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Local Observables in Quantum Theory

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Abstract. The Pauli theory of electrons is formulated in the language of multivector calculus. The advantages of this approach are demonstrated in an analysis of local observables. Planck's constant is shown to enter the theory only through the magnitude of the spin. Further, it is shown that, when obtained as a limiting case of the Pauli theory, the Schrödinger theory describes a particle with constant local spin. An important consequence of this result is the realization that the usual interpretations of the Dirac and Schrödinger theories are mutually inconsistent in certain details.

Introduction

Most experimental tests of quantum theory pertain only to "global observables" such as the total energy, total momentum, and total angular momentum. Nevertheless, a purely local quantity, the position probability density ρ, plays a central role in the theory. And the very existence of ρ entails the existence of other "local observables" describing spacial distributions of energy, momentum and spin. The Dirac theory of the electron and its nonrelativistic limit the Pauli theory make detailed predictions about local observables. A study of local observables ought to lead to finer tests of quantum mechanics and perhaps provide clues to an even deeper theory. Yet, surprisingly, theoretical studies of local observables are relatively few and altogether insufficient to lead to any firm conclusions. So naturally no experimental efforts have been addressed expressly to the subject.

This paper is concerned with a general theoretical analysis of local observables in the Pauli theory. The analysis is greatly simplified and clarified by utilizing the multivector calculus as explained in Ref. 1. No attempt is made here to analyze specific experimental implications of the theory. Rather, the object is to display the general properties of local observables in a form suitable as a starting point for more specific studies in the future.

Several other studies[2−4] are more or less similar to the one carried out here. Those studies interpret the Pauli theory in terms of various models such as a fluid of spinning particles or an elastic medium. This paper invokes no such unconventional interpretations. It is enough to show that the Pauli theory lead to definite equations for local observables. The same equations are obtained no matter what the interpretation. However, the definition of local observables are by no means unambiguous, though certain choices seem to be more natural than others. No final conclusions can be made until further physical information is brought to bear on the matter.

The work of Takabayasi[2] is closest to the present study, but it differs significantly in method, in the formulation of equations, and in interpretation of results. Takabayasi proceeds by translating expressions in spinor calculus into equivalent expressions in tensor calculus. The need for such translations is eliminated by the multivector approach, which is consequently simpler and more direct. The equations obtain by all methods must, of course, be equivalent, or someone has made a mistake. But the form that the equations take is greatly influenced by the method, as a comparison of references[2−4] readily reveals. Accordingly, it is by no means a trivial task to cast the equations for local observables in their most perspicuous form. multivector calculus makes possible an especially compact formulation which helps bring to the fore the physical and geometrical content of the theory.

The present study brings to light some features of the Pauli theory which may be of far-reaching importance to quantum theory in general. The text shows that if the local spin is assumed to be constant, then the Pauli theory reduces *exactly* to the Schrödinger theory. Therefore it is possible to conclude that *spin is already in the Schrödinger theory;* that is, that Schrödinger theory is simply the nonrelativistic theory of an electron with constant spin. At first sight such a conclusion may seem preposterous, for Schrödinger knew nothing of spin when he framed his equation. But it is no more preposterous than the incredible fact, that Schrödinger wrote down his equation and solved the first problems of modern quantum theory without any mention of probability; though it now appears that probability was already in the theory, for it was proved possible to adopt Born's interpretation of $\psi^\dagger\psi$ as a probability density without the slightest modification of Schrödinger's work, indeed, Born's interpretation is now an indispensable part of the theory.

The place where spin resides in the Schrödinger theory can be explicitly pointed out. The limit of the Pauli theory reveals that the distinctive factor $\mathbf{i}\hbar$ in the Schrödinger theory is exactly twice the spin. Crucial to this conclusion is the realization that the Pauli theory implies that $\mathbf{i}$ is a unitary bivector. The usual matrix formulation of the theory hides this fact very well indeed. It was first uncovered from the Dirac theory in Ref. 5.

Having identified the factor $\mathbf{i}\hbar$ correctly, one can regard the Pauli theory simply as a generalization of the Schrödinger theory which allows the bivector $\mathbf{i}$ to be a function of position and time. The Dirac theory takes account of relativity, but contrary to a widespread opinion, it says nothing fundamental about spin that is not already in the Pauli theory. As others have noted, and as can be easily be conclude from the treatment to follow, when properly formulated, the Pauli theory even predicts the correct gyromagnetic ratio of the electron. This should help put to rest once and for all the old canard that "spin is a relativistic phenomena."

It is significant that in Schrödinger theory $\mathbf{i}$ and $\hbar$ always appear together in the product $\mathbf{i}\hbar$. The significance is made manifest in the study of the Pauli theory to follow. There it is shown that *Planck's constant $\hbar$ enters the theory only as twice the magnitude of the electron's spin.* This strongly suggests that there is a general connection of spin to the appearance both of Planck's constant and of "complex numbers" in quantum theory. It should especially be noted that this idea has arisen only from insistence on the internal consistency of quantum theory as it exists today; it has not been imposed on the theory by external considerations.

While only widely accepted features of quantum theory are presumed in the present considerations, the conclusions are strongly at variance with the usual interpretation of quantum theory. It should be clear, for example, that if Planck's constant appears in electron theory only in connection with the spin, then any discussion of "uncertainty relations" which fails to mention spin must be profoundly deficient. Just the same, no novel modifications of quantum theory will be ventured here.

1. The Pauli Theory and the Schrödinger Theory

The Pauli theory is here formulated in the language of multivector calculus as set forth in Ref. 1. The notations and conventions of Ref. 1 are much utilized here, mostly without comment. The relation of this approach to the more usual matrix is formulation is discussed in Appendix A. The "Pauli wave function" $\psi = \psi(\mathbf{x}, t)$ is a spinor-valued function. Therefore, as explained in Ref. 1, it can be written in the form

$$\psi = \rho^{1/2} U \,, \tag{1.1}$$

where

$$UU^\dagger = U^\dagger U = 1 \,, \tag{1.2}$$

so

$$\psi\psi^\dagger = \psi^\dagger\psi = \rho \,. \tag{1.3}$$

The scalar function $\rho = \rho(\mathbf{x}, t)$ is the usual particle position probability density. So it is subject to the normalization condition

$$\int d^3x\,\rho = 1 \,. \tag{1.4}$$

The Pauli equation for a particle with mass m and charge e (an electron) can be written in the form

$$\partial_t \psi \mathbf{i}\hbar = (2m)^{-1} [P_{\mathrm{op}} - \frac{e}{c}\,\mathbf{A}]^2 \psi + e\phi\psi \,. \tag{1.5}$$

Here the quantity $\mathbf{i}$ is the constant bivector

$$\mathbf{i} = i\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 \,. \tag{1.6}$$

As usual, ϕ and $\mathbf{A}$ are, respectively, the electromagnetic scalar and vector potentials. The operator $P_{\rm op}$ is defined by the expression

$$P_{\rm op}\psi = -\nabla\psi\mathbf{i}\hbar\,. \tag{1.7}$$

By eliminating $P_{\rm op}$ from (1.5), the Pauli equation can be expanded into the more explicit form

$$\partial_t\psi\mathbf{i}\hbar = (2m)^{-1}\big[-\hbar^2\nabla^2 + \frac{e^2}{c^2}\,\mathbf{A}^2\big]\psi + (e\hbar/mc)\mathbf{A}\cdot\nabla\psi\mathbf{i}\ (e\hbar/2mc)(\nabla\mathbf{A})\psi\mathbf{i} + e\,\phi\psi\,. \tag{1.8}$$

The coefficient of the term with $(\nabla\mathbf{A})\psi\mathbf{i}$ determines the magnitude of the electron's magnetic moment, so it is worth noting how that number is related to the other terms in (1.8) by the form of Eq. (1.5).

By itself, the Pauli equation is hardly a physical theory It must be supplemented by additional *physical* assumptions which relate ψ to observable quantities. The first prescription of this kind was made by Schrödinger in his initial formulation of quantum theory. He identified the total energy E of an electron in a stationary state with the eigenvalue of an eigenvalue problem. When carried over into the Pauli theory, his prescription gives the eigenvalue problem

$$\partial_t\psi\mathbf{i}\hbar = E\psi\,. \tag{1.9}$$

Born added the *physical interpretation* of $\rho = \psi^\dagger\psi$. Finally, Pauli introduced the electron spin into "wave mechanics." In the present formulation, this appears in the identification of

$$\rho S = \tfrac{1}{2}\hbar\psi\mathbf{i}\psi^\dagger = \rho\,\tfrac{1}{2}\hbar U i\boldsymbol{\sigma}_3 U^\dagger\,. \tag{1.10}$$

with *spin density.*

The bivector S is here called the *local spin.* The spin is more commonly represented by a vector $\mathbf{s}$ dual to S. Thus,

$$S = i\mathbf{s}\,, \tag{1.11}$$

so, from (1.10),

$$\mathbf{s} = \tfrac{1}{2}\hbar U\boldsymbol{\sigma}_3 U^\dagger\,. \tag{1.12}$$

Since angular momentum is fundamentally a bivector quantity, S may be considered more fundamental than $\mathbf{s}$. However, both quantities are used here; the relation (1.11) makes it a nearly trivial task to interchange them.

Most applications of the Pauli theory hardly go beyond the use of Schrödinger's prescription to find energy eigenvalues of the Pauli equation. They only begin to explore the physical content of the theory. But the Pauli theory as given above is incomplete; general expressions for energy and momentum density are still needed. A straight-forward generalization of (1.9) leads to the following expression for the *total energy density:*

$$\rho E = \hbar(\partial_t\psi\mathbf{i}\psi^\dagger)_S = \tfrac{1}{2}\hbar(\partial_t\psi\mathbf{i}\psi^\dagger - \psi\mathbf{i}\partial_t\psi^\dagger)\,. \tag{1.13}$$

So the election has a total energy or an average, depending on interpretation, of

$$\langle\, E\,\rangle = \int d^3x\,\rho E\,. \tag{1.14}$$

Replacement of the time derivatives in (1.13) by space derivatives leads to the corresponding components of the *total* (or *canonical*) *momentum density:*

$$\rho P_k = -\hbar(\partial_k\psi\mathbf{i}\psi^\dagger)_S = -\tfrac{1}{2}\hbar(\partial_k\psi\mathbf{i}\psi^\dagger - \psi\mathbf{i}\partial_k\psi^\dagger)\,. \tag{1.15}$$

The consistency of (1.13) with (1.15) is obvious in the Dirac theory where they are seen to be components of a single stress-energy tensor. The difference in sign is accounted for by the metric of space-time, but it can also be seen to be correct in the analysis to follow.

The total momentum density $\rho\mathbf{P} = \rho P_k\boldsymbol{\sigma}_k$ can be decomposed into a kinetic momentum density $\rho\mathbf{p}$ and a potential momentum density $\rho e\mathbf{A}/c$, that is

$$P_k\boldsymbol{\sigma}_k = \mathbf{P} = \mathbf{p} + (e/c)\mathbf{A}\,. \tag{1.16}$$

This decomposition corresponds exactly to the separation of total energy into kinetic energy plus potential energy.

As is shown in subsequent sections, the entire physical content of the Pauli theory can be expressed in terms of the *local energy* E, the *local kinetic momentum* $\mathbf{p}$, and the *local spin* $\mathbf{s}$. The Schrödinger theory can be obtained explicitly as a special case of the Pauli theory. First note that

$$\nabla\mathbf{A} = \nabla\cdot\mathbf{A} + \nabla\wedge\mathbf{A} = \nabla\cdot\mathbf{A} + i\,\mathbf{B}\,, \tag{1.17}$$

where, as usual, $\mathbf{B} = \nabla\times\mathbf{A}$ is the external magnetic field. So on the right side of (1.8), one can distinguish the term

$$(e\hbar/2mc)(\nabla\wedge\mathbf{A})\psi\mathbf{i} = (e\hbar/2mc)\mathbf{B}\,\psi\,\boldsymbol{\sigma}_3 = -(e/mc)\mathbf{B}\,\mathbf{s}\,\psi\,. \tag{1.18}$$

Now note that if this term can be neglected, and if $\mathbf{i}$ commutes with ψ, then (1.8) is identical to the Schrödinger equation. Moreover, by (1.10),

$$S = \tfrac{1}{2}\hbar U\mathbf{i}U^\dagger = \tfrac{1}{2}\mathbf{i}\hbar\,. \tag{1.19}$$

The work in subsequent sections clearly shows that neglect of the term (1.18) in the Pauli equation is a sufficient, though not a necessary condition for constancy of the local spin expressed in (1.19). Therefore in the absence of a magnetic field the Schrödinger theory is identical to the Pauli theory, and the constant "imaginary factor" $\mathbf{i}\hbar$ is exactly twice the spin.

2. Spinning Frames

The unitary spinor field $U = U(\mathbf{x},t)$ determines a frame field of orthonormal vectors $\mathbf{e}_k$ by the equations

$$\mathbf{e}_k = U\boldsymbol{\sigma}_3 U^\dagger \qquad (k = 1,2,3)\,. \tag{2.1}$$

Comparison with (1.12) shows that the vector $\mathbf{e}_3 = \hat{\mathbf{s}}$ is just the direction of the local spin vector. By themselves, the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ do not have any ordinary physical interpretation. However, their product is the direction of the local spin bivector

$$\mathbf{e}_1\mathbf{e}_2 = U\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 U^\dagger = \hat{S} = i\hat{s} = i\mathbf{e}_3\,. \tag{2.2}$$

At any given point $\mathbf{x}$ of space, the frame $\{\mathbf{e}_k\}$ rotates with an angular velocity Ω_t; thus, by (2.1),

$$\partial_t\mathbf{e}_k = \Omega_t\cdot\mathbf{e}_k \tag{2.3}$$

where

$$\Omega_t = 2(\partial_t U)U^\dagger = -2U(\partial_t U^\dagger)\,. \tag{2.4}$$

In the same way it follows that

$$\partial_t S = \tfrac{1}{2}[\Omega_t, S] = \tfrac{1}{2}(\Omega_t S - S\,\Omega_t)\,. \tag{2.5}$$

Now observe from (1.13) that, with the help of (2.4), the local energy can be written in the following forms

$$E = \hbar(\partial_t U\mathbf{i}U^\dagger)_S = \Omega_t\cdot S = \tfrac{1}{2}(\Omega_t S + S\Omega_t)\,. \tag{2.6}$$

By adding (2.5) and (2.6), one finds an expression for Ω_t in terms of observables:

$$\Omega_t = (E + \partial_t S)S^{-1} = S^{-1}(E - \partial_t S)\,. \tag{2.7}$$

The term $(\partial_t S)S^{-1} = (\partial_t \hat{S})\hat{S}$ in (2.7) describes the rate at which the spin plane changes direction. The other term shows that the magnitude of the angular velocity in the spin plane is precisely $E/|\,S\,| = 2E/\hbar$. In this way the local energy can be given a simple geometric interpretation. No attempt will be made here to ascribe some physical significance to this feature of the theory. Any such attempt win have to go beyond the usual interpretation of quantum mechanics.

The local momentum is related to displacements in space in a manner entirely analogous to the relation of energy to displacements in time just discussed. Thus,

$$\boldsymbol{\sigma}_j \cdot \nabla \mathbf{e}_k = \partial_j \mathbf{e}_k = \Omega_j \cdot \mathbf{e}_k\,, \tag{2.8}$$

$$\Omega_j = 2(\partial_j U)U^\dagger = -2U(\partial_j U^\dagger)\,, \tag{2.9}$$

$$\partial_j S = \tfrac{1}{2}[\Omega_j, S]\,, \tag{2.10}$$

$$P_j = -\hbar(\partial_j U \mathbf{i} U^\dagger)_S = -\Omega_j \cdot S\,, \tag{2.11}$$

$$\Omega_j = (\partial_j S - P_j)S^{-1} = -S^{-1}(\partial_j S + P_j)\,. \tag{2.12}$$

The Pauli theory allows exceptions to the above relations at nodes of the wave function (i.e., where $\rho = 0$); for at a nodal point it is possible for the derivative of ψ in some direction to be finite even though the derivative of U is singular. However, finite results can be obtained from any expression by multiplying by all appropriate density factor.

3. Compatibility Conditions

It has been shown that the local dynamic quantities energy, momentum, and spin are all determined by the unitary spinor U and its derivatives. As Takabayasi[2] first pointed out, it follows that the derivatives of these dynamical quantities are not mutually independent, but are related by certain "compatibility conditions." These conditions are derived below.

Write (2.9) in the form

$$\partial_j U = \tfrac{1}{2}\Omega_j U\,. \tag{3.1}$$

Differentiate, to get

$$\partial_k \partial_j U = \tfrac{1}{2}(\partial_k \Omega_j + \tfrac{1}{2}\Omega_j \Omega_k)U\,. \tag{3.2}$$

But

$$\partial_k \partial_j U = \partial_j \partial_k U\,; \tag{3.3}$$

so

$$\partial_j \Omega_k - \partial_k \Omega_j = \tfrac{1}{2}[\Omega_j, \Omega_k]\,. \tag{3.4}$$

Thus, the derivatives of the angular velocities are not mutually independent. These "compatibility conditions" can be expressed in terms of observables by using (2.12). Thus,

$$\partial_j \Omega_k = (\partial_j \partial_k S - \partial_j P_k)S^{-1} + (\partial_k S - P_k)(\partial_j S)S^{-2}\,; \tag{3.5}$$

so

$$\partial_j \Omega_k - \partial_k \Omega_k = (\partial_k P_j - \partial_j P_k)S^{-1} + \big\{[\partial_k S, \partial_j S] + P_j \partial_k S - P_k \partial_j S\big\}S^{-2}\,; \tag{3.6}$$

Also

$$\Omega_j \Omega_k = (P_j P_k - \partial_j S \partial_k S + P_j \partial_k S - P_k \partial_j S)S^{-2}\,; \tag{3.7}$$

so

$$\tfrac{1}{2}[\Omega_j, \Omega_k] = \big\{\tfrac{1}{2}[\partial_k S, \partial_j S] + P_j \partial_k S - P_k \partial_j S\big\}S^{-2}\,. \tag{3.8}$$

Substitution of (3.6) and (3.8) into (3.4) yields

$$(\partial_k P_j - \partial_j P_k) = \tfrac{1}{2}[\partial_j S, \partial_k S]S^{-1} = (S^{-1}\partial_j S \partial_k S)_S\,, \tag{3.9}$$

or

$$(\partial_k P_j - \partial_j P_k) S = \tfrac{1}{2}[\,\partial_j S, \partial_k S\,]\,. \tag{3.10}$$

Equation (3.9) shows that the curl of the local canonical momentum is determined entirely by the local spin. This important result can be re-expressed in coordinate-free form as follows:

$$\begin{aligned}
\boldsymbol{\sigma}_k\boldsymbol{\sigma}_j(\partial_k P_j - \partial_j P_k)S &= 2(\nabla \wedge \mathbf{P})S \\
&= \tfrac{1}{2}\boldsymbol{\sigma}_k\boldsymbol{\sigma}_j[\,\partial_j S, \partial_k S\,] \\
&= \boldsymbol{\sigma}_k\boldsymbol{\sigma}_j \partial_k \mathbf{s} \wedge \partial_j \mathbf{s} \\
&= \boldsymbol{\sigma}_k \wedge \boldsymbol{\sigma}_j \partial_k \mathbf{s} \partial_j \mathbf{s} \\
&= (\boldsymbol{\sigma}_k \wedge \boldsymbol{\sigma}_j \wedge \partial_k \mathbf{s} + \partial_k s_j \boldsymbol{\sigma}_k - \partial_k s_k \boldsymbol{\sigma}_j)\partial_j \mathbf{s} \\
&= -(\nabla \wedge \mathbf{s}) \wedge \nabla \mathbf{s} + [\nabla(\mathbf{s}\cdot\nabla)]\mathbf{s} - (\nabla\cdot\mathbf{s})\nabla\mathbf{s}\,.
\end{aligned}$$

Employing the identities

$$(\nabla \mathbf{s}\cdot\nabla)\mathbf{s} - (\nabla\wedge\mathbf{s})\nabla\cdot\mathbf{s} = -(\nabla\wedge\mathbf{s})\wedge\nabla\mathbf{s} - \mathbf{s}\cdot(\nabla^2\mathbf{s}) + (\nabla\wedge\mathbf{s})^2$$

and

$$\mathbf{s}\cdot(\nabla^2\mathbf{s}) + (\nabla\cdot\mathbf{s})^2 = -\nabla\cdot[\mathbf{s}\cdot\nabla\mathbf{s} = -(\nabla\cdot\mathbf{s})\mathbf{s}] + (\nabla\wedge\mathbf{s})^2$$

one obtains

$$\begin{aligned}
&= -2(\nabla\wedge\mathbf{s})\wedge\nabla\mathbf{s} - \mathbf{s}\cdot(\nabla^2\mathbf{s}) - (\nabla\cdot\mathbf{s})^2 + (\nabla\wedge\mathbf{s})^2 \\
&= -2i(\nabla\times\mathbf{s})\cdot\nabla\mathbf{s} + \nabla\cdot[\mathbf{s}\cdot\nabla\mathbf{s} - (\nabla\cdot\mathbf{s})\mathbf{s}] \\
&= 2(\nabla\cdot\mathbf{s})\cdot\nabla S - \nabla\cdot(S\wedge\nabla S - S\nabla\wedge S)\,.
\end{aligned}$$

So

$$(\nabla\wedge\mathbf{P})S = (\nabla\cdot S)\cdot\nabla S - \tfrac{1}{2}\nabla\cdot(S\wedge\nabla S - S\nabla\wedge S)\,, \tag{3.11}$$

or

$$\begin{aligned}
(\nabla\times\mathbf{P})\mathbf{s} &= (\nabla\wedge S)\wedge\nabla\mathbf{s} + \tfrac{1}{2}\mathbf{s}\cdot(\nabla^2\mathbf{s}) + \tfrac{1}{2}(\nabla\cdot\mathbf{s})^2 - \tfrac{1}{2}(\nabla\wedge\mathbf{s})^2 \\
&= (\nabla\times S)\cdot\nabla i\mathbf{s} - \tfrac{1}{2}\nabla\cdot(\mathbf{s}\cdot\nabla\mathbf{s} - \mathbf{s}\nabla\cdot\mathbf{s})\,.
\end{aligned} \tag{3.12}$$

In addition to (3.4), there are the following compatibility conditions involving time derivatives:

$$\partial_t\Omega_k - \partial_k\Omega_t = \tfrac{1}{2}[\,\Omega_t, \Omega_k\,]\,. \tag{3.13}$$

Using (2.7) and (2.12) to express this in terms of observables, one gets

$$\begin{aligned}
\partial_t P_k + \partial_k E &= \tfrac{1}{2}[\partial_k S, \partial_t S]S^{-1} = (S^{-1}\partial_k S\partial_t S)_S \\
&= [(\partial_t S)S^{-1}]\cdot\partial_k S = \Omega_t\cdot\partial_k S\,.
\end{aligned} \tag{3.14}$$

4. Physical Content of the Pauli Equation

It has been shown that certain relations among observables are already implied by their definitions in terms of a spinor. Further relations are implied by the Pauli equation. The exact nature of these relations can be determined by reexpressing the Pauli equation in terms of local observables. To accomplish this, multiply (1.8) on the right by ψ^{-1} and consider the various terms separately.

First, use (1.1) and (1.2)

$$\begin{aligned}
\partial_t\psi\mathbf{i}\hbar\psi^{-1} &= (\partial_t\psi)U^\dagger U\mathbf{i}U^\dagger\rho^{-1/2}\hbar = 2(\partial_t\psi)\psi^{-1}S \\
&= [\partial_t\ln\rho + 2(\partial_t U)U^\dagger]S\,.
\end{aligned}$$

So, by (2.4)

$$\partial_t\psi\mathbf{i}\hbar\psi^{-1} = 2(\partial_t\psi)\psi^{-1}S = (\partial_t \ln\rho + \Omega_t)S\,. \tag{4.1}$$

Second, use (2.9) and (2.12) to show

$$-\partial_k\psi\mathbf{i}\hbar = (P_k - W_k)\psi\,, \tag{4.2}$$

with the new quantity W_k defined by

$$W_k = \rho^{-1}\partial_k(\rho S)\,. \tag{4.3}$$

Differentiate (4.2) once again to get

$$-\hbar^2(\partial_j\partial_k\psi)\psi^{-1} = -2(\partial_j P_k - \partial_j W_k)S + P_k P_j - (P_k W_j + W_k P_j) + W_k W_j\,.$$

Hence

$$-\hbar^2(\nabla^2\psi)\psi^{-1} = -2(\nabla\cdot\mathbf{P} + \partial_k W_k)S + \mathbf{P}^2 - 2P_k W_k + \sum_k W_k^2\,. \tag{4.4}$$

Now we used (4.2) to get

$$(\mathbf{A}\cdot\nabla\psi\mathbf{i}\hbar)\psi^{-1} = -\mathbf{A}\cdot\mathbf{P} + A_k W_k\,. \tag{4.5}$$

So, with the help of (4.4), (4.5), and (1.16),

$$\begin{aligned}
&[(-\hbar^2\nabla^2 + (e^2/c^2)A^2\psi + 2(e\hbar/c)\mathbf{A}\cdot\nabla\psi\mathbf{i} + (e\hbar/c)(\nabla\cdot\mathbf{A})\psi\mathbf{i}]\psi^{-1}\\
&\qquad = 2\big\{-\nabla\cdot[\mathbf{P} - (e/c)\mathbf{A}] + \partial_k W_k\big\} + [\mathbf{P} - (e/c)\mathbf{A}]^2 - 2[P_k - (e/c)A_k]W_k + \sum_k W_k^2\\
&\qquad = 2(-\nabla\cdot\mathbf{p} + \partial_k W_k)S + \mathbf{p}^2 - 2p_k W_k + \sum_k W_k^2
\end{aligned} \tag{4.6}$$

Finally, use (4.1), (4.5), and (4.6) to write the Pauli equation (1.8) in the form

$$(\partial_t \ln\,\rho + \Omega_t)S = (e/mc)i\mathbf{B}S + e\phi + (2m)^{-1}[2(-\nabla\cdot\mathbf{p} + \partial_k W_k)S + \mathbf{p}^2 - 2p_k W_k + \sum_k W_k^2\,]\,, \tag{4.7}$$

or after multiplying by S^{-1} and rearranging,

$$\begin{aligned}
\partial_t \ln\,\rho + \Omega_t &= -\nabla\cdot(\mathbf{p}/m) - (\mathbf{p}/m)\cdot\nabla\ln\,\rho\\
&\quad + \big\{-(\mathbf{p}/m)\cdot\nabla S + (\mathbf{p}^2/2m) + (2m)^{-1}[2(\partial_k W_k)S + \sum_k W_k^2\,] + e\phi\big\}\times S^{-1} + (e/mc)i\mathbf{B}\,.
\end{aligned} \tag{4.8}$$

The scalar part of (4.8) is just the local conservation law for ρ:

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{p}/m) = 0\,. \tag{4.9}$$

The bivector part of (4.8) is

$$\Omega_t = \big\{-(\mathbf{p}/m)\cdot\nabla S + (\mathbf{p}^2/2m) + (2m)^{-1}[2(\partial_k W_k)S + \sum_k W_k^2\,] + e\phi\big\}\times S^{-1} + (e/mc)i\mathbf{B}\,. \tag{4.10}$$

Since $\Omega_t = (E + \partial_t S)S^{-1}$, (4.10) gives

$$\begin{aligned}
E &= (\mathbf{p}^2/2m) + (2m)^{-1}[2(\partial_k W_k)\cdot S + \sum_k W_k^2\,] + e\phi + (e/mc)(i\mathbf{B})\cdot S\\
&= (\mathbf{p}^2/2m) + (2m)^{-1}\big\{S^2[2(\nabla^2\ln\,\rho + (\nabla\ln\,\rho)^2] + S\cdot(\nabla^2 S)\big\} + e\phi + (e/mc)(i\mathbf{B})\cdot S
\end{aligned} \tag{4.11}$$

and

$$[\partial_t + (\mathbf{p}/m)\cdot\nabla]S = \tfrac{1}{2}[m^{-1}\partial_k W_k + (e/mc)i\mathbf{B}, S\,]\,. \tag{4.12}$$

When re-expressed in terms of the local spin vector, (4.12) becomes

$$[\partial_t + (\mathbf{p}/m)\cdot\nabla]\mathbf{s} = [m^{-1}\partial_k W_k + (e/mc)i\mathbf{B}]\cdot\mathbf{s} = (e/mc)\mathbf{s}\times\mathbf{B} + m^{-1}\mathbf{s}\times[\nabla^2\mathbf{s} + (\nabla\ln\,\rho)\cdot\nabla\mathbf{s}\,]\,. \tag{4.13}$$

The above derivation shows that Eqs. (4.9), (4.11), and (4.12) express the full physical content of the Pauli equation in terms of local observables. This physical content consist of the local conservation law (4.9) required to uphold the interpretation of ρ together with the "constitutive equation" (4.10) which express Ω_t as a specific function of observables. It is worth reiterating that Planck's constant appears in these equations only in the magnitude of the spin. Even in the "Schrödinger limit," when the local spin is constant and the magnetic interaction is negligible, (4.11) shows that the spin appears in the energy through the factor $S^2 = -s^2 = -\hbar^2/4$.

5. Local Conservation Laws

Since E and $(\partial_t S)S^{-1}$ are known from the Pauli equation gives the momentum conservation law immediately. The problem is only to write the equation in its most perspicuous form. First write (3.14) as a vector equation:

$$\partial_k \mathbf{P} + \nabla E = \boldsymbol{\sigma}_k \Omega_t \cdot \partial_k S \,. \tag{5.1}$$

From (4.10), remembering $S \cdot \partial_k S = 0$,

$$\boldsymbol{\sigma}_k \Omega_t \cdot \partial_k S = \boldsymbol{\sigma}_k [(-(\mathbf{p}/m) \cdot \nabla S) S^{-1} + m^{-1} \partial_j W_j + (e/mc) i\mathbf{B}] \cdot \partial_k S \,. \tag{5.2}$$

By utilizing (3.9), the first term on the right of (5.2) can be written

$$\begin{aligned}
\boldsymbol{\sigma}_k (\mathbf{p}/m)(S^{-1}\partial_j S \partial_k S) &= \boldsymbol{\sigma}_k (\mathbf{p}_j/m)(\partial_k p_j - \partial_j p_k) \\
&= -(\mathbf{p}/m) \cdot (\nabla \wedge \mathbf{P}) \\
&= -(\mathbf{p}/m) \cdot [\nabla \wedge \mathbf{p} + (e/c)\nabla \wedge \mathbf{A}] \\
&= -(\mathbf{p}/m) \cdot \nabla \mathbf{p} + \nabla(\mathbf{p}^2/2m) + (e/mc)\mathbf{p} \times \mathbf{B} \,.
\end{aligned} \tag{5.3}$$

By using (5.2), (5.3), (4.11), and recalling $\mathbf{E} = -\nabla\phi - c^{-1}\partial_t \mathbf{A}$, after a simple regrouping of terms (5.1) can be written in the form

$$\begin{aligned}
[\partial_t + (\mathbf{p}/m) \cdot \nabla]\mathbf{p} = e[\mathbf{E} + (\mathbf{p}/m) \times \mathbf{B}] - (e/m)\boldsymbol{\sigma}_k S \cdot (\partial_k i\mathbf{B}) \\
- m^{-1}[\boldsymbol{\sigma}_k (\partial_k \partial_j W_j) \cdot S + \tfrac{1}{2}\nabla \sum_j W_j^2 \,] \,.
\end{aligned} \tag{5.4}$$

The last term in (5.4) can be simplified by noting the identity

$$\rho^{-1}\partial_j(\rho S \cdot \partial_k W_j) = S \cdot (\partial_k \partial_j W_j) + \tfrac{1}{2}\partial_k \sum_j W_j^2 \,. \tag{5.5}$$

Now introduce the notation

$$\begin{aligned}
T_{kj} &= -m^{-1}\rho S \cdot \partial_j W_k = m^{-1}\rho \mathbf{s} \cdot \partial_j [\rho^{-1}\partial_k(\rho \mathbf{s})] \\
&= -m^{-1}\rho S \cdot (\partial_j \partial_k S + S \partial_j \partial_k \ln\, \rho) = T_{jk} \,.
\end{aligned} \tag{5.6}$$

Also write

$$\mathbf{T}_k = \boldsymbol{\sigma}_j T_{kj} \tag{5.7}$$

and

$$\mathbf{f} = e[\mathbf{E} + (\mathbf{p}/m) \times \mathbf{B}] - (e/mc)\boldsymbol{\sigma}_k S \cdot (\partial_k i\mathbf{B}) \,. \tag{5.8}$$

So, at]ast, (5.4) can be written

$$\rho D_t \mathbf{p} \equiv \rho[\partial_t + (\mathbf{p}/m) \cdot \nabla]\mathbf{p} = \rho \mathbf{f} + \partial_k \mathbf{T}_k \,. \tag{5.9}$$

The physical significance of the various terms in (5.9) are readily identified. The left side of (5.9) describes the time rate of change of momentum in a volume $\mathcal{V}_p$ moving along a "momentum streamline" determined by the continuity equation (4.9). If $\mathbf{n} = n_k\boldsymbol{\sigma}_k$ is the unit outward normal to the boundary $\partial\mathcal{V}_p$ of $\mathcal{V}_p$, then $\mathbf{T} = T_k\boldsymbol{\sigma}_k$ is the momentum flux through that boundary. The T_{jk} are components of the corresponding stress tensor. The term $\rho\mathbf{f}$ is the external (body) force on the volume element. The last term on the right of (5.8) is the so-called Stern-Gerlach force. It can alternatively, be written

$$\begin{aligned} -(e/mc)\boldsymbol{\sigma}_k S \cdot (\partial_k i\mathbf{B}) &= \boldsymbol{\sigma}_k \boldsymbol{\mu} \cdot (\partial_k \mathbf{B}) \\ &= \boldsymbol{\mu} \cdot \nabla\mathbf{B} - \boldsymbol{\mu} \cdot (\nabla \wedge \mathbf{B}) \\ &= \boldsymbol{\mu} \cdot \nabla\mathbf{B} + c^{-1}\boldsymbol{\mu} \times \partial_t \mathbf{E}\,, \end{aligned} \tag{5.10}$$

where the magnetic moment μ is defined by

$$\boldsymbol{\mu} = (e/mc)\mathbf{s}\,. \tag{5.11}$$

The density of total angular momentum is $\rho(\mathbf{x} \wedge \mathbf{p} + S)$. But since, as (5.6) shows, the stress tensor is symmetric, the orbital angular momentum and the spin are separately conserved. In fact, the conservation equation for the spin has already been written down in (4.12). However, (4.12) is more readily interpreted if it is written in the form

$$\rho D_t \mathbf{s} \equiv \rho[\partial_t + (\mathbf{p}/m) \cdot \nabla]S = (e\rho/2mc)[i\mathbf{B}, S] + \partial_k M_k\,, \tag{5.12}$$

where

$$\begin{aligned} M_k &= (2m)^{-1}[W_k, \rho S] = (\rho/m)(\partial_k S)S \\ &= (\rho/m)\mathbf{s} \wedge \partial_k \mathbf{s} = i(\rho/m)\mathbf{s} \times \partial_k \mathbf{s} = iM_k\,. \end{aligned} \tag{5.13}$$

The left side of (5.12) gives the time rate of change of the spin angular momentum in $\mathcal{V}_p$. The first term on the right of (5.12) gives the torque. And $M_n \equiv n_k A_k$ is clearly the spin flux through $\partial\mathcal{V}_p$. The vector form of (5.12) is

$$\rho D_t \mathbf{s} = \rho\boldsymbol{\mu} \times \mathbf{B} + \partial_k M_k\,. \tag{5.14}$$

The momentum flux $\mathbf{T}_k$ and the spin flux $\mathbf{M}_k$ should be classed with local variables along with the momentum density and the spin density. The expressions (5.6) and (5.13) may be regarded as "constitutive equations" which relate T_k and M_k to other observables. Note that these equations vanish with vanishing spin.

6. Charge Current

Aside from any question of experimental test, the theory of local observables just developed may appear to be fairly complete and satisfactory state. For a closed set of local conservation laws for probability, momentum, and angular momentum have been formulated, and an explicit expression has been found for the total energy in terms of other observables. However, the proper identification of local observables is not so straight-forward a matter as the preceding analysis might make it seem. For instance, because of the continuity Eq. (4.9), one might suppose that $\mathbf{p}/m$ is the local flow velocity of probability or charge. Indeed, this supposition is universally made in the Schrödinger theory and usually made in the Pauli theory. However, it is inconsistent with the usual interpretation of the Dirac theory. The nonrelativistic limit of the Dirac theory yields the expression for local observables already given above, but for the local flow velocity $\mathbf{v}$ of probability or charge it gives the expression

$$\rho[m\mathbf{v} - (e/c)\mathbf{A}] = -[(\nabla\psi)\mathbf{i}\hbar\psi^\dagger]_V\,. \tag{6.1}$$

Now $\mathbf{v}$ can be expressed as a function of other observables. By (4.2)

$$-\nabla\psi\mathbf{i}\hbar = [\mathbf{P} - \rho^{-1}\nabla(\rho S)]\psi\,. \tag{6.2}$$

So,

$$m\rho\mathbf{v} = \rho\mathbf{p} - \nabla\cdot(\rho S) = \rho\mathbf{p} + \nabla\times\rho\mathbf{s}\,. \tag{6.3}$$

It is not necessary to appeal to the Dirac theory to introduce $\mathbf{v}$ into the Pauli theory. This velocity enters the Dirac theory by way of an independent physical assumption. It is no less tenable to introduce $\mathbf{v}$ directly into the Pauli theory by (6.3) or (6.1). The problem then is to see if the quantity makes sense on physical grounds. The following observations should bring the issue into sharper focus:

Note that the divergence of (6.3) gives

$$\nabla\cdot(\rho m\mathbf{v}) = \nabla\cdot(\rho\mathbf{p})\,. \tag{6.4}$$

Hence, by (4.9),

$$\partial_t\rho + \nabla\cdot\rho\,\mathbf{v} = 0\,. \tag{6.5}$$

So the suggested interpreted for $\mathbf{v}$ is consistent with conservation of probability. To substantiate the distinction between $\mathbf{v}$ and $\mathbf{p}/m$ physical considerations must be introduced which go beyond anything mentioned i this paper.

It should be observed that the distinction between $\mathbf{v}$ and $\mathbf{p}/m$ persists even in the Schrödinger theory, for when the local spin is constant, (6.3) becomes

$$m\mathbf{v} = \mathbf{p} + S\cdot\nabla\ln\,\rho = \mathbf{p} - \mathbf{s}\times\nabla\ln\,\rho\,. \tag{6.6}$$

This shows that the usual interpretations of observables in the Schrödinger theory are inconsistent with those in the Dirac theory, for the probability current is taken to be $\rho\mathbf{p}/m$ in the one theory and $\rho\mathbf{v}$ in the other. An analysis of the physical significance of this difference will be carried out at another time.

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Appendix A: Matrix Form of the Pauli Theory

To show that the multivector formulation of the Pauli theory is equivalent to the usual matrix formulation, multivectors may be replace by their matrix represnetations according to the prescription given in Sec. 2 of Ref. 1. Then the column spinor ψ is obtained directly from (1.1) by operating on the eigenvector u of the "Pauli matrix" σ_3; thus,

$$\Psi = \psi u\,. \tag{A.1}$$

where

$$\sigma_3 u = u\,, \qquad u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,. \tag{A.2}$$

Also, since $\mathbf{i}\sim i\sigma_3$ and i commutes with all elements of the algebra,

$$\partial_t\psi\mathbf{i}\hbar u = i\hbar\partial_t\psi\,\sigma_3\,u = i\hbar\partial_t\Psi\,. \tag{A.3}$$

Note how $\boldsymbol{\sigma}_3$, which this paper has shown to be so important to the interpretation of the theory, is "swallowed up" by u; it remains in the theory, however, hidden in the choice of matrix representation. By multiplying (1.8) on the right by u and utilizing (1.18) together with the usual (bad) notation $\boldsymbol{\sigma}\cdot\mathbf{B}$ for $\mathbf{B} = B_i\sigma_i$, one obtains

$$i\hbar\,\partial_t\Psi = \frac{1}{2mc}\Big(-\hbar^2\nabla^2 + \frac{2e\hbar}{c}\,\mathbf{A}\cdot\nabla + \frac{e\hbar}{c}\,\nabla\cdot\mathbf{A} + \frac{e^2}{c^2}\,\mathbf{A}^2\Big)\Psi + e\,\phi\Psi - \frac{e\hbar}{2mc}\,\boldsymbol{\sigma}\cdot\mathbf{B}\Psi\,. \tag{A.4}$$

This is precisely the usual matrix form of the Pauli equation.

Correspondence between observables in the formulations is most easily established by introducing the "density matrix":

$$\Psi\Psi^\dagger = \psi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\psi^\dagger = \tfrac{1}{2}\psi(I+\sigma_3)\psi^\dagger\,, \tag{A.5}$$

where

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{A.6}$$

The trace of the density matrix gives the probability density,

$$\Psi^\dagger\Psi = \tfrac{1}{2}\mathrm{Tr}\,\Psi\Psi^\dagger = \tfrac{1}{2}\mathrm{Tr}\,[\psi^\dagger\psi(I+\sigma_3)] = \rho \tag{A.7}$$

since in the matrix theory

$$\psi^\dagger\psi = \psi^\dagger\psi = \rho I\,, \tag{A.8}$$

instead of (1.3). Also observe

$$\Psi^\dagger\sigma_k\Psi = \tfrac{1}{2}\mathrm{Tr}\,(\sigma_k\Psi\Psi^\dagger) = \tfrac{1}{2}\mathrm{Tr}\,(\sigma_k\psi\sigma_3\psi^\dagger) = \rho s_k\,. \tag{A.9}$$

Here the $s_k = (\boldsymbol{\sigma}_k\mathbf{s})_S = \boldsymbol{\sigma}_k\cdot\mathbf{s}$ are just the components of the local spin vector. In a similar manner it can be shown that the expression (1.15) for the canonical momentum density, is equivalent to the "real part" of $\Psi^\dagger i\hbar\partial_k\Psi$.

It can be shown quite generally that the operation $\frac{1}{2}\mathrm{Tr}$ in the matrix theory corresponds to the taking of scalar plus pseudoscalar parts in the multivector theory (see Appendix D of Ref. 6).

Appendix B: Relation to Classical Theory

Schrödinger was led to his "wave equation" by completing an analogy between optics and mechanics in which the Hamilton-Jacobi equation was taken as the mechanical analog of the eikonal equation. Schrödinger's reasoning is equally valid if it is rephrased in the language of hydrodynamics. In fact, the hydrodynamic description may well be more apt than the optical analogy because, if the Born interpretation of $\rho=\psi^\dagger\psi$ is strictly adhered to, in the limit of vanishing spin the Pauli theory goes directly into the classical theory with *no change interpretation.* But before this can be demonstrated, the relation of Hamilton-Jacobi theory to hydrodynamics must be understood. The Hamilton-Jacobi equation for a point charge (electron) can be written

$$-\partial_t W = (2m)^{-1}[\nabla W - (e/c)\mathbf{A}]^2 + e\phi\,. \tag{B.1}$$

By itself, this equation is insufficient to describe the motion of the charge. It must be supplemented by an equation relating the "action" function W to mechanical quantities. The appropriate equation is

$$\nabla W = m\mathbf{v} + (e/c)\mathbf{A}\,. \tag{B.2}$$

With this ∇W be eliminated from (B.1) to yield

$$-\partial_t W = \tfrac{1}{2}m\mathbf{v}^2 + e\phi\,. \tag{B.3}$$

Equations (B.2) and (B.3) are more fundamental than (B.1), being expressions for the total energy and momentum of the particle. But problems are solved by using (B.1) to find W.

Now it is possible to interpret the energy-momentum equations (B.2) and (B.3) as governing the motion of an ideal fluid. By "ideal," it is meant here that the particles (or elements) of the fluid do not interact. Solution of the equations yields a family of trajectories, the streamlines of the fluid. Each streamline is the trajectory of a single particle, identical, as shown below, to a trajectory more commonly obtained from the Lorentz force.

To the energy-momentum equations, fluid dynamics adds the equation of continuity,

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0\,. \tag{B.4}$$

If ρ is the relative particle density, then at all times

$$\int d^3x\,\rho = 1\,, \tag{B.5}$$

and the total mass of the fluid is m. The total charge of the fluid is e, for Eqs. (B.2) and (B.3) entail that the fluid density has constant charge-to-mass ratio e/m. Equations (B.4) and (B.5) can also be applied to the fluid dynamics opf a single point particle, if rho is interpreted as position probability density.

The energy-momentum equations (B.2) and (B.3) differ from the usual equations of fluid dynamics in not depending on the density of the fluid. Therefore they can be solved for particle trajectories without reference to the equation of continuity. This difference is, of course, due to the fact that for a real fluid density dependence of the energy arises from interparticle interactions. The ideal fluid envisaged here describes, not the motion of a system of interacting particles, but a family of possible trajectories for a single particle.

Electromagnetic field are usually defined by the accelerations they induce on idealized "test charges" (the electromagnetic field produced by a test charge being assumed negligible). Alternatively, an electromagnetic field can be described by its effect on an "ideal test fluid." The instantaneous motion of a single charge is sufficient only to determine the electric field at a point. To determine also the magnetic field at a point, the independent motions of at least three charges must be given. To determine the fields throughout a region, the motions of an infinite number of test charges must be known. This idealized circumstance is most simply described with a "test fluid." By taking a curl of (B.2), one finds that the vorticity of the fluid is a direct measure of the magnetic field. Thus,

$$\nabla \times \mathbf{v} = (e/mc)\mathbf{B}\,, \tag{B.6}$$

where, of course, $\mathbf{B} = \nabla \times \mathbf{A}$.

The fluid viewpoint subsumes the particle viewpoint. This may be seen by taking the time in the form derivative of (B.2) and the gradient of (B.3), then eliminating derivatives of the "energy-momentum potential," W, to obtain

$$m(d\mathbf{v}/dt) = m(\partial_t + \mathbf{v}\cdot\nabla)\,\mathbf{v} = e(\mathbf{E} + c^{-1}\mathbf{v}\times\mathbf{B})\,. \tag{B.7}$$

Here, as usual, $\mathbf{E} = -c^{-1}\partial_t\mathbf{A} - \nabla\phi$. Thus, Euler's equation (B.7) for the streamlