In: *J. Math. Phys.* Vol. 14, No. 7, July 1973

Local observables in the Dirac theory

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Abstract

By a new method, the Dirac electron theory is completely reexpressed as a set of conservation laws and constitutive relations for local observables, describing the local distribution and flow of mechanical quantities. The coupling of the electromagnetic field to the electron is shown to be determined by the definitions of the observables rather than by the Dirac equation. Planck's constant appears in the equations only in connection with the electron spin. The equations are most readily interpreted by assuming that the electron is a structureless point charge, the spin and magnetic moment arising from the dynamics of electron motion.

INTRODUCTION

Most experimental tests of the Dirac electron theory pertain only to "global observables" such as average energy, average momentum, and average angular momentum. The theory actually contains much more detailed implications about "local observables" which describe a spacial distribution and flow of charge, mass, energy-momentum, and angular momentum. The local observables of the Dirac theory have some peculiar properties of which Dirac was undoubtedly unaware when the theory was first formulated. These peculiarities deserve careful study; first, because they can be expected to lead to particularly sensitive tests of the theory if they can be subjected to experimental scrutiny; and second, because they furnish valuable theoretical clues about the interpretation and significance of the theory.

A systematic analysis of the properties of local observables happens to be surprisingly difficult by conventional methods. This paper uses an unconventional mathematical formalism developed in Refs. [1] and [2] to achieve a compact reformulation of the complete Dirac theory in terms of local observables. The new method greatly simplifies the derivation of conservation laws and "relations" among local observables. The results obtained are complete in the sense that the relations found among the observables determine the theory uniquely. It is believed that all significant relations among local observables found by previous authors are derived here, though they usually appear in quite different form and often as only part of a more general relation. Because of the considerable difference in method and viewpoint, comparison of results is sometimes

quite tedious, though it can always be carried out by the method explained in Appendix A. Since the work of Takabaysi [3] is much more extensive than anything else in the literature on local observables, special effort was made to compare it with the present work; exact agreement has been found even on comparing some of the more complicated and esoteric formulas, though the two approaches have not been compared in every detail.

The difficult problem of subjecting the theory of local observables to experimental test will not be broached in this paper, though it is hoped that the present formulation of the theory will facilitate the task. What the theory needs most is an experimental test of the peculiar noncollinearity of velocity and momentum predicted by eq. (5.8) below. The problem of devising such a test has engaged the attention of Costa de Beauregard for many years; he has reviewed the problem in Ref. [4], including some significant progress that has recently been made. Reasoning by analogy with the Dirac theory, Costa de Beauregard was lead to suggest tests for asymmetry in the free space electromagnetic energy-momentum tensor for circularly polarized light. This asymmetry implies that the energy flux in a polarized light beam is not collinear with the momentum density, the difference being due to the "photon spin." In a clever experiment, Imbert [5] showed that this difference manifests itself as a lateral displacement of a reflected light beam with a magnitude and direction in complete agreement with theoretical predictions. This success makes it all the more likely that the analogous asymmetry in the Dirac theory can somehow be tested experimentally.

It should be emphasized that the asymmetry just alluded to does not reveal itself in the Dirac theory when observables are defined in terms of operators in the usual manner. The operator definitions refer only to global observables, the local features being suppressed by averaging (i.e., integrating over all space). The local theory is more detailed, defining mechanical quantities such as momentum and angular momentum densities as definite functions of position and time. One local observable, the probability density determined by the Dirac wave function, is already widely used to predict the electron charge distribution in an atom. The local distributions of other mechanical quantities have not as yet been associated with any experimentally accessible effects.

To achieve a complete theory of local observables it is necessary to go beyond the original assumptions of Dirac. It is important to realize that the definitions of observables in Dirac theory have far-reaching consequences quite independent of the exact form of the Dirac equation. The specification of observables in Dirac's original paper [6] was incomplete in several respects. His crucial assumption about observables was made by adopting, without comment, the operator definition of energy already used in Schrodinger's theory. This was all he needed to predict the energy levels of stationary atomic states. Dirac's initial assumptions were not sufficient to prove local conservation of probability. But this deficiency was soon rectified [7] by defining a 'probability current,' which, in accordance with the Dirac equation, has vanishing divergence and so describes

a locally conserved quantity. This definition is actually a new physical assumption, as the current specifies a local flow (or at least, a "probable" or "local average" flow) of electron charge and mass, and tacitly attributes a constant ratio of charge to mass density for the "smeared out" electron.

After the local distribution and flow of electron charge was specified by defining a conserved current, the local distribution and flow of energy-momentum was specified by Tetrode's definition [8] of an energy-momentum tensor. It is remarkable that these definitions above completely determine a local distribution and flow of angular momentum, since in general continuum theory

Once a complete set of local observables has been defined, the wave function can be eliminated and the Dirac theory completely reformulated as a set of equations for local observables. Reformulation proved to be a surprisingly difficult task. Many authors achieved partial results. But it was not until 1957 that Takabayasi [3] developed a systematic approach and carried the reformulation to completion. Unfortunately, physicists have derived little benefit from his work, most probably because of its great complexity. However, much of this complexity is unnecessary, because it arises from redundancy in the admixture of matrix and tensor algebras in the mathematical formalism usually used to express the Dirac theory. Such redundancy is eliminated in this paper by employing the "space-time algebra" developed in Refs. [1] and [2].

Because of the much simpler method employed here, it has proved possible to work out the relations among local observables in considerably more detail than was done previously, especially in Sec. 6 below. It should be mentioned that the many relations among local observables derived below are not mutually independent. No attempt has been made to select one particular complete set of relations as more fundamental than another, because the desirability of any particular selection is determined by the use to which it will be put. However, a careful distinction has been made between those relations which are determined by the Dirac equation and those which are not.

The particular properties of the energy-momentum tensor which are determined by the Dirac equation are ascertained in Sec. 2. However, the main aim of Secs. 1 through 4 is to determine those properties of local observables which follow from the definitions of the probability current by (1.3) and (1.4) and the energy-momentum tensor (2.3) as specific functions of the wave function (1.1), without appeal to the Dirac equation. Since local conservation is essential to the notion of probability density, the eq. (1.4) which expresses local conservation of probability is taken as part of the definition of the probability current even though it can be derived from the Dirac equation (e.g., see Appendix B). Of course, the mere fact that the Dirac equation implies that the "current" (1.3) has vanishing divergence does not entail that that particular quantity describes the local flow of probability. It seems best, then, to say merely that the Dirac equation is consistent with the identification of (2.26) as probability current.

In Sec. 5 the Dirac equation is completely reformulated in terms of local observables. It will be noted that the electromagnetic potential does not ap-

pear in the resulting eq. (5.7). Therefore, the Dirac equation by itself implies nothing about electromagnetic interactions of the electron. Moreover, the reformulated Dirac equation does not yield equations of motion for local observables. Rather, it functions as a "constitutive equation," expressing, as shown by (5.8), the local momentum as a definite function of the local velocity and spin. The is equations of motion for the local observables are just the general conservation laws ascertained in Secs. 1 and 2. But these equations are underdetermined and cannot be solved without the additional "constitutive relations" provided by Dirac equation. Since these "constitutive relations" do not involve the electromagnetic field, it can be concluded that the coupling of the electron to the electromagnetic field is already completely determined by the identification of (2.26) as the equation for energy-momentum conservation.

The relation of the spin density to the local circulation of charge in the Dirac theory is studied in Secs. 6 and 7. Section 8 discusses the bearing of local observables on the interpretation of quantum mechanics.

1 WAVE FUNCTION

This paper continues the reformulation and analysis of Dirac's theory begun in Ref. [1]. For the most part definitions, conventions, and results appearing in Ref. [1] are adopted here with little or no comment. The relation to the usual formulation of Dirac theory is discussed in Appendix A.

In Ref. [1] it was established that the Dirac "wave function" $\psi = \psi(x)$ can be written in the canonical form

$$\psi = \rho^{1/2} e^{i(1/2)\beta} R \,. \tag{1.1}$$

The wave function does not have a direct physical significance, and a crucial part of the Dirac theory is to relate ψ to observable quantities. The canonical decomposition (1.1) greatly facilitates this task and, in addition, makes the geometric content of the theory explicit.

The spinor $R = R(x)$ describes a Lorentz rotation at every point x of spacetime which takes an orthonormal frame of constant vectors γ_α into the frame $e_\alpha = e_\alpha(x)$ according to the equation

$$e_\alpha = R\gamma_\alpha \widetilde{R} \,, \qquad (\alpha = 0,\, 1,\, 2,\, 3) \,. \tag{1.2}$$

The vectors e_0 and e_3 have direct physical interpretations in the Dirac theory. To emphasize its interpretation the vector $v \equiv e_0 = v(x)$ is called the *(local) particle velocity* (at x). In agreement with established parlance, the quantity

$$\psi\gamma_0\widetilde{\psi} = \rho R\gamma_0\widetilde{R} = \rho e = pv \tag{1.3}$$

is called the *probability current* and ρ is called the *proper probability density*. Local conservation of probability is expressed by the equation

$$\Box \cdot (\rho v) = 0 \,. \tag{1.4}$$

With the identification of $m\rho$ as *proper mass density*, (1.4) expresses local conservation of mass. Equation (1.4) is a consequence of the Dirac equation, but it must be emphasized that the physical interpretation given to ρv is an independent postulate of the Dirac theory.

The vector

$$s \equiv \frac{\hbar}{2}\, e_3 = s(x) \tag{1.5}$$

is called the (local) spin (vector), and the bivector

$$S \equiv isv = \frac{\hbar}{2}\, ie_3e_0 = \frac{\hbar}{2}\, e_2e_1 = \frac{\hbar}{2}\, R\gamma_2\gamma_1\widetilde{R} \tag{1.6}$$

called the (local) spin (bivector). Either s or S can be used to describe the spin of the electron; S is preferable because angular momentum is fundamentally a bivector quantity, but s has the advantage that vectors are sometimes easier to manipulate than bivectors. At any rate, (1.6) makes it easy to switch from one to the other. The physical interpretation of S as angular momentum need not be introduced into the Dirac theory as an independent postulate; it follows from the definition of the energy-momentum tensor given in the next section.

The e_α describe intrinsic properties of the electron and so are independent of any coordinate system associated with an observer. In contrast, the choice of the γ_α in (1.2) is a mere convention, though once that choice is made, the R that gives the e_α is uniquely determined. The choice of the γ_α is disguised in the usual matrix formulation of the Dirac theory as a choice of matrix representation. A change in the matrix representation is equivalent to a change in the choice of the γ_α. The γ_α can always be related to a set of inertial (Cartesian) coordinates $\{x_\alpha\}$ by the equations $\gamma_\alpha = \partial_\alpha x$ or $\gamma^\alpha = \Box x^\alpha$.

Equations (1.2) can be solved for R (see Sec. 17 of Ref. [2]). One obtains

$$R = (\widetilde{A}A)^{-1/2}A\,, \tag{1.7}$$

where

$$A = e_\alpha\gamma^\alpha\,. \tag{1.8}$$

This is one way of exhibiting the dependence of R on the e_α, but no further use of it will be made in this paper. The important point is that once the convenient arbitrary choice of the γ_μ has been made, then the Lorentz rotation R is determined by eight (scalar) parameters. Five of these are determined by specifying the velocity and the spin directions. This much determines the plane of e_1 and e_2; the additional parameter χ, which is needed to fix the directions of e_1 and e_2 in the plane, is the phase of the Dirac wave function. the angular momentum tensor is not determined by the energy-momentum tensor. Thus no further assumptions are needed to determine a complete set of local observables for the Dirac theory.

The Dirac wave function ψ is completely determined by specifying the set of eight independent quantities:

$$\{\rho,\, \beta,\, v,\, S \text{ or } s,\, \chi\} \tag{1.9}$$

Of these, ρ, v, S have straightforward physical interpretations. Comments on the interpretation of β will be deferred until the role of β in the Dirac theory has been studied in some detail. The observable significance of χ is indirect, and it will not be convenient to make explicit use of χ in equations for local observables. Rather, the energy-momentum vector, to be introduced later, will contain an implicit dependence on the gradient of χ.

2 ENERGY-MOMENTUM TENSOR

One of the key assumptions in Dirac's initial paper [6] is that the total energy E of the electron in a stationary state is to be obtained from the equation

$$(-1)^{1/2}\hbar\partial_t\Psi = E\,\Psi\,, \tag{2.1}$$

where Ψ is a column spinor. Superficially, (2.1) appears to be identical to an assumption made by Schrodinger in his first papers on quantum theory. But something new is present because of the spinor character of Ψ. This becomes apparent when, by using the spinor Ψ, (2.1) is reexpressed in a form which is independent of any matrix representation:

$$\partial_t\psi\gamma_2\gamma_1\hbar = E\psi\,. \tag{2.2}$$

The equivalence of (2.2) and (2.1) is easily established by the method of Appendix A. Comparison of (2.1) and (2.2) then reveals that the root of minus one on the left of (2.1) has a geometrical significance; it may be regarded as a representation of the spacelike bivector $\gamma_1\gamma_2 = i\gamma_3\gamma_0 = i\sigma_3$, which itself is a particular root of minus one since $(\gamma_2\gamma_1)^2 = -1$. Moreover, this root of minus one has a physical significance, for according to (1.6) it determines the direction of the electron spin. In fact, it is only through (2.1) or (2.2) and its generalizations that electron spin finds its way into the Dirac theory.

To complete the formulation of the Dirac theory, (2.1) must be generalized to give an expression for the energy even when the electron is not in a stationary state, if only because the wavefunction cannot be an eigenfunction of the energy in all inertial systems unless it is a plane wave. Besides, (2.1) determines only the energy density, which is but one component of a complete energy-momentum tensor.

The most straightforward guess at the required generalization of (2.1) is given by

$$\begin{aligned} T_{\mu\nu} &\equiv \left\{\gamma_0\widetilde{\psi}\gamma_\mu(\partial_\nu\psi\gamma_2\gamma_2\hbar - eA_\nu\psi)\right\}_S \\ &= \hbar(\gamma_\mu\partial_\nu\psi i\gamma_3\widetilde{\psi})_S - e\rho v_\mu A_\nu\,. \end{aligned} \tag{2.3}$$

The second term on the right has been added to remove the contribution of the electromagnetic field to the electron's energy-momentum and so (hopefully) to produce a tensor that describes only intrinsic properties of the electron. To get the last line of (2.3), the definition of velocity (1.3) has been used.

That (2.3) is a reasonable generalization of (2.1) can be shown by computing the average energy in "the inertial system of γ_0," namely

$$\langle\, E \,\rangle = \langle\, E(t) \,\rangle = \int d^3x\, (T_{00} + e\rho v_0 A_0) = \int d^3x\, (\gamma_0 \widetilde{\psi} \gamma_0 \partial_0 \psi \gamma_2 \gamma_1 \hbar)_S \,, \tag{2.4}$$

where the integral is taken over the spacelike hypersurface of points x satisfying the equation $a \cdot \gamma_0 = x_0 = t$. If (2.1) is satisfied, then

$$\langle\, E \,\rangle = \int d^3E\, (\gamma_0 \widetilde{\psi} \gamma_0 \psi)_S = E \int d^3x\, \rho_0 = E \,, \tag{2.5}$$

where

$$\rho_0 \equiv \rho\, v_0 = \rho\, v \cdot \gamma = (\psi \gamma_0 \psi \gamma_0)_S \tag{2.6}$$

is the particle probability density in the inertial system of γ_0. It is very important to note the appearance of v in (2.5) and (2.3), for it shows that consistency of (2.3) with (2.1) requires the conservation law (1.4) and the interpretation previously given to v.

Appendix A shows that (2.3) is equivalent to Tetrode's definition of the electron's energy-momentum tensor. Though this tensor has been much studied with matrix methods, something may yet be learned by analyzing its properties with the methods of space-time algebra.

An energy-momentum tensor is a linear vector function of a vector variable. To be more specific, let $T(n, x)$ denote a flux of energy-momentum through a hypersurface with normal n at the space-time point x. Suppressing the x dependence, one says that $T(n)$ is an energy-momentum tensor. Since T is a linear function of $n = n^\mu \gamma_\mu$,

$$T(n) = n^\mu T(\gamma_\mu) = n^\mu T_\mu \,. \tag{2.7}$$

So an energy-momentum tensor is completely characterized by the vectors $T_\mu = T(\gamma_\mu)$, which specify the flux in four independent directions. The components of the T_μ are Inversely,

$$T_{\mu\nu} \equiv T_\mu \cdot \gamma_\nu \,. \tag{2.8a}$$

Inversely,

$$T_\mu = T_{\mu\nu} \gamma^\nu \,. \tag{2.8b}$$

With this, the $T_{\mu\nu}$ for the Tetrode tensor can be determined from (2.3).

The conservation law (1.4) implies the existence of *velocity streamlines*, timelike trajectories with tangents $v = v(x)$ describing the local flow of the "probability fluid." The *proper energy-momentum density* given by

$$\rho p = T(v) = v^\mu T_\mu \tag{2.9}$$

describes the flow of energy-momentum along the velocity streamline. The vector p deserves a special name, because it is one of the most fundamental quantities of the Dirac theory. The name *local energy-momentum* or simply name *momentum* is appropriate. Now T_μ can be decomposed in the form

$$T_\mu = \rho v_\mu p + N_\mu \,. \tag{2.10}$$

Because of (2.9), $N(v) = v^\mu N_\mu = 0$, so the N_μ describe the flow of energy momentum normal to the velocity streamline.

Certain special properties of the Tetrode tensor are determined by the Dirac equation. These properties are most easily ascertained by studying the "transposed" tensor defined from (2.3) by

$$\begin{aligned} \overline{T}_\mu &= \gamma^\nu T_{\mu\nu} = \gamma^\nu \{\hbar\gamma_\nu \cdot (\partial_\mu \psi i\gamma_3 \widetilde{\psi})_V - e\rho v_\nu A_\mu\} \\ &= \hbar(\partial_\mu \psi i\gamma_3 \widetilde{\psi})_V - e\rho\, v_\nu A_\mu \,. \end{aligned} \tag{2.11}$$

First observe that

$$\hbar(\partial_\mu \psi i\gamma_3 \widetilde{\psi})_V = \frac{\hbar}{2}\left(\partial_\mu \psi i\gamma_3 \widetilde{\psi} - \psi i\gamma_3 \partial_\mu \widetilde{\psi}\right), \tag{2.12}$$

Also, note that the definition of the spin vector (1.5) implies $\frac{1}{2}\hbar\psi\gamma_3\widetilde{\psi} = \rho s$, so

$$\partial_\mu(i\rho s) = \frac{\hbar}{2}(\partial_\mu \psi i\gamma_3 \widetilde{\psi} + \psi i\gamma_3 \partial_\mu \widetilde{\psi}) \,. \tag{2.13}$$

Hence

$$\hbar(\Box\psi)i\gamma_3\widetilde{\psi} = \hbar\gamma^\mu(\partial_\mu \psi i\gamma_3 \widetilde{\psi})_V + \Box(i\rho s) \,. \tag{2.14}$$

Next, recall the form of Dirac equation given in Ref. [5]. With a convenient change in sign convention and explicit introduction of Planck's constant, it can be written

$$\hbar\Box\psi i\gamma_3\gamma_0 = m\psi\gamma_0 + eA\psi \,. \tag{2.15}$$

Multiply this on the right by $\gamma_0\widetilde{\psi}$ and use $\psi\widetilde{\psi} = \rho e^{i\beta}$ as well as (1.3) to get

$$\hbar\Box\psi i\gamma_3\widetilde{\psi} = m\rho e^{i\beta} + eA\rho v \,. \tag{2.16}$$

Finally, combine (2.9), (2.11), (2.14), and (2.16) to obtain

$$\gamma^\mu \overline{T}_\mu = T_\mu \gamma^\nu m\rho e^{i\beta} + i\Box\rho s \,. \tag{2.17}$$

The pseudoscalar part of (2.17) yields

$$\square \cdot (\rho s) = -m\rho \sin\beta \,. \tag{2.18}$$

This says nothing, at least directly, about the Tetrode tensor. But the scalar part of (2.17) gives the trace of the Tetrode tensor:

$$T^\mu_\mu = T_\mu \cdot \gamma^\mu = m\rho \cos\beta \,. \tag{2.19}$$

and the bivector part of (2.17) is

$$\gamma^\mu \wedge \overline{T}_\mu = T_\mu \wedge \gamma^\nu = i(\square \wedge \rho s) = -\square \cdot (i\rho s) \,. \tag{2.20}$$

tensor, as can be seen by expressing it in component form:

This specifies the antisymmetric part of the Tetrode

$$\begin{aligned} (\gamma_\mu \wedge \gamma_\nu) \cdot (T_\beta \wedge \gamma^\beta) &= (\gamma_\mu \wedge \gamma_\nu) \cdot (\gamma^\alpha \wedge \gamma^\beta) T_{\beta\alpha} = T_{\mu\nu} - T_{\nu\mu} \\ &= i\gamma_\mu \wedge \gamma_\nu \wedge \square \wedge (\rho s) = -\epsilon_{\mu\nu\alpha\beta} \partial^\alpha (\rho s^\beta) \,. \end{aligned} \tag{2.21}$$

where $\epsilon_{\mu\nu\alpha\beta} = -i\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\alpha \wedge \gamma_\beta$.

By definition, the divergence of the correct energy-momentum tensor must equal the density of force acting on the electron. The divergence of the Tetrode tensor is determined by the Dirac equation. To evaluate it, first note that

$$\partial_\mu T^\mu = \partial_\mu \overline{T}^\mu \,, \tag{2.22}$$

because the divergence of the antisymmetric part of the Tetrode tensor vanishes, as follows immediately from (2.16) or (2.18). With the help of (2.12), the divergence of (2.11) can be written

$$\partial_\mu \overline{T}^\mu = \hbar(\square^2 \psi i\gamma_3 \widetilde{\psi})_V - e\partial(\rho v A^\mu) \,. \tag{2.23}$$

To express the first term on the right on (2.23) in terms of observables, take the gradient of the Dirac equation (2.15) and multiply on the right by $\gamma_0\widetilde{\psi}$ to get

$$\hbar(\square^2\psi) i\gamma_3\widetilde{\psi} = m(\square\psi)\widetilde{\psi} + e(\square A\psi)\gamma_0\widetilde{\psi} \,.$$

Again using the Dirac equation, rewrite the right side of this expression to get

$$\hbar\square^2\psi i\gamma_3\widetilde{\psi} = \hbar^{-1}(e^2A^2 - m^2) i\rho s + e(\square A)\rho v + 2e(A\cdot\square\psi)\gamma_0\widetilde{\psi} \,. \tag{2.24}$$

The vector part of this equation is

$$\begin{aligned} (\hbar\square^2\psi i\gamma_3\widetilde{\psi})_V &= \frac{\hbar}{2}(\square^2\psi i\gamma_3\widetilde{\psi} - \psi i\gamma_3 \square^2\widetilde{\psi}) \\ &= \rho e(\square \wedge A)\cdot v + e\rho v \square\cdot A + eA\cdot\square(\rho v) \\ &= \rho e F\cdot v + e\partial_\mu(\rho v A^\mu) \,, \end{aligned} \tag{2.25}$$

where $F = \Box \wedge A$. Substitution into (2.23) and use of (2.10) yields

$$\partial_\mu T^\mu = \rho \dot{p} + \partial_\mu N^\mu = \rho e F \cdot v = \rho f \,, \tag{2.26}$$

where a dot has been used to represent the "proper time" derivative along the streamline by writing $v \cdot \Box p = \dot{p}$. The local force $f = eF \cdot v$ is just the familiar Lorentz force. This further confirms the compatibility of interpretations given to T^μ and v. The striking fact is that (2.26) has exactly the form that classical electrodynamics gives for the effect of an electromagnetic field F acting on a charged current. Note that there are no multipole force terms, such as would arise if the electron had some complicated local structure. Thus, according to (2.26) the effect of external electromagnetic forces acting on the electron is exactly what one would expect from classical electrodynamics. The peculiar features of the Dirac theory reside in the specific nature of the T^μ.

The results above already suffice to show that the Dirac electron possesses an angular momentum which does not appear to be induced by external forces. Observe that, since $\partial_\mu x = \gamma_\mu$, (2.26) implies

$$\partial_\mu (T^\mu \wedge x) = T^\mu \wedge \gamma_\mu + \rho f \wedge x \,. \tag{2.27}$$

And note that, by virtue of (2.20),

$$T^\mu \wedge \gamma_\mu = -\partial_\mu S^\mu \,, \tag{2.28}$$

where

$$\begin{aligned} S^\mu &= \rho i s \wedge \gamma^\mu = \rho (is) \cdot \gamma^\mu = \rho (s \wedge v) \cdot \gamma^\mu \\ &= \rho v^\mu S + \rho S \cdot \gamma^\mu v \,. \end{aligned} \tag{2.29}$$

So, with the definition

$$J^\mu \equiv T^\mu \wedge x + S^\mu \,, \tag{2.30}$$

(2.27) can be written

$$\partial_\mu J^\mu = \rho f \wedge x \,. \tag{2.31}$$

The right side of this equation can be identified as the local torque, so J^μ can be interpreted as the flux of angular momentum through a hypersurface with normal γ^μ. Moreover, J^μ consists of an orbital part $J^\mu \wedge x$ and an "intrinsic part" S^μ. The angular momentum flow along the velocity streamline is described by the *proper angular momentum density*,

$$J(v) = v_\mu J^\mu = \rho (p \wedge x + S) \,, \tag{2.32}$$

where $S = \rho^{-1} v_\mu S^\mu = is \wedge v$ is seen to be precisely the local spin, as advertised in Sec. 1.

The right side of (2.29) is a decomposition of the spin angular momentum density S^μ into a part $\rho v^\mu S$ which describes angular momentum flow along the streamline and a part

$$M^\mu \equiv \rho S \cdot \gamma^\mu v = \rho \tfrac{1}{2} [\, S, \gamma^\mu v \,] , \tag{2.33}$$

which describes angular momentum flow normal to the streamline. Using this, and the corresponding decomposition (2.10) for T_μ, (2.28) can be written in the form

$$\rho S + \rho p \wedge v = \gamma_\mu \wedge N^\mu - \partial_\mu M^\mu , \tag{2.34}$$

where, of course, $\dot{S} = v \cdot \Box S$.

Equations (1.4), (2.26), and (2.34) are local conservation laws for mass, energy-momentum and "intrinsic angular momentum," respectively. But they constitute a determinate set of equations only when "constitutive equations" have been specified which relate the basic local observables ρ, v, S and p and expresses the fluxes N_μ, and M_μ in terms of them. In the general form given above with a more general form for the local force than is given in (2.26), the local conservation laws hold for any classical relativistic theory of continuous media as well as for the Dirac theory. The peculiar features of the Dirac theory are found not in the conservation laws, but in the form it gives to the constitutive relations and boundary conditions.

3 LOCAL MOMENTUM AND ANGULAR VELOCITY

From (1.2) it follows that

$$\gamma_\mu \cdot \Box e_\alpha = \partial_\mu e_\alpha = \tfrac{1}{2} [\, \Omega_\mu, e_\alpha \,] = \Omega_\mu \cdot e_\alpha \tag{3.1}$$

where

$$\Omega_\mu \equiv 2(\partial_\mu R)\widetilde{R} \, . \tag{3.2}$$

Equation (3.1) says that on displacement in the γ_μ direction, the frame $\{e_\mu\}$ rotates with "angular velocity" Ω_μ. To see what physical significance such a rotation might have, the angular velocity must be expressed as a function of local observables. To this end, introduce quantities P_μ, and q_μ by the equation

$$P_\mu + i q_\mu = \frac{\hbar}{2} \left(\partial_\mu R \gamma_2 \gamma_1 \widetilde{R} - R \gamma_2 \gamma_1 \partial_\mu \widetilde{R} \right) . \tag{3.3}$$

Also, use (1.6) to obtain

$$\partial_\mu S = \frac{\hbar}{2} \left(\partial_\mu R \gamma_2 \gamma_1 \widetilde{R} + R \gamma_2 \gamma_1 \partial_\mu \widetilde{R} \right) \tag{3.4}$$

and

$$\hbar\partial_\mu R\gamma_2\gamma_1\partial_\mu\widetilde{R} = (\partial_\mu R\widetilde{R})(\hbar R\gamma_2\gamma_1\widetilde{R}) = \Omega_\mu S\,. \tag{3.5}$$

The sum of (3.3) and (3.4) yields

$$P_\mu + iq_\mu + \partial_\mu S = \Omega_\mu S = \hbar\partial_\mu R\gamma_2\gamma_1\widetilde{R}\,. \tag{3.6}$$

The scalar part of (3.6) is

$$P_\mu = \Omega_\mu \cdot S = \hbar(\partial_\mu R\gamma_2\gamma_1\widetilde{R})_S\,, \tag{3.7}$$

which shows that P_μ, measures the component of the angular velocity in the local spin plane.

The pseudoscalar part of (3.6) yields

$$q_\mu = -i(\Omega_\mu \wedge S) = -\Omega_\mu \cdot (iS) = \hbar(\partial_\mu R\gamma_0\gamma_3\widetilde{R})_S\,, \tag{3.8}$$

which shows that q_μ measures the component of the angular velocity in the plane orthogonal to the spin plane. Finally, the bivector part of (3.6) is

$$\partial_\mu S = \tfrac{1}{2}(\Omega_\mu S - S\Omega_\mu) \equiv \tfrac{1}{2}[\,\Omega_\mu, S\,]\,, \tag{3.9}$$

which measures the rate at which the spin plane changes direction on displacement along γ_μ.

To find an expression for Ω_μ in terms of observables solve (3.6).

$$\begin{aligned}\Omega_\mu &= (\partial_\mu S + P_\mu + iq_\mu)S^{-1} \\ &= s(\partial_\mu v)vs^{-1} + (\partial_\mu s)s^{-1} + q_\mu vs^{-1} + P_\mu S^{-1}\,,\end{aligned} \tag{3.10}$$

where $S^{-1} = |\,S\,|^{-2}\widetilde{S} = is^{-1}v$ and $s^{-1} = -|\,s\,|^{-2}s$. The quantity q_μ can be expressed in terms of the spin and velocity by relating (3.10) to the derivative of the velocity.

$$\partial_\mu v = \Omega_\mu \cdot v = s^{-1}(v \cdot \partial_\mu s) + \partial_\mu v - q_\mu s^{-1}\,.$$

Hence

$$q_\mu = v \cdot \partial_\mu s = -s \cdot \partial_\mu v \tag{3.11}$$

or

$$q = \gamma^\mu q_\mu = -v \cdot (\Box \wedge s) + v \cdot \Box s = -s \cdot \Box v + s \cdot (\Box \wedge v)\,. \tag{3.12}$$

An expression similar to (3.11) can be found for P_μ from

$$\partial_\mu e_1 = \Omega_\mu e_1 = (e_1 \cdot \partial_\mu v)v + (e_1 \cdot \partial_\mu s)s^{-1} + P_\mu \frac{2}{\hbar}e_2\,.$$

Thus,

$$P_\mu = -\frac{\hbar}{2} e_2 \cdot \partial_\mu e_1 = \frac{\hbar}{2} e_1 \cdot \partial_\mu e_2 \,. \tag{3.13}$$

Equation (3.11) says that q_μ measures the rate of rotation in the (vs) plane, while (3.13) says that P_μ measures the rate of rotation in the spin plane though no physical significance has been attributed to e_1 and e_2 separately. Since P_μ itself is an observable, (3.10) along with (3.11) gives the complete relation of Ω_μ to observables.

A change of "phase" of the wave function by an amount $\hbar^{-1}\Lambda$, i.e., the spinor transformation

$$R \to R e^{-\gamma_2 \gamma_1 \Lambda/\hbar}$$

induces, by (3.2), the change of angular velocity

$$\Omega_\mu \to \Omega_\mu + \partial_\mu \Lambda S^{-1} \,,$$

and by (3.2), the transformation

$$P_\mu \to P_\mu + \partial_\mu \Lambda \,. \tag{3.14}$$

This shows that P_μ depends on the phase only through its gradient.

The physical significance of P_μ can be ascertained by relating it to the energy-momentum tensor. Use (1.1) and (3.2) to write

$$\partial_\mu \psi = \tfrac{1}{2}\{\partial_\mu \ln(\rho e^{i\beta}) + \Omega_\mu\}\psi \,. \tag{3.15}$$

Multiply this on the left by $\hbar i\gamma_3 \widetilde{\psi}$ and use $\frac{1}{2}\hbar\psi i\gamma_3\widetilde{\psi} = \rho S v$ together with (3.6) to obtain

$$\begin{aligned} \hbar\partial_\mu \psi i\gamma_3 \widetilde{\psi} &= \{\partial_\mu \ln(\rho e^{i\beta}) + \Omega_\mu\}(\rho S v) \\ &= \{P_\mu + i q_\mu + \partial_\mu S + S\partial_\mu \ln(\rho e^{i\beta}) + \Omega_\mu\}\rho v \\ &= \{P_\mu + i q_\mu + \partial_\mu S + W_\mu\}\rho v \,, \end{aligned} \tag{3.16}$$

where the bivector W_μ is defined by

$$W_\mu \equiv (\rho e^{i\beta})^{-1}\partial_\mu(\rho e^{i\beta} S) = \partial_\mu S + S(\partial_\mu \ln \rho + i\partial_\mu \beta) \,. \tag{3.17}$$

The vector part of (3.16) is

$$\hbar(\partial_\mu \psi i\gamma_3 \widetilde{\psi})_V = \rho(v P_\mu - v \cdot W_\mu) \,. \tag{3.18}$$

Hence,

$$\hbar\gamma_\nu \cdot (\partial_\mu \psi i\gamma_3 \widetilde{\psi})_V = \rho(v_\nu P_\mu + (v\Lambda\gamma_\nu)\cdot W_\mu) \,. \tag{3.19}$$

Write

$$P_\mu = p_\mu + eA_\mu \tag{3.20}$$

and compare (3.19) with (2.3) and (2.10) to get

$$T_{\mu\nu} = \rho\, v_\mu p_\nu + N_{\mu\nu}\,, \tag{3.21}$$

where, with the help of (3.17),

$$\begin{aligned} N_{\mu\nu} &= N_\mu \cdot \gamma_\nu = \rho\,(v \wedge \gamma_\mu)\cdot W_\nu \\ &= \rho\,(v \wedge \gamma_\mu)\cdot \partial_\nu S - \rho\, s_\mu \partial_\nu \beta\,. \end{aligned} \tag{3.22}$$

This shows that the p_μ introduced in (3.20) are exactly the components of local momentum introduced by (2.9). Equations (3.7) and (3.20) thus show how the local momentum is related to rotations in the local spin plane. In addition, (3.22) expresses the components $N_{\mu\nu}$ of the momentum flux tensor in terms of local observables.

4 INTEGRABILITY CONDITIONS

The fundamental local observables v, S, and p are all determined by the single spinor field R and its derivatives. Since R itself is completely parametrized by v, S and phase χ, p must be completely determined by these quantities and their derivatives. Since p depends on the phase only through its gradient, the curl of p must be a function only of the spin, the velocity, and their derivatives. This function can be found in the following systematic way.

Write (3.2) in the form

$$\partial_\mu R = \tfrac{1}{2}\Omega_\mu R\,. \tag{4.1}$$

Differentiate, to get

$$\partial_\nu\partial_\mu R = \tfrac{1}{2}(\partial_\nu\Omega_\mu + \tfrac{1}{2}\Omega_\mu\Omega_\nu)R\,. \tag{4.2}$$

But

$$\partial_\mu\partial_\nu R = \partial_\nu\partial_\mu R\,. \tag{4.3}$$

So

$$\partial_\nu\Omega_\mu - \partial_\mu\Omega_\nu = \tfrac{1}{2}[\,\Omega_\nu, \Omega_\mu\,]\,. \tag{4.4}$$

Thus the derivatives of the angular velocities are not mutually independent. These "integrability conditions" can be expressed as relations among observables by using (3.6). One obtains

$$\partial_\mu P_\nu - \partial_\nu P_\mu + i(\partial_\nu q_\nu - \partial_\mu q_\nu) = \tfrac{1}{2}[\,\partial_\nu S, \partial_\mu S\,]S^{-1}\,. \tag{4.5}$$

The bivector part of (4.5) gives nothing new since it is just a consequence of the fact that S^2 is a constant. The pseudoscalar part of (4.5) gives an expression for the curl of q_μ, but that is of little interest since the relation of q_μ to the spin and velocity is already completely exhibited by (3.11). However, the scalar part of 4.5) gives the following valuable relations, first derived in a different form by Takabayasi [3]:

$$\begin{aligned} \partial_\mu P_\nu - \partial_\nu P_\mu &= \tfrac{1}{2}[\,\partial_\nu S, \partial_\mu S\,]\cdot S^{-1} \\ &= (S^{-1}\partial_\nu S\partial_\mu S)_S = S\cdot(\partial_\nu v\wedge\partial_\mu v + \partial_\nu s\wedge\partial_\mu s^{-1}) \\ &= i\,s\wedge v\wedge[\,\partial_\nu v\wedge\partial_\mu v + \partial_\nu s\wedge\partial_\mu s^{-1}\,]\,. \end{aligned} \tag{4.6}$$

Since $P_\mu = p_\mu + eA_\mu$,

$$\partial_\nu P_\mu - \partial_\mu P_\nu + eF_{\mu\nu} = (\partial_\nu S\partial_\mu S)\cdot S^{-1}\,. \tag{4.7}$$

where

$$F_{\mu\nu} = (\gamma_\mu\wedge\gamma_\nu)\cdot F = (\gamma_\mu\wedge\gamma_\nu)\cdot(\Box\wedge A) = \partial_\mu A_\nu - \partial_\nu A_\mu$$

are the components of the electromagnetic field.

It is worth emphasizing that (4.7) or (4.6) depends on the definitions of the local observables only and not at all on the Dirac equation. Since it relates different observables, (4.7) can be regarded as a kind of constitutive relation. A constitutive relation that gives the full dependence of p on v and S is obtained from the Dirac equation in the next section.

5 PHYSICAL CONTENT OF THE DIRAC EQUATION

The conclusions of Sec. 2 are worth repeating. Equations (1.4), (2.26), and (2.34) are precisely the conservation laws of mass, energy-momentum, and angular momentum which are expected to hold for all physical theories. The use of the Dirac equation to obtain these laws merely serves to show that the Dirac theory is consistent with general principles. It can now be shown that the real physical content of the Dirac equation is to be found in the fact that it provides *constitutive relations* among the local density, velocity, momentum, and spin.

When these relations are put along side those relations which follow from the definitions of the local observables in terms of the Dirac wave function, then the conservation laws become determinate differential equations describing the time evolution of the local observables.

To express the Dirac equation (2.15) as a relation among local observables, first multiply it on the right by $\widetilde{\psi}$ to obtain

$$\hbar(\Box\psi)\gamma_2\gamma_1\widetilde{\psi} = m\rho v + eA\rho e^{i\beta}\,. \tag{5.1}$$

Next use (3.15) and (3.6) to obtain

$$\begin{aligned}\hbar(\partial_\mu\psi)\gamma_2\gamma_1\widetilde{\psi} &= \{\partial_\nu(\rho e^{i\beta}) + \Omega_\mu \rho e^{i\beta}\}\,\frac{\hbar}{2}\,R\gamma_2\gamma_1\widetilde{R} \\ &= \partial_\nu(\rho e^{i\beta})S + (P_\nu + iq_\mu + \partial_\mu S)\rho e^{i\beta} \\ &= (P_\nu + iq_\mu)\rho e^{i\beta} + \partial_\mu(\rho e^{i\beta}S)\,. \end{aligned} \tag{5.2}$$

This implies

$$\hbar(\Box\psi)\gamma_2\gamma_1\widetilde{\psi} = (P + qi)\rho e^{i\beta} + \Box(\rho e^{i\beta}S)\,. \tag{5.3}$$

Finally, equate (5.1) to (5.3) and use $p = P - eA$ to get

$$\rho e^{-i\beta}(p - iq) = \rho m v - \Box(\rho e^{i\beta}S)\,. \tag{5.4}$$

The pseudovector part of (5.4) yields

$$\rho(p\sin\beta + q\cos\beta) = \Box\cdot(\rho e^{i\beta}iS)\,. \tag{5.5}$$

The vector part of (5.4) is

$$\rho(p\cos\beta - q\sin\beta) = \rho m v - \Box\cdot(\rho e^{i\beta}S)\,. \tag{5.6}$$

This quantity, multiplied by e/m, is commonly known as the *Gorden current*. The last term in (5.6), then, is the divergence of a magnetic moment density. This is consistent with the identification of the magnetic moment in (6.28) below.

Equation (5.4) displays the physical content of the Dirac equation as a relation among local observables. The Dirac equation can be recovered from (5.4) by writing the local observables as functions of the Dirac wave function and its derivatives, but it should be remembered that these expressions are physical assumptions quite independent of the Dirac equation.

The "Dirac relations" among local observables are better expressed by multiplying (5.4) on the right to get

$$\rho(p - iq) = \rho m e^{i\beta}v - \Box(\rho S) + i(\Box\beta)\rho S\,. \tag{5.7}$$

The vector part of (5.7) gives the momentum density as a function of velocity and spin.

$$\rho p = \rho m v\cos\beta - \Box\cdot(\rho S) + \rho(iS)\cdot\Box\beta\,. \tag{5.8}$$

This is the simplest way to express the general noncollinearity of velocity and momentum in the Dirac theory. Note that a valid physical interpretation of β must account for the strange factor $\cos\beta$, which reduces the contribution of the "mass density" to the energy-momentum density. The last term $(iS)\cdot\Box\beta = (v\wedge s)\cdot\beta = vs\cdot\Box\beta - sv\cdot\Box\beta$ shows a dependence of momentum on the rate of change of β in the $v\wedge s$ plane.

The trivector part of (5.7) yields

$$-\rho q = \rho m \sin\beta v + \Box\cdot(\rho s v) + (\Box\beta)\cdot(\rho S)\,. \tag{5.9}$$

This can be reduced to simpler terms by using (2.18) and (3.12) to get

$$S\cdot\Box\beta = s\cdot(\Box\wedge v) - v\cdot\Box s + s\cdot\Box v - v\cdot(\Box\wedge s)\,. \tag{5.10}$$

Since $S\cdot\Box\beta = (is\wedge v)\cdot\Box\beta = is\wedge v\wedge\Box\beta = -s\cdot(iv\wedge\Box\beta)$, the first equation (5.10) can be written

$$v\cdot\Box = \cdots = s\cdot(\Box\wedge v) + s\cdot(iv\wedge\Box\beta)\,. \tag{5.11}$$

This is an equation for the rate of change of spin along a velocity streamline, and so exhibits explicitly the physical content of the trivector part of (5.7).

A number of important auxiliary formulas are easily obtained from (5.8) by utilizing algebraic properties of the velocity and spin:

$$\begin{aligned} p\cdot v &= m\cos\beta - (v\wedge\Box)\cdot s - (iS)\cdot(v\wedge\Box\beta) \\ &= m\cos\beta - S\cdot(\Box\wedge v + iv\wedge\Box\beta)\,, \end{aligned} \tag{5.12}$$

$$v\cdot\Box\beta = \dot\beta = p\cdot s^{-1} + S\cdot(\Box\wedge s^{-1}) = p\cdot s^{-1} + is^{-1}\wedge v\wedge\Box s\,, \tag{5.13}$$

$$p\cdot S = \rho^{-1}S\cdot(\Box\cdot\rho S) = -S\cdot(S\cdot\Box\ln\rho) + S\cdot(\Box\cdot S)\,, \tag{5.14}$$

$$\begin{aligned} v\wedge p &= \rho^{-1}[\,\Box\cdot(\rho S)\,]\wedge v - iS\dot\beta \\ &= \rho^{-1}\Box\cdot(\rho is) - \dot S + (S\cdot\Box)\wedge v - iS\dot\beta\,. \end{aligned} \tag{5.15}$$

Equation (5.12) is an expression for the *local energy* $p\cdot v$ that flows along a streamline. In the first term the rest mass is reduced by the factor $\cos\beta$. The remaining terms involve the "normal gradient" $v\wedge\Box$ which shows that their contribution to the local energy is determined by the flow of S and β onto the streamline.

By comparing (5.15) with (2.10) and (2.20) one finds

$$\rho^{-1}\gamma_\mu\wedge N^\mu = \dot S - (S\cdot\Box)\wedge v + iS\dot\beta\,. \tag{5.16}$$

The same result can be obtained with more effort by direct evaluation from (3.22).

6 PROPER FLOWS

The reformulation of the Dirac theory as a set of conservation laws and constitutive equations for local observables has already been completed. But further insight into the theory can be obtained by casting some of the equations into different forms. It is particularly interesting to study the flow of local observables along a streamline. This can be approached systematically by studying the *proper angular velocity* Ω, i.e., the angular velocity along a streamline:

$$\Omega \equiv 2\dot{R}\widetilde{R} = 2(v\cdot\Box R)\widetilde{R} = v^\mu \Omega_\mu \,. \tag{6.1}$$

Here the Ω_μ are just the angular velocities defined by (3.2).

An expression for Ω in terms of observables can be obtained directly from the Dirac equation by utilizing the identity

$$\Omega \equiv 2\dot{R}\widetilde{R} = \{(\Box R)\widetilde{R}, v\} - \Box v \,. \tag{6.2}$$

The curly brackets denote anticommutator. The identity can be established by noting that

$$(\Box R)\widetilde{R}vR = \Box R\gamma_0 = \Box(vR) = (\Box v)R - v\Box R + 2v\cdot\Box R \,.$$

Now write the Dirac equation (3.15) in the form

$$\hbar(\Box\psi)i\gamma_3\gamma_0\widetilde{\psi} = 2(\Box\psi)\widetilde{\psi}S = m\rho v + eA\rho e^{i\beta} \,. \tag{6.3}$$

Also note that

$$2(\Box\psi)\widetilde{\psi} = [\,\Box \ln\rho + (\Box\beta)i + 2(\Box R)\widetilde{R}\,]\rho e^{i\beta} \,. \tag{6.4}$$

Hence, from (6.3) and (6.4)

$$2(\Box R)\widetilde{R} = -\Box\ln\rho + i\Box\beta + (me^{i\beta}v + eA)S^{-1} \,. \tag{6.5}$$

So

$$\begin{aligned} \tfrac{1}{2}\{2(\Box R)\widetilde{R}, v\} &= \{2(\Box R)\widetilde{R}\}\cdot v \\ &= -v\cdot\Box\ln\rho + v\cdot(i\Box\beta) + v\cdot(mv\cos\beta + eA)S^{-1} \end{aligned} \tag{6.6}$$

Finally, substitute (6.6) into (6.2) to get

$$\Omega = -\Box\wedge v + v\cdot(i\Box\beta) + v\cdot(mv\cos\beta + eA)S^{-1}. \tag{6.7}$$

By (6.7), the proper time derivatives of the velocity and the spin are

$$\dot{v} = \Omega\cdot v = v\cdot(\Box\wedge v)\,, \tag{6.8}$$

$$\dot{s} = \Omega\cdot s = s\cdot(\Box\wedge v) + s\cdot[\,v\cdot(\Box\beta i)\,]\,, \tag{6.9}$$

$$\dot{S} = \tfrac{1}{2}[\,Q, S\,] = \tfrac{1}{2}[\,S, \Box\wedge v\,] + \tfrac{1}{2}[\,S, v\cdot(\Box\beta i)\,]\,. \tag{6.10}$$

Equation (6.8) is a mere identity, which depends only on the fact that v^2 is constant. Equation (6.9) is identical with (5.11), and, of course, (6.10) follows from (6.8) and (6.9), though it is handier to get it from (6.7). Clearly, these equations are not of much help unless a useful expression for $\Box \wedge v$ can be found.

Before proceeding further, it is worthwhile to examine the "classical limit," i.e., the limit in which the magnitude of the spin $|\, s \,| = \frac{1}{2}\hbar$ regarded a negligibly small quantity. In that limit, the compatibility conditions (4.6) can be written

$$\Box \wedge P = 0\,. \tag{6.11}$$

This implies that

$$P = \Box\chi\,. \tag{6.12}$$

lt is easy to see that the scalar χ is the phase of the Dirac wave function, so the classical limit amounts to a kind of "eikonal approximation" to the Dirac equation.

In the classical limit, Eq. (2.18) becomes $\sin\beta = 0$, which implies $\cos\beta = \pm 1$. In the same limit, the Dirac equation in the form (5.7) reduces to

$$p = \pm mv = \Box\chi - eA\,. \tag{6.13}$$

Clearly, the two signs correspond to limits describing particles with opposite charge.

The square of (6.13) is just the Hamilton-Jacobi equation for a classical "test charge":

$$(\Box\chi - eA)^2 = m^2\,. \tag{6.14}$$

Given the external potential A, one solves (6.14) to get χ. But equation (6.13) is still needed to get the velocity field from χ.

The curl of (6.13) is

$$\pm m\Box \wedge v = -e\Box \wedge A = -eF\,. \tag{6.15}$$

So, from (6.7) the proper angular velocity is just

$$\Omega = -\Box \wedge v = \pm\,\frac{e}{2m}\,F\,. \tag{6.16}$$

When this is substituted into (6.8), one obtains the Lorentz force.

But, more generally, (6.16) gives a spinor form for the Lorentz force

$$\dot{R} = \pm\,\frac{e}{2m}\,FR\,. \tag{6.17}$$

The solution of this equation is a one parameter family of Lorentz transformations $R = R(\tau)$ describing the rotation of the e_μ, as they "move" along a

streamline. In particular, this describes the rotation of the spin. So the spin does not simply disappear in the "classical limits" to the Dirac theory; only the effect of the spin on the motion of the particle disappears; an effect of the motion on the spin remains.

Now, to see what can be said about $\Box \wedge v$ without any approximation, write (5.7) in the form

$$(P + qi) - eA = mve^{-i\beta} - \gamma^\nu W_\nu \,, \tag{6.18}$$

where W_μ is defined by (3.17). The gradient of (6.18) is

$$(\Box P + \Box qi) - e\Box A = m[\,\Box v - i(\Box\beta)v\,]e^{-i\beta} + \gamma^\mu\gamma^\nu\partial_\mu W_\nu \,. \tag{6.19}$$

The bivector part of (6.19) is

$$\begin{aligned} m[\,\Box \wedge v + iv \wedge \Box\beta\,] \;\; &= \;\; -eF + \partial_\mu W^\mu \\ &\quad + \tfrac{1}{2}[\,\partial_\mu S, \partial_\nu S\,] + (\Box \wedge P + \Box \wedge qi)\,. \end{aligned} \tag{6.20}$$

The dependence on momentum in the last term can be eliminated by using the integrability conditions (4.5) in the form

$$\Box \wedge P + \Box \wedge qi = \tfrac{1}{4}\gamma^\mu\gamma^\nu[\,\partial_\nu S, \partial_\mu S\,]S^{-1}\,. \tag{6.21}$$

Also, a little calculation shows that

$$\partial_\mu W_\nu - \partial_\nu W_\mu + \tfrac{1}{2}[\,\partial_\nu S, \partial_\mu S\,]S^{-1} = z[\,W_\mu, W_\nu\,]S^{-1}\,, \tag{6.22}$$

so the last two terms of (6.20) can be combined. When this has been done, (6.20) can be written

$$\Box \wedge v + iv \wedge \Box\beta = -\frac{e}{m}\,Fe^{i\beta} + C\,, \tag{6.23}$$

where

$$mC = e^{i\beta}\big(\partial_\mu W^\mu + \tfrac{1}{4}[\,\gamma^\mu\gamma^\nu, [W_\mu, W_\nu\,]S^{-1}\,]\big)\,. \tag{6.24}$$

Equation (6.23) is very important because it *completely* describes the interaction of the external electromagnetic field F with the "Dirac observables" in a manifestly gauge invariant form. Section 5 shows that the electromagnetic field is absent from the basic relations among observables implied by the Dirac equation. The electromagnetic field is related to the momentum solely by the integrability conditions (4.8). Equation (6.22) expresses the implications of the integrability conditions in a form independent of p. And note that C in (6.24) is manifestly independent of velocity. The appropriate use of (6.23) is to eliminate $\Box \wedge v$ from equations to reveal explicitly the influence of the external field F on relations among observables.

The dependence of Ω on F is revealed by substituting (6.23) into (6.7) to get

$$\Omega = \frac{e}{m} Fe^{i\beta} - C + v \cdot (mv \cos\beta + eA)S^{-1} . \tag{6.25}$$

This gives immediately equations for the velocity and spin which show the dependence on F explicitly:

$$\dot{v} = \frac{e}{m} (Fe^{i\beta}) \cdot v + v \cdot C , \tag{6.26}$$

$$\dot{S} = \tfrac{1}{2}[\, F, \frac{e}{m} Se^{i\beta}\,] + \tfrac{1}{2}[\, S, C\,] . \tag{6.27}$$

In (6.27) the commutivity of $e^{i\beta}$ with bivectors was used to obtain a form that suggests that

$$\mu = \frac{|\, e\,|}{m} Se^{i\beta} \tag{6.28}$$

be interpreted as the local magnetic moment. And, in fact,

$$|\, \mu\,| = \frac{|\, e\,|}{m} |\, S\,| = \frac{|\, e\,|\, \hbar}{2m} \tag{6.29}$$

is the famous magnitude of the electron's magnetic moment, which was the first striking consequence of Dirac's theory. But the interpretation of the magnetic moment is not so simple a matter as (6.27) and (6.28) suggest, for (6.25) and (6.26) indicate that the factor $e^{i\beta}$ belongs with the F and not with the S.

Other important relations can be derived from (6.25) by using (3.6) to get

$$\begin{aligned} v \cdot (p + eA) + iv \cdot q + \dot{S} &= \Omega S \\ &= \frac{e}{m} Fe^{i\beta} S - CS + m\cos\beta + ev \cdot A . \end{aligned} \tag{6.30}$$

The bivector part of (6.30) just gives equation (6.27). But, with the help of (3.11), the pseudoscalar part of (6.30) gives

$$\begin{aligned} v \cdot q &= -v \cdot \cdots = s \cdot \dot{v} = i\left[\, C \wedge S - \frac{e}{m} (Fe^{i\beta}) \wedge S\,\right] \\ &= \frac{e}{m} (Fe^{i\beta}) \cdot (s \wedge v) + S \cdot (v \wedge s) . \end{aligned} \tag{6.31}$$

The scalar of (6.30) is

$$p \cdot v = m\cos\beta + \left(\frac{e}{m} Se^{i\beta}\right) \cdot F - C \cdot S . \tag{6.32}$$

The same result can be obtained by substituting (6.23) into (5.12). Equation (6.32) explicitly shows the effect of an external field F on the local energy.

Equation (2.26) describes the rate of change of momentum along a streamline, but it does not reveal the full effect of the electromagnetic field F on the momentum flow, because it does not show how the momentum flux N^μ is affected by F. The influence of F on the flux can be found by beginning with the expression (3.22) for $N_{\mu\nu}$ and evaluating $\partial_\mu N^\mu$ with the help of (2.18) and the integrability conditions in the form (6.23). Thus,

$$\begin{aligned}\partial^\mu N_{\mu\nu} &= \rho S\cdot(\square\wedge\partial_\nu v)-(\partial_\nu v\wedge\square)\cdot(\rho S)-\rho s\cdot\square\partial_\nu\beta-(\partial_\nu\beta)\square\cdot\rho s\\ &= -\rho S\cdot\partial_\nu\Big(\frac{e}{m}Fe^{i\beta}-C\Big)+\rho(\partial_\nu v\wedge\square\beta)\cdot(iS)-(\partial_\nu v\wedge\square)\cdot(\rho S)\\ &\quad -m\partial_\nu\cos\beta\\ &= -\rho\Big(S\cdot\partial_\nu\frac{e}{m}(Fe^{i\beta}-C)+(\partial_\nu v\wedge\gamma^\mu)\cdot W_\mu+m\,\partial_\nu\cos\beta\Big)\,. \quad (6.33)\end{aligned}$$

Substitution of this expression into (2.26) yields

$$\begin{aligned}\dot p=\frac{e}{m}F\cdot v &+ \frac{e}{m}\,\square(Fe^{i\beta})\cdot\not S-\square C\cdot\not S\\ &+ \square(v\wedge\gamma^\mu)\cdot\not W_\mu+m\square\cos\beta\,. \quad (6.34)\end{aligned}$$

Here a slash mark has been used to denote quantities which are not differentiated by the gradient operating from the left.

The second term on the right of (6.34) is a force of the Stern-Gerlach type. But note that the β is differentiated whereas the S is not, so it is not quite the usual form for the force due to a magnetic moment given by (6.28), Moreover, this force is not a body force; rather, as the derivation shows, it expresses only the effect of the external field on the local momentum flux.

It is also important to realize that the influence of F on the spin is entirely due to its effect on the spin flux. This can be shown by beginning with (2.34) and evaluating $\partial_\mu M^\mu$ with the help of (6.23). Thus,

$$\begin{aligned}\partial_\mu M^\mu &= \rho\tfrac12[\,S,\square\wedge v\,]+\tfrac12[\,v\wedge\square,\rho S\,] \quad (6.35)\\ &= \rho\tfrac12\big[\,F,\frac{e}{m}Se^{i\beta}\,\big]+\rho\tfrac12[\,S,C\,]+\tfrac12[\,v\wedge\gamma^\mu,W_\mu\,]\,.\end{aligned}$$

Substitution of this into (2.34) and comparison with (6.27) shows

$$pv\wedge p+\gamma_\mu\wedge N^\mu=\gamma_\mu\wedge T^\mu=\tfrac12[\,v\wedge\gamma^\mu,W_\mu\,]\,. \quad (6.36)$$

It is not difficult to establish that this relation follows from the Dirac equation without appeal to the integrability conditions.

The above facts about the influence of external fields on the local observables have important bearing on the interpretation of the Dirac theory.

7 WEYSSENHOFF MOTION

The velocity streamlines of the Dirac theory compose a congruence of timelike curves in space-time, and Eqs. (2.26) and (2.34) describe the flow of momentum

and angular momentum along these curves. The flow along one streamline is coupled to that along its neighbors by the flux of spin through the walls of a comoving volume element. To get some conception of the streamlines in the Dirac theory, suppose that along a particular stream line the net flux of momentum and angular momentum through the walls of a comoving volume element vanishes. This supposition can be stated mathematically by

$$\partial_\mu N^\mu = 0\,, \tag{7.1}$$

$$\gamma_\mu \wedge N^\mu = \partial_\mu M^\mu\,. \tag{7.2}$$

The equations of motion for a streamline satisfying these conditions are decoupled from those of its neighbors. However, the extent to which such a decoupling can be justified either rigorously or as some approximation to the Dirac theory is a difficult and unsolved problem.

Along the "decoupled" streamline just described, Eqs. (2.26) and (2.34) take the simple form

$$\dot{p} = f\,, \tag{7.3}$$

$$\dot{S} = v \wedge p\,. \tag{7.4}$$

The relation between velocity can be found by multiplying (7.4) by v and solving to get

$$p = v(p \cdot v + \dot{S}) = (p \cdot v)v + v \cdot \dot{S}\,. \tag{7.5}$$

Of course these equations must be supplemented by the conditions that v and S are orthogonal and have constant magnitude.

Equations (7.3), (7.4), and (7.5) were obtained from a model of a fluid "with spin" by Wessenhoff [9]; they have been studied by many authors as a "classical approximation" to the Dirac theory; details can be found in books by Corben [10] and Halbwachs [11]. Of course there is nothing surprising in the fact that these equations can be obtained from a classical model; they require for their validity only general conservation laws and the assumptions that decouple the streamlines. The classical models become unphysical when they assume that a continuous system can be shrunk to a point particle obeying (7.3) and (7.4). For insight into the Dirac theory it is sufficient to suppose that the equations describe only a single streamline.

The interesting point is that the equations for the decoupled streamline can be solved. In the absence of external forces the streamline is a generalized helix. This helical motion persists in the presence of a constant magnetic field and, as Corben [10] has repeatedly emphasized, gives the correct gyromagnetic ratio for the electron, the simple number for which the Dirac theory is most

famous. Thus, however dubious the decoupling assumptions, the decoupled equations retain some of the main features of the Dirac theory. And their solutions suggest that the magnetic moment of the electron is not due to any structure of the electron; it appears to be a dynamical effect, arising from the general tendency to execute helical motion because of the noncollinearity of velocity and momentum.

8 INTERPRETATION OF THE DIRAC THEORY

The set of local observables adopted in this paper is complete and thoroughly satisfactory in the sense that the entire *mathematical content* of the Dirac theory can be expressed as a system of determinate equations for these quantities. On the other hand, the physical content of the Dirac theory depends on the *physical interpretation* accorded to the local observables, that is, on the correspondence of the mathematical quantities called observables with quantities measured experimentally. Unfortunately, this correspondence is not so well established as to eliminate the possibility that the local observables have been incorrectly identified or that the Dirac theory is incorrect in some of its more detailed implications.

There does not seem to be any reason to doubt that the local velocity (1.3) has been correctly identified. The identification adopted here leads to a reasonable interpretation of the energy levels in hydrogen, which seems to be the reason it was originally accepted by the Dirac. To say that it has been universally adopted by practitioners of the Dirac theory would not be much of an exaggeration.

The correctness of identifying the Tetrode tensor (2.3) with the energy-momentum tensor can be questioned. It is easy to write down different tensors which yield identical values for global observables such as the energy levels of stationary atomic states. Each such tensor defines a different physical theory distinguished by the particular local distribution and flow of energy-momentum it predicts. No such distinction has as yet been tested experimentally. Nevertheless, there are good reasons to favor the Tetrode tensor.

Tetrode's definition of the energy-momentum tensor is adopted in this paper on the theoretical ground that it is the most straightforward generalization of the operator definitions of energy and momentum taken over by Dirac from Schrodinger theory. Perhaps a better reason for adopting the Tetrode tensor is the fact that it leads to the classical Lorentz force (2.26); this seems to have been Tetrode's original reason. Also, on close inspection it becomes clear that no alternative to the local momentum (2.9) determined by the Tetrode tensor could lead to simpler equations for local observables. Finally, it should be mentioned that the successful prediction of the Imbert effect ([4],[5]) was

made from a nonsymmetric electromagnetic energy-momentum tensor obtained by analogy from the Tetrode tensor. Thus the Tetrode tensor is supported by indirect experimental evidence. The challenge is to test it directly.

In the absence of any experimental or theoretical reason to doubt that the identification of local observables has been correctly made, it is necessary to develop a coherent interpretation of the Dirac theory which is consistent with the present identifications of local observables. Any satisfactory interpretation of the Dirac theory must explain the prominent role played by spin. In Dirac's original paper [6] the appearance of spin is rather mysterious, since no mention of spin was made in his basic assumptions. It is still widely believed that spin emerged as a consequence of relativity, though this has been refuted by many authors (e.g., Ref. [12]). With a complete theory of local observables in hand, it is not difficult to ascertain the key assumption by which Dirac (implicitly) introduced spin into his theory. The derivation of the local conservation laws in Sec. 2 leads to the identification of the local spin in Eq. (2.23). Following the argument backwards, one discovers that spin was introduced by the definition of energy (2.1), in particular by the factor $(-1)^{1/2}\hbar$ in that definition. The usual matrix formulation of the Dirac theory hides the relation of $(-1)^{1/2}\hbar$ to spin very well indeed. But the relation was uncovered in Ref. [1] when matrices were dispensed with. The relation can be explained succinctly as follows: In the Dirac theory the spin is a skew symmetric tensor, or, equivalently, the bivector defined by (1.6); the usual matrix representation of this bivector has eigenvalue $\frac{1}{2}(-1)^{1/2}\hbar$, as is shown by Eq. (A.1). Thus, the factor $(-1)^{1/2}\hbar$ which appears in the matrix formulation of the Dirac theory is a representation of the spin tensor by its eigenvalue; its true identity is revealed by reformulating the theory in terms of observables.

It is important to note that $(-1)^{1/2}$ and $\hbar$ always occur together as a factor $(-1)^{1/2}\hbar$ in the original basic equations of Dirac, specifically, in the definition of the energy operator (2.1) and in the Dirac equation (A.5) of Appendix A. It follows that *Planck's constant is inseparably related to the spin in the Dirac theory.* So it should be no surprise that examination of Secs. 1 to 5 reveals that Planck's constant appears in the equations for local observables only as twice the magnitude of the local spin.

The ubiquitous connection between spin and Planck's constant obviously must persist in any nonrelativistic approximation to the Dirac theory, including the Schrodinger theory. Indeed, as shown explicitly in Ref. [13], the Schrodinger equation is identical to the Pauli equation in the absence of magnetic fields, and spin appears in the Schrodinger theory as the innocent factor $(-1)^{1/2}\hbar$. It follows that every appearance of Planck's constant in the Schrodinger theory is directly related to the existence of spin. This fact is difficult to reconcile with the usual interpretation of uncertainty relations derived from Schrodinger's equation. Though Planck's constant has a prominent place in the uncertainty relations, none of the usual interpretations seem capable of accounting for its connection with spin. It is strange that uncertainty relations for position and

momentum, which are presumed to be to the interpretation of quantum theory, are not derivable from the Dirac theory without suppressing spin.

The Dirac theory poses another difficulty for the usual interpretation of the uncertainty relations. The noncollinearity of local momentum and velocity suggests that uncertainty relations for momentum and position are not, as is usually assumed, equivalent to uncertainty relations for velocity and position. Indeed, when the nonrelativistic limit is correctly carried out it can be shown that the inequivalence of velocity and momentum persists even in the Schrodinger theory. Details will be given elsewhere.

Because of these problems with the interpretation of the uncertainty principle, it is difficult to reconcile the details of the Dirac theory with the socalled Copenhagen interpretation of quantum theory in general. But there is an alternative interpretation which appears to be more congenial to the Dirac theory. It is called the *statistical interpretation* of quantum mechanics in a review by Ballentine [13]. According to the statistical interpretation, an electron is always to be regarded as a particle, and the Dirac theory describes an ensemble average of its motion. The probability density in the Dirac theory specifies the relative probability that the electron is located at a given place at a given time.

The simplest model of an electron compatible with the statistical interpretation is a structureless point charge. Of course it is impossible to derive any such model from the mathematical formulation of the Dirac theory. But it is easy to see that the model is consistent with many properties of local observables implied by the mathematical structure of the theory. First, if the charge and mass of the electron are actually localized at a single point, then the "smearing out" of the electron by any kind of averaging process must produce a distribution with constant charge to mass ratio, in agreement with the Dirac theory. Second, if the charge of the electron were not localized at a point, then terms describing the coulomb interaction between different elements of charge would appear in the local conservation laws and constitutive equations derived in Sec. 2 and 5; no such terms exist; but this is to be expected if the density in the Dirac theory describes only the probable location of a single point charge. Third, the fact that the local electromagnetic interaction is described in the Dirac theory by the Lorentz force is just what would be expected for a point charge. Finally, if the electron is assumed to be a structureless point charge, then the electron spin and magnetic moments must arise from some peculiar dynamical property of the average electron motion. These last two points deserve elaboration.

The coupling of a Dirac electron to the electromagnetic field is completely described by Eq. (2.26). But (2.26) is exactly the "classical" expression for the electromagnetic force on a local distribution of charge, the socalled Lorentz force; moreover, the angular momentum conservation law (2.31) shows that the Dirac electron has no local intrinsic magnetic moment such as would appear in higher multipole moments of the local charge distribution if the spin of the electron were associated with some local structure of the electron. Thus, the electromagnetic interaction in the Dirac theory differs in no way from that given

by the "classical" theory of a local charge distribution.

It follows that the magnetic moment of the electron arises from the circulation of the local charge distribution. But what about the well-known theorem that the "classical" circulation of a fluid with constant charge to mass ratio and with angular momentum equal to the known spin of the electron cannot give rise to the known magnetic moment of the electron? That theorem does not apply! Because it implicitly assumes that the local momentum flow is collinear with the local flow of charge. This assumption certainly does not hold in the Dirac theory, nor, in fact, is it required even in classical theories.

Exactly how in the Dirac theory the local spin is related to a local circulation of charge giving rise to the observed magnetic moment of the electron is difficult to ascertain. But the problem is attacked in Sec. 6 where a general expression for the curl of the local velocity is obtained Eq. (6.23)]. Unfortunately, the physical significance of the complicated term (6.24) is difficult to fathom, injecting some uncertainty in the conclusions that can be drawn. Nevertheless, it seems that the contribution of the electromagnetic field to the circulation of charge is completely and explicitly revealed by Eq. (6.24). Indeed, the equation is shown to lead to the well-known value for the electron magnetic moment in Eqs. (6.26), (6.27), and (6.32). However, the relations among local observables in these equations are given in more detail than in corresponding equations in the literature, and it appears to be no simple matter to interpret them fully.

Valuable insight into the relation of charge circulation to magnetic moment is given by approximation to the Dirac theory briefly discussed in Sec. 7. The resulting "Weyssenhoff motion" explicitly shows the correct gyromagnetic ratio for the electron as arising from a generalized "helical motion" which is a consequence of the noncollinearity of velocity and momentum. It seems reasonable, therefore, to suppose that the electron magnetic moment is but one consequence of the general noncollinearity of local velocity and momentum. But a great deal more study will be necessary before firm conclusions can be drawn.

There seems to be no alternative to the point charge model of an electron which is capable of interpreting the details of the Dirac theory just mentioned. Therefore, the hypothesis that the Dirac theory describes some sort of average motion of a structureless point charge ought to be examined very carefully.

ACKNOWLEDGMENT

I am indebted to R. Gurtler for checking some of the calculations.

A APPENDIX A: MATRIX FORM OF THE DIRAC THEORY

There are a number of ways to establish the equivalence of the formalism used in this paper to the matrix formalism usually used to express the Dirac theory. Though this equivalence has already been established in Refs. [1] and [2], a brief discussion of how to translate expressions from one formalism to the other should be helpful. The simplest method is to replace the vectors γ_μ, directly by their representations as 4×4 matrices.

One can represent the vector γ_0 by a hermitian matrix and the vectors γ_i $(i = 1, 2, 3)$ by antihermitian matrices. Some writing is saved by using the same symbols γ_μ for both the vectors and their matrix representations. But, when this is done, the symbol $i = \gamma_0\gamma_1\gamma_2\gamma_3$ for the unit pseudoscalar should be replaced by the symbol $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ usually used for the antihermitian matrix which represents it, so as to avoid confusion with the uninterpreted unit imaginary usually symbolized by $i = \sqrt{-1}$ in matrix theory.

To express the Dirac wavefunction as a column spinor, introduce a unit column spinor u which is simultaneously an "eigenvector" of the matrices γ_0 and $\gamma_5\gamma_3\gamma_0 = \gamma_2\gamma_1$ with eigenvalues 1 and i respectively; i.e., write

$$\gamma_0 u = u \qquad \text{and} \qquad \gamma_2\gamma_1 u = iu\,. \tag{A.1}$$

This can be done, for example, with the matrices

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma_2\gamma_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \tag{A.2}$$

Now, considering the spinor ψ or Eq. (1.1) as a matrix operator, one obtains a corresponding column spinor by operating on u:

$$\Psi = \psi u\,. \tag{A.3}$$

The above specifications suffice to relate expressions in the "space-time algebra" to expressions in the matrix algebra. For example, in the space-time algebra the Dirac equation can be written

$$\hbar\Box\psi\gamma_2\gamma_1 - eA = m\psi\gamma_0\,. \tag{A.4}$$

Considering this as a matrix equation, multiplying by u on the right and using (A.1) and (A.3), one obtains the usual matrix form of the Dirac equation:

$$(i\hbar\Box - eA)\Psi = \gamma^\mu(i\hbar\partial_\mu - eA_\mu)\Psi = m\Psi\,. \tag{A.5}$$

Equation (A.5) may look simpler than (A.4), but actually it is not, because some of its properties depend implicitly on the choice of matrix representation. Equation (A.4) is independent of any matrix representation.

Translation of expressions for observables from one language to the other requires an understanding of the role of hermitian conjugation in the theory. Observe that hermitian conjugation of γ_μ can be expressed as a multiplicative operation:

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 \,. \tag{A.6}$$

Let M be a linear combination of the γ_μ and their products with *real coefficients*. It follows from (A.7) that the hermitian conjugate of M is

$$M^\dagger = \gamma_0 \widetilde{M} \gamma_0 \,, \tag{A.7}$$

where the tilde means reverse order of multiplication of all products of the γ_μ. Thus, from (1.1), one obtains

$$\begin{aligned} \widetilde{\psi} &= \widetilde{R} e^{\frac{1}{2}\beta\gamma_5} \rho^{\frac{1}{2}} = \rho^{\frac{1}{2}} e^{\frac{1}{2}\beta\gamma_5} \widetilde{R} \,, & \text{(A.8)} \\ \psi^\dagger &= \gamma_0 \widetilde{\psi} \gamma_0 = \rho^{\frac{1}{2}} e^{-\frac{1}{2}\beta\gamma_5} R^\dagger \,. & \text{(A.9)} \end{aligned}$$

The operation (A.7) is essentially different and must be distinguished from the complex conjugation of a scalar in matrix algebra. The latter has no physical significance if it cannot be reduced to the former, as is shown by the fact that it has been completely eliminated from the geometric language used in this paper.

From (A.2) it is easily ascertained that

$$u u^\dagger = \tfrac{1}{4}(1 + \gamma_0)(1 - i\gamma_2\gamma_1) \,. \tag{A.10}$$

So

$$\begin{aligned} \Psi \widetilde{\Psi} \gamma_0 &= \psi u u^\dagger \psi^\dagger \gamma_0 = \psi u u^\dagger \widetilde{\psi} \\ &= \tfrac{1}{4}\{\psi\widetilde{\psi} + \psi\gamma_0\widetilde{\psi} - i\psi\gamma_2\gamma_1\widetilde{\psi} - i\psi\gamma_5\gamma_3\widetilde{\psi}\} \\ &= \tfrac{1}{4}\rho\{e^{\beta\gamma_5} + v + ie^{\beta\gamma_5}\hat{S} - i\gamma_5\hat{s}\} \,. \end{aligned} \tag{A.11}$$

To get this last line, the canonical form (1.1) for ψ and the definitions of Sec. 1 have been taken over into the matrix algebra; also, $S = R\gamma_1\gamma_2\widetilde{R}$, $R\gamma_0\gamma_2\gamma_1\widetilde{R} = R\gamma_5\gamma_3\widetilde{R} = \gamma_5 R\gamma_3\widetilde{R} = \gamma_5\hat{s}$. From (A.11), it follows that for any matrix M

$$\begin{aligned} \Psi^\dagger\gamma_0 M \Psi &= \mathrm{Tr}\,\{\Psi^\dagger\gamma_0 M\Psi\} = \mathrm{Tr}\,\{M\Psi\Psi^\dagger\gamma_0\} \\ &= \tfrac{1}{4}\{M[\,\psi\widetilde{\psi} + \psi\gamma_0\widetilde{\psi} - i\psi\gamma_2\gamma_1\widetilde{\psi} - i\psi\gamma_0\gamma_2\gamma_1\widetilde{\psi}\,]\} \\ &= (M\psi\widetilde{\psi})_S + (M\psi\gamma_0\widetilde{\psi})_S - i(M\psi\gamma_2\gamma_1\widetilde{\psi})_S - i(M\psi\gamma_0\gamma_2\gamma_1\widetilde{\psi})_S \,. \end{aligned} \tag{A.12}$$

The trace of a matrix in the Dirac matrix algebra is equal to four times the scalar part of the corresponding multivector in the space-time algebra, so with

this understood, the last line of (A.12) has the same form and value in both languages. This greatly facilitates translation from one language to the other. Thus, from (A.12) and the last time of (A.11), one easily obtains the following variety of equivalent expressions:

$$\begin{aligned}\Psi^\dagger\gamma_0\gamma_\mu\Psi &= \mathrm{Tr}\,\{\gamma_\mu\Psi\Psi^\dagger\gamma_0\} = \tfrac{1}{4}\mathrm{Tr}\,\{\gamma_\mu\psi\gamma_0\widetilde{\psi}\} \\ &= (\widetilde{\psi}\gamma_0\gamma_\mu\psi)_S = (\gamma_\mu\psi\gamma_0\widetilde{\psi})_S = \rho(\gamma_\mu v)_S = \rho\gamma_\mu\cdot v = \rho v_\mu\,,\end{aligned} \tag{A.13}$$

$$\begin{aligned}i\Psi^\dagger\gamma_0\gamma_\mu\gamma_5\Psi &= i\mathrm{Tr}\,\{\gamma_\mu\gamma_5\Psi\Psi^\dagger\gamma_0\} = \frac{i^2}{4}\,\mathrm{Tr}\,\{\gamma_\mu\gamma_5\psi\gamma_5\gamma_3\widetilde{\psi}\} \\ &= \tfrac{1}{4}\mathrm{Tr}\,\{\gamma_\mu\psi\gamma_3\widetilde{\psi}\} = \rho(\gamma_\mu\hat{s})_S = \rho\gamma_\mu\cdot\hat{s} = \rho\,\hat{s}_\mu\,.\end{aligned} \tag{A.14}$$

This establishes the equivalence of the usual expressions for velocity and spin in the matrix language with those adopted in this paper. In going from the left to the right sides of (A.13) and (A.14), use has been made of the fact that the trace of an odd product of the matrices γ_μ, vanishes, which is equivalent to the fact that an odd product of the vectors γ_μ has no scalar part. Note that the sole function of the i on the left side of (A.14) is to cancel the i which (A.11) shows to be hidden in the matrix representation.

In matrix language the components of the Tetrode tensor are

$$T_{\mu\nu} = \frac{i\hbar}{2}\,\{\Psi^\dagger\gamma_0\gamma_\mu\partial_\nu\Psi - \partial_\nu\Psi^\dagger\gamma_0\gamma_\mu\Psi\} - eA\Psi^\dagger\gamma_0\gamma_\mu\Psi\,. \tag{A.15}$$

Using (A.10) as before, one obtains

$$\begin{aligned}i\Psi^\dagger\gamma_0\gamma_\mu\partial_\nu\Psi &= i\mathrm{Tr}\,\{\gamma_\mu(\partial_\nu\Psi)\Psi^\dagger\gamma_0\} \\ &= \frac{i}{4}\,\mathrm{Tr}\,\{\gamma_\mu\partial_\nu\psi[\,1+\gamma_0 - i\gamma_2\gamma_1 - i\gamma_5\gamma_3\,]\widetilde{\psi}\} \\ &= i(\gamma_\mu\partial_\nu\psi\gamma_0\widetilde{\psi})_S + (\gamma_\mu\partial_\nu\psi\gamma_5\gamma_3\widetilde{\psi})_S\,.\end{aligned} \tag{A.16}$$

Two terms vanished in proceeding to the last line of (A.7) because they are odd, which follows from the fact that both ψ and $\partial_\nu\psi$ are even. Similarly,

$$\begin{aligned}i\partial_\nu\Psi^\dagger\gamma_0\partial_\mu\Psi &= i(\gamma_\mu\psi\gamma_0\partial_\nu\widetilde{\psi})_S + (\gamma_\mu\psi\gamma_5\gamma_3\partial_\nu\widetilde{\psi})_S \\ &= i(\gamma_\mu\partial_\nu\psi\gamma_0\widetilde{\psi})_S - (\gamma_\mu\partial_\nu\psi\gamma_5\gamma_3\widetilde{\psi})_S\,.\end{aligned} \tag{A.17}$$

The last line of (A.17) follows by using the fact that the scalar part of a product is unchanged by reversing the order of multiplication. Subtracting (A.17) from (A.16) and using (A.13) one finds that (A.15) can be written

$$T_{\mu\nu} = \hbar(\gamma_\mu\partial_\nu\psi\gamma_5\gamma_3\widetilde{\psi})_S - e\rho\,v_\mu A_\nu\,, \tag{A.18}$$

which is the form used in the test above.

B APPENDIX B: DIVERGENCE OF THE J_μ

Equation (5.17) of Ref. [1] contains an error and should be amended to read

$$\Box \cdot J_\mu = -2m \sin\beta\, e_3 \cdot J_\mu + 2ei(e_3 \wedge e_0 \wedge J_\mu \wedge A)\,. \tag{B.1}$$

The two sentences following that equation should be corrected accordingly.

In the interest of completeness, it may be worthwhile to give the simple derivation of (B.1) directly from the Dirac equation. So multiply the Dirac equation (2.15) on the right by $i\gamma_0\gamma_3\gamma_\mu\widetilde{\psi}$ to get

$$\begin{aligned}
\hbar(\Box\psi)\gamma_\mu\widetilde{\psi} &= -im\psi\gamma_3\gamma_\mu\widetilde{\psi} - eiA\psi\gamma_0\gamma_3\gamma_\mu\widetilde{\psi} \\
&= -im\rho e^{i\beta}R\gamma_3\gamma_\mu\widetilde{R} - e\rho iAR\gamma_0\gamma_3\gamma_\mu\widetilde{R} \\
&= -im\rho e^{i\beta}e_3e_\mu - e\rho iA\psi e_0e_3e_\mu\,.
\end{aligned}$$

Write $J = \psi\gamma_\mu\widetilde{\psi}$ and take the scalar part of this expression

$$\begin{aligned}
\Box \cdot J_\mu &= (\Box J_\mu)_S = 2[\,(\Box\psi)\gamma_\mu\widetilde{\psi}\,]_S \\
&= -\frac{2}{\hbar}\,[\,im\rho e^{i\beta}e_3J_\mu + e\,A\,ie_3e_0J_\mu\,]_S\,, \\
\Box \cdot J_\mu &= \frac{2m}{\hbar}\,\sin\beta\, e_3\cdot J_\mu - \frac{2e}{\hbar}\,(ie_3e_0J_\mu A)_S\,.
\end{aligned} \tag{B.2}$$

The last term on the right can be written in several different ways:

$$(ie_3e_0J_\mu A)_S = i(e_3 \wedge e_0 \wedge J_\mu \wedge A) = (e_2\,e_1J_\mu A)_S = (e_2e_1)\cdot(J_\mu \wedge A)\,.$$

Equation (B.2) with $\hbar = 1$ is seen to agree with (B.1) except for a sign which comes from using a different sign convention in the Dirac equation.

The divergences of the probability and spin currents given by Eqs. (1.4) and (2.18) are seen to agree with (B.2) when $\mu = 0$ and 3, respectively.

References

[1] D. Hestenes, *J. Math. Phys.* **8**, 798 (1967).

[2] D. Hestenes, *Space-Time Algebra* (Gordon and Breach, New York, 1966).

[3] T. Takabayasi, *Prog. Theor. Phys. Suppl.* **4**, 1 (1957). Summarized in *Nuovo Cimento* **3**, 233 (1956). Elaborated in *Nuovo Cimento* **7**, 118 (1958). Takabayasi and Costa de Beauregard, Ref. [4], give references to earlier work on local observables. The equivalence of the parameter β with Takabayasi's parameter θ was pointed out by G. Casanova, *C.R. Acad. Sci.* **B 266**, 1551 (1968). Some of the mathematical expressions in Secs. 3, 5, and 6 of this paper were derived independently by R. Boudet, private communication and *C.R. Acad. Sci.* **A 272**, 767 (1971); also R. Boudet and P. Quilichini, *C.R. Acad. Sci.* **A 268**, 724 (1969).

[4] O. Costa de Beauregard, *Found. Phys.* **2**, 111 (1972).

[5] C. Imbert, *Phys. Rev.* **D 5**, 787 (1972).

[6] P. A. M. Dirac, *Proc. R. Soc. Lond.* **117**, 610 (1928).

[7] G. Breit, *Proc. Natl. Acad. Sci. USA* **14**, 553 (1928).

[8] H. Tetrode, Z. Phys. **49**, 858 (1928).

[9] J. Wessenhoff and A. Raabe, *Acla Phys. Pol.* **9**, 7 (1947).

[10] H. Corben, Classical and Quantum Theories of Spinning Particles (Holden-Day, San Francisco, 1948).

[11] F. Halbwachs, *Theorie relativiste des fluids a spin* (Gauthier-Villars, Paris, 1960).

[12] D. Hestenes and R. Gurtler, *Am. J. Phys.* **39**, 1028 (1971).

[13] L. Ballentine, *Rev. Mod. Phys.* **42**, 358 (1970).