did not produce the complete and coherent physical interpretation that I was looking for. For me, it did clarify the structure of the Dirac theory and the problems of interpretation, though, so far as I know, it has not influenced anyone else. To someone who is not conversant with STA my ten pages of computation and definitions may look unnecessarily complicated, but, in a brilliant *tour de force*, it took Takabayasi [10] more than one hundred pages to do much the same thing with standard matrix and tensor methods. I regard that as an impressive demonstration of the mathematical power of STA.

I had nearly finished this work in 1971 when Costa de Beauregard visited me in Arizona. Accompanied by his brilliant student Christian Imbert, he was on a lecture tour advertising results of their joint theoretical and experimental research on *noncollinearity of velocity and momentum* in electron theory and optics. He was surprised and gratified by my immediate positive response, as they had met mostly skepticism and disbelief on the rest of their trip. I was already steeped in questions about observables in Dirac theory, and here they arrive with fresh insights and experimental tests to boot! [11, 12] I learned from them the importance of asymmetry in the energy-momentum tensor and the feasibility of experiments to test for it. I learned to see this asymmetry as a consequence of intrinsic spin and an expression of noncollinearity of velocity and momentum.

The question of asymmetry in the energy-momentum tensor is a prime example of the critical role for definitions of observables in the interpretation of quantum mechanics. How does one know, for example, whether the Dirac current or the Gordon current (Table II) is correctly identified as a probability current or a charge current, as both are conserved? I got it wrong in [9], but corrected it in [13], where it is pointed out that such mistakes are rife in the literature. Here, taking the formulation and analysis of conservation laws in the Dirac theory for granted, we concentrate on the definitions of observables and their physical interpretation.

Table II lists standard matrix expressions for basic observables equated to their STA counterparts. *Kinetic energy-momentum operators*  $\underline{p}_\mu$  are defined as usual in the matrix theory by

$$\underline{p}_\mu = i'\hbar\partial_\mu - eA_\mu. \quad (19)$$

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**TABLE II: Observables of the energy-momentum operator, relating real and matrix versions.**

Energy-momentum tensor	$T^{\mu\nu} = T^\mu \cdot \gamma^\nu = \langle \gamma_0 \tilde{\psi} \gamma^\mu \underline{p}^\nu \psi \rangle$ $= \tilde{\Psi} \gamma^\mu \underline{p}^\nu \Psi$
Kinetic energy density	$T^{00} = \langle \psi^\dagger \underline{p}^0 \psi \rangle = \Psi^\dagger \underline{p}^0 \Psi, \quad \psi^\dagger = \gamma_0 \tilde{\psi} \gamma_0$
Kinetic momentum density	$T^{0k} = \langle \psi^\dagger \underline{p}^k \psi \rangle = \Psi^\dagger \underline{p}^k \Psi$
Angular Momentum tensor	$J^{\nu\alpha\beta} = [T^\nu \wedge x + i\rho(s \wedge \gamma^\nu)] \cdot (\gamma^\beta \wedge \gamma^\alpha)$ $= T^{\nu\alpha} x^\beta - T^{\nu\beta} x^\alpha - \frac{i\hbar}{2} \tilde{\Psi} \gamma_5 \gamma_\mu \Psi \epsilon^{\mu\nu\alpha\beta}$
Gordon current	$K_\mu = \frac{e}{m} \langle \tilde{\psi} \underline{p}_\mu \psi \rangle = \frac{e}{m} \tilde{\Psi} \underline{p}_\mu \Psi$

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The corresponding STA definition is

$$\underline{p}_\mu = \underline{i} \hbar \partial_\mu - e A_\mu, \quad (20)$$

where the underbar signifies a “linear operator” and the operator  $\underline{i}$  signifies right multiplication by the bivector  $\gamma_2 \gamma_1$ , as defined by

$$\underline{i} \psi = \psi \gamma_2 \gamma_1. \quad (21)$$

The importance of (20) can hardly be overemphasized. Above all, it embodies the fruitful “minimal coupling” rule, a fundamental principle of gauge theory that fixes the form of electromagnetic interactions. In this capacity it played a crucial heuristic role in the original formulation of the Dirac equation, as is clear when the equation is written in the form

$$\gamma^\mu \underline{p}_\mu \psi = \psi \gamma_0 m. \quad (22)$$

In light of our previous explanation for the origin of spin in Dirac theory, we can pinpoint the definition of  $\underline{p}_\mu$  as the crucial assumption that introduced spin. It behooves us to examine what STA can tell us about the geometrical and physical significance of this fundamental operator.

First a word about the relation of operators to observables in quantum mechanics. As an impressionable student, I had the privilege of attending Richard Feynmann’s course on quantum electrodynamics. His impious ridicule of traditional verities helped embolden me to question received knowledge for myself. In particular, his tirade against axiomatic formulations of quantum mechanics emblazoned the following words on my memory:

“If anyone tells me that ‘to every observable there corresponds a hermitian operator for which the eigenvalues correspond to observed values,’ *I will defeat him! I will cut his feet off!*” (emphasis his!).



He supported his position by showing that associating observables with functions of Dirac matrices leads to physical absurdities, particularly in interpreting commutation relations. Later on I realized that STA explained why. For example, for  $\mu = 1, 2, 3$  the matrices  $\gamma_\mu \gamma_0$  are hermitian operators that are sometimes identified as velocity operators [23]. The fact that they anticommute is taken to mean that velocities in orthogonal directions cannot be simultaneously measured, and the fact that each has eigenvalues  $\pm 1$  is taken to mean that the electron's instantaneous velocity is the speed of light (we are using natural units with  $c = 1$ ) so it must be fluctuating rapidly (zitterbewegung) to produce the much lower average velocity that is observed. On the contrary, STA shows that the  $\gamma_\mu$  should be regarded as mere vectors, and a glance at Table I shows them as operators only in the trivial sense that basis vectors can be regarded as operators that pick out components of a given vector.

With that cautionary note, it can be asserted without question that the physical interpretation of standard quantum mechanics is crucially dependent on meaning ascribed to the *kinetic energy-momentum operators*  $\underline{p}_\mu$ , specifically through their role in defining the components  $T^{\mu\nu}$  of the electron *energy-momentum tensor* as given in Table II.

Accordingly, the energy-momentum flux in direction  $\gamma^\mu$  is given by

$$T^\mu = T(\gamma^\mu) = T^{\mu\nu} \gamma_\nu \quad (23)$$

The flux along a Dirac streamline with tangent  $v = e_0$  is

$$T(v) = v_\mu T^\mu = \rho p. \quad (24)$$

This defines a local “*expected*” *momentum vector*  $p = p(x)$ . It can be regarded as a statistical prediction for the momentum of the electron at the spacetime point  $x$ . It can be identified with the Gordon current (Table II) only when  $\sin \beta = 0$  [3], one of the many ways that  $\beta$  complicates physical interpretation. In that case,  $\rho p$  is a conserved current with streamlines of momentum flow, just like the Dirac current  $e\rho v$  is assumed to give streamlines of charge flow. Therefore, noncollinearity of  $p$  and  $v$  means that charge and energy flows are not concurrent.

When the vector field  $p = p(x)$  is uniformly constant, it is the vector eigenvalue of the energy-momentum operator, that is,

$$\underline{p}\psi = p\psi. \quad (25)$$

The eigenfunction, of course, is a plane wave. As a rule, this is the only way that the vector  $p$  appears in conventional quantum mechanics.

In general, the momentum  $p$  is not collinear with the local velocity  $v = v(x)$ , because it includes a contribution from the spin. A measure of this noncollinearity is  $p \wedge v$ . As I discovered in [9] and summarized in [3], analysis and interpretation of local conservation laws is considerably simplified by expressing them in terms of  $v$  and  $p$ . For example, the angular momentum density in Table II assumes the perspicuous form:

$$J(v) = \rho(p \wedge x + S), \quad (26)$$

where, with the spacetime point represented by vector  $x$ ,  $p \wedge x$  is recognized as the expected orbital angular momentum and  $S = isv$  is the spin bivector defined in (19).

We still need to ascertain precisely how the *kinetic momentum*  $p$  is related to the wave function. For that purpose we employ the invariant decomposition of the wave function  $\psi$  in (10). By differentiating  $R\tilde{R} = 1$ , it is easy to prove that derivatives of the rotor  $R$  must have the form

$$\partial_\mu R = \frac{1}{2} \Omega_\mu R, \quad (27)$$

where  $\Omega_\mu = \Omega_\mu(x)$  is a bivector “rotational velocity”. Accordingly, action on  $\psi$  of the energy momentum operator (20) can be put in the form

$$\underline{p}_\nu \psi = [\partial_\nu(\ln \rho + i\beta) + \Omega_\nu] S \psi - e A_\nu \psi. \quad (28)$$

Inserting this in the definition for the energy-momentum tensor in Table II, we obtain the explicit expression

$$T_{\mu\nu} = \rho [v_\mu(\Omega_\nu \cdot S - e A_\nu) - (\gamma_\mu \wedge v) \cdot (\partial_\nu S) - s_\mu \partial_\nu \beta], \quad (29)$$

where the derivative of the spin bivector is given by

$$\partial_\mu S = \frac{1}{2}(\Omega_\mu S - S \Omega_\mu). \quad (30)$$

From this we find, by (24), the momentum components

$$p_\nu = \Omega_\nu \cdot S - e A_\nu. \quad (31)$$

This remarkable equation reveals that (apart from the  $A_\nu$  contribution) the *momentum has a kinematical meaning related to the spin*: It is completely determined by the component of  $\Omega_\nu$  in the spin plane. In other words, it describes the *rotation rate* of the frame  $\{e_\mu\}$  in the spin plane or, if you will “about the spin axis.” But we have identified the angle of rotation in this plane with the phase of the wave function. Thus, the momentum describes the rate of phase change of the wave function in all directions. The component of (31) along the Dirac streamline can be interpreted as an energy in the local electron rest frame, given by

$$p \cdot v = \Omega \cdot S - e A \cdot v. \quad (32)$$

A physical interpretation for this geometrical fact will be offered later.

As a general observation about the structure of observables, we note that STA disabuses us of the conventional belief that representation of observables by bilinear functions of the wave function, as shown in Tables I and II, is unique to quantum mechanics. On the contrary, equation (12) and its subsequent interpretation shows that bilinearity is a consequence of employing the spin representation of a Lorentz transformation. Reference [3] shows that STA makes the same spin representation equally useful and powerful in relativistic classical mechanics, and, not so incidentally, it simplifies and clarifies the classical limit of the Dirac equation. Thus, bilinearity of observables is not an essential difference between classical and quantum mechanics.

## V. Spin and Zitterbewegung

At last we are ready to grapple with the most profound insight and the deepest mystery in the real Dirac theory: *The inseparable connection between quantum mechanical phase and spin!* This flies in the face of conventional wisdom that phase is an essential feature of quantum mechanics, while spin is a mere detail that can often be ignored. We have seen that it is a rigorous feature of real Dirac theory, though it remains hidden in the matrix formulation. To understand the physical significance of this feature, indeed, to provide a physical interpretation for the whole theory, we need to make some ontological commitment about the nature of the electron. To my knowledge, the most promising commitment is to assume that the *electron is a structureless point particle with a continuous history in spacetime.* Though the assumption of a unique continuous history has been rejected by many physicists, it has been vigorously defended by David Bohm [21] and many others. Ultimate justification will depend on its success in interpreting the theory and what the theory can tell us about the histories. Perceptive readers will have noticed tacit assumptions about the electron throughout this paper. Now it is necessary to be more explicit.

The Dirac current  $\rho v$  assigns a unit timelike vector  $v(x)$  to each spacetime point  $x$  where  $\rho \neq 0$ . As already mentioned, we interpret  $v(x)$  as the *expected proper velocity* of the electron at  $x$ , that is, the velocity predicted for the electron if it happens to be at  $x$ . The velocity  $v(x)$  defines a local reference frame at  $x$  called the (local) *electron rest frame*. The *proper probability density*  $\rho = (\rho v) \cdot v$  can be interpreted as the probability density in the rest frame. By a well known theorem, the probability conservation law (15) implies that through each spacetime point there passes a *unique integral curve* (or *streamline*) that is tangent to  $v$  at each of its points. In any spacetime region where  $\rho \neq 0$ , a solution of the Dirac equation determines a *family of streamlines* that fills the region with exactly one streamline through each point. The streamline through a specific point  $x_0$  is the expected history of an electron at  $x_0$ , that is, it is the optimal prediction for the history of an electron that actually is at  $x_0$  (with relative probability  $\rho(x_0)$ , of course!). Parametrized by proper time  $\tau$ , the streamline  $x = x(\tau)$  is determined by the equation

$$\frac{dx}{d\tau} = v(x(\tau)). \quad (33)$$

Motion along a Dirac streamline  $x = x(\tau)$  is determined by the kinematical rotor factor  $R = R(x(\tau))$  in the Dirac wave function (10). The rotor determines the *comoving frame*  $\{e_\mu = R\gamma_\mu\tilde{R}\}$  on the streamline with velocity  $e_0 = v = v(x(\tau))$ , while the spin vector  $s$  and bivector  $S$  are given as before by (18) and (19). In accordance with (27), the directional derivative of  $R$  along the streamline has the form

$$\dot{R} = v \cdot \partial R = \frac{1}{2}\Omega R, \quad (34)$$

where the overdot indicates differentiation with respect to proper time, and

$$\Omega = v^\mu \Omega_\mu = \Omega(x(\tau)) \quad (35)$$

is the rotational velocity of the frame  $\{e_\mu\}$ ; thus,

$$\dot{e}_\mu = v \cdot \partial e_\mu = \Omega \cdot e_\mu. \quad (36)$$

These equations are identical in form to equations for the classical theory of a relativistic rigid body with negligible size given in [3] and shown to be derivable as a classical limit of the Dirac equation. The only difference between classical and quantum theory is in the functional form of  $\Omega$ . Our main task, therefore, is to investigate what Dirac theory tells us about  $\Omega$  and what that has to do with spin.

The origin of spin has been a great mystery since the inception of quantum mechanics. Many students of Dirac theory, including Schroedinger [22, 23] and Bohm [21], have suggested that the spin of a Dirac electron is generated by localized particle circulation that Schroedinger called *zitterbewegung* (= trembling motion). To study that possibility, classical models of the electron as a point particle with spin were first formulated by Frenkel [14] and Thomas [15], improved by Mathisson [16], and given a definitive form by Weysenhoff [17, 18]. They are of interest here for the insight they bring to the interpretation of Dirac theory. They are also of practical interest, for example, in the study of spin precession [24, 25] and tunneling [26, 20].

In Weysenhoff's analysis [18] the classical models fall into two distinct classes, differentiated by the assumption that the electron's spacetime history is timelike in one and lightlike in the other. The timelike case has been studied by many investigators [19]. Ironically, the lightlike case, which Weysenhoff regarded as far more interesting, seems to have been ignored. We shall see that both versions appear naturally in the real Dirac theory, and physical assumptions are needed to discriminate between them.

We begin our analysis by examining what STA can tell us about a free particle; then we extend it to a more comprehensive interpretation of Dirac theory.

We noted in (25) that, for a free particle with given momentum  $p$ , the wave function  $\psi$  is an eigenstate of the "energy-momentum operator." This reduces the Dirac equation to the algebraic equation

$$p\psi = \psi\gamma_0 m. \quad (37)$$

The solution is a *plane wave* of the form

$$\psi = (\rho e^{i\beta})^{\frac{1}{2}} R = \rho^{\frac{1}{2}} e^{i\beta/2} R_0 e^{-\gamma_2 \gamma_1 p \cdot x / \hbar}, \quad (38)$$

where the kinematical factor  $R$  has been decomposed to explicitly exhibit its spacetime dependence on a phase satisfying  $\partial(p \cdot x) = p$ . Inserting this into (37) and solving for  $p$  we get

$$p = m e^{i\beta} R \gamma_0 \tilde{R} = m v e^{-i\beta}. \quad (39)$$

This implies  $e^{i\beta} = \pm 1$ , so

$$e^{i\beta/2} = 1 \text{ or } i, \quad (40)$$

and  $p = \pm mv$  corresponding to two distinct solutions. One solution appears to have negative energy  $E = p \cdot \gamma_0$ , but that can be rectified by changing the sign in the phase of the “trial solution” (38).

Thus we obtain two distinct kinds of plane wave solutions with positive energy  $E = p \cdot \gamma_0$ :

$$\psi_- = \rho^{\frac{1}{2}} R_0 e^{-\gamma_2 \gamma_1 p \cdot x / \hbar}, \quad (41)$$

$$\psi_+ = \rho^{\frac{1}{2}} i R_0 e^{+\gamma_2 \gamma_1 p \cdot x / \hbar}. \quad (42)$$

These can be interpreted as *electron* and *positron* wave functions.

Determining the comoving frame  $\{e_\mu\}$  for the electron solution (41), we find that the velocity  $v = R_0 \gamma_0 \tilde{R}_0$  and the spin  $s = \frac{1}{2} \hbar R_0 \gamma_3 \tilde{R}_0$  are constant, but, for  $k = 1, 2$ ,

$$e_k(\tau) = e_k(0) e^{-p \cdot x / S} = e_k(0) e^{e_2 e_1 \omega \tau}, \quad (43)$$

where  $\tau = v \cdot x$  is the proper time along the streamline and frequency  $\omega$  is given by

$$\omega = \frac{2m}{\hbar} = 1.6 \times 10^{21} \text{ s}^{-1}. \quad (44)$$

Thus, the streamlines are straight lines along which the spin is constant, and  $e_1$  and  $e_2$  rotate about the “spin axis” with the ultrahigh frequency (44) as the electron moves along the streamline generalizes to arbitrary solutions of the Dirac equation, so it should be telling us about some general property of the electron. To get a better idea about what that might be, we examine the general class of *unimodular solutions* to the Dirac equations, so-called because they assume constant  $\rho$  and  $\beta = 0$ .

Obviously, our simple and transparent geometrical picture of comoving vectors  $e_1$  and  $e_2$  rapidly rotating about the spin vector  $e_3$  as the electron moves along a streamline generalizes to arbitrary solutions of the Dirac equation, so it should be telling us about some general property of the electron. To get a better idea about what that might be, we examine the general class of *unimodular solutions* to the Dirac equations, so-called because they assume constant  $\rho$  and  $\beta = 0$ .

Consider a unimodular free particle solution of the form

$$\psi = e^{\frac{1}{2} \Omega \tau} R_0, \quad (45)$$

where

$$\Omega = \omega R_1 \gamma_1 \gamma_2 \tilde{R}_1 \quad (46)$$

is a constant spacelike bivector,  $R_0$  is a constant rotor and  $\tau = \hat{p} \cdot x$ , where  $\hat{p}$  is a unit vector. This reduces to the positive energy plane wave solution (41)

when  $R_1 = R_0$ , but otherwise it gives us something new. Inserting it into the Dirac equation, we get the algebraic relation

$$\frac{\hbar}{2}\hat{p}\Omega\psi\gamma_2\gamma_1 = m\psi\gamma_0. \quad (47)$$

Multiplication on the right by  $\tilde{\psi}$  and on the left by  $\hat{p}$  gives us an elegant equation relating spin, momentum and velocity:

$$\Omega S = pv. \quad (48)$$

Its scalar part

$$p \cdot v = \Omega \cdot S. \quad (49)$$

is a constant of motion that can be interpreted as energy in “the electron’s rest frame.” Since

$$\dot{S} = \frac{1}{2}(\Omega S - S\Omega), \quad (50)$$

the bivector part of (48) gives us

$$\dot{S} = p \wedge v. \quad (51)$$

This is precisely Wessenhoff’s classical equation for angular momentum conservation [17], the noncollinearity of  $v$  and  $p$  compensating for the precessing spin.

The velocity  $v$  precesses with the *z*bw frequency  $\omega$  and fixed angle relative to the constant momentum vector  $p$ . The streamline is readily found to be a timelike helix with fixed pitch and axis aligned with  $p$ . As the velocity oscillates, or better, wobbles with the same frequency as the phase, let us refer to (45) as the *wobble* (plane wave) *solution*.

We are justified in regarding the wobble solution as a plane wave, because it is constant on hypersurfaces with normal  $p$ . Although it is a simple and natural solution in STA, to my knowledge its equivalent matrix form has not appeared in the literature. To see what that form would be, write (45) in the form

$$\psi = R_1 e^{\gamma_1 \gamma_2 p \cdot x / \hbar} \tilde{R}_1 R_0. \quad (52)$$

Then break  $U \equiv \tilde{R}_1 R_0 = U_+ + U_-$  into parts

$$U_{\pm} = \frac{1}{2}(U \mp \gamma_2 \gamma_1 U \gamma_2 \gamma_1) \quad (53)$$

that commute/anticommute with  $\gamma_2 \gamma_1$ , to get a “Fourier analysis” of the wobble solution:

$$\psi = R_1 U_+ e^{-\gamma_2 \gamma_1 p \cdot x / \hbar} + R_1 U_- e^{+\gamma_2 \gamma_1 p \cdot x / \hbar}. \quad (54)$$

Then we see that the wobble solution is a particular superposition  $\psi = \psi_+ + \psi_-$  of positive and negative energy solutions. The velocity is given by

$$v = \psi \gamma_0 \tilde{\psi} = \psi_+ \gamma_0 \tilde{\psi}_+ + \psi_- \gamma_0 \tilde{\psi}_- + \psi_+ \gamma_0 \tilde{\psi}_- + \psi_- \gamma_0 \tilde{\psi}_+, \quad (55)$$

which exhibits the *zbw* oscillations as “interference” between  $\psi_+$  and  $\psi_-$  states in the last two terms. This explains how Schrödinger could obtain a circulating electron state by superposition and attribute it to interference.

It also raises interesting theoretical issues about quantizing the one particle theory to incorporate particle creation and annihilation. The standard approach identifies positive and negative energy states  $\psi_{\pm}$  as particle/antiparticle states respectively, and so quantizes them separately. But suppose that, for a free electron, the wobble state  $\psi$  is a more fundamental representation than  $\psi_+$ . The standard theory must then introduce pair creation and annihilation to represent it. Indeed, physicists often describe *zbw* as a rapid irregular motion of the electron due to pair creation/annihilation [27], though that does not account for the regularity of the spin associated with it.

Alternatively, if the wobble state  $\psi$  replaced  $\psi_+$  as the basic electron state in quantization, then Feynman diagrams associated with wobble would be modified or disappear from QED. The resulting theory should be equivalent to the standard one, because it is based on the same field equations. But the physical interpretation might be quite different.

As described so far, the wobble solution is not a satisfactory model of *zbw*, because its amplitude is variable, and it does not make the connection between spin and phase that we are looking for. To do better, we augment Dirac theory with an ontological assumption that completes the kinematical interpretation of the wave function and observables.

Mindful that the velocity attributed to the electron is an independent assumption imposed on the Dirac theory from physical considerations, we recognize that a kinematical explanation for spin can be achieved by giving the electron a component of velocity in the spin plane. An obvious choice is to identify, or if you will, *define* the vector  $e_2$  as a component of velocity in the spin plane, so the entire *electron velocity* is given by the null vector

$$u = v + e_2 = e_0 + e_2 = R(\gamma_0 + \gamma_2)\tilde{R}. \quad (56)$$

This gives the vector  $e_2$  a physical interpretation that was hitherto missing and gives the electron phase a kinematical meaning as *zbw rotation angle*. It follows that the vector  $e_1$  gives the direction of a *zbw radius vector*.

For the sake of consistency, we need to change our definition of spin angular momentum. To see how, we reconsider the wobble wave function (45) and its reduction of the Dirac equation to the algebraic equation (48). Multiplying the latter by  $(1 + \gamma_0\gamma_2)$  we get

$$\frac{\hbar}{2}\hat{p}\Omega\psi(\gamma_0 + \gamma_2)\gamma_1 = m\psi(\gamma_0 + \gamma_2). \quad (57)$$

As before, this gives us an algebraic relation among the new observables:

$$\Omega\Sigma = pu, \quad (58)$$

where we have introduced a new general definition of spin angular momentum:

$$\Sigma = \frac{1}{2}\hbar R(\gamma_0 + \gamma_2)\gamma_1\tilde{R} = \frac{1}{2}\hbar ue_1 = mu \wedge r, \quad (59)$$

and

$$r = \frac{\hbar}{2m} e_1 \quad (60)$$

is a radius vector for circular *zbw* with diameter equal to a Compton wavelength, at least for a free particle. Note that  $\Sigma$  is a null bivector, and the right side of (59) has the form of an orbital angular momentum.

The solution of (58) proceeds in essentially the same way as before, except that the electron paths turn out to be lightlike helices centered on Dirac streamlines [2]. Therefore, the *zbw* persists even if wobble is eliminated by adjusting  $\Omega$  as explained in connection with (46). Thus, in this model *zbw* and wobble are different things. In fact, the equations for electron paths are the same as in Weyssenhoff's second (and favorite) model for a classical free particle with spin. The big difference is that our equations are tied unequivocally to solutions of the Dirac equation.

Note that the *zbw* interpretation ensures that the lightlike velocity  $u$  is never collinear with the momentum  $p$ , even in the plane wave case. Note also, by (60), that the electron mass is a measure of curvature (or pitch) in the helical world line, which might be attributed to electron self-interaction, but that is beyond the purview of Dirac theory.

Although the frequency and radius ascribed to the *zbw* are the same here as in Schroedinger's work, its role in the theory is quite different. The present approach associates the *zbw* phase and frequency with the phase and frequency of the complex phase factor in the electron wave function. This is the central feature of the *the zitterbewegung interpretation* of quantum mechanics, although in previous accounts [2] the necessity of representing the spin by the null bivector (59) was not recognized.

The strength of the *zbw* interpretation lies first in its coherence and completeness in Dirac theory and second in the intimations it gives of more fundamental physics. It will be noted that the *zbw* interpretation is completely general, because the definitions (56) and (59) of *zbw* velocity and spin are compatible with any solution of the Dirac equation. One need only recognize that the Dirac velocity and spin can be interpreted as averages over a *zbw* period, as expressed by

$$v = \bar{u} \quad \text{and} \quad S = \bar{\Sigma}. \quad (61)$$

Since the period is on the order of  $10^{-21}$ s, it is  $v$  and  $S$  rather than  $u$  and  $\Sigma$  that best describe the electron in most experiments. However, (61) suggests that the Dirac current describes only the average charge flow, while *zbw* oscillations are associated with the magnetic moment.

Perhaps the strongest theoretical support for the *zbw* interpretation is the fact that it is fundamentally geometrical; it completes the kinematical interpretation of the rotor  $R$  in the canonical form (9) for the wave function, so all components of  $R$ , even the complex phase factor, characterize features of the electron history. The *zbw* interpretation also brings to light geometric meaning for the mysterious "quantum energy-momentum operators"  $p_\mu$  by relating them



to the computation of phase rotation rate with the equation  $p_\nu = \Omega_\nu \cdot S - eA_\nu$ . This suggests that energy is stored in the *zbw*.

I believe it fair to say that the *zbw* concept provides the most complete physical interpretation of Dirac theory that is available. It does not resolve all the mysteries of quantum mechanics, but it suggests new directions for research. In particular, it suggests the existence of a substructure in quantum mechanics that should be amenable to some kind of experimental test if it is ontologically real. Join me in the search!

**Note.** The papers listed in the references that deal with spacetime algebra and real Dirac theory are available on line at <http://modelingnts.la.asu.edu> or <http://www.mrao.cam.ac.uk/~clifford/>.

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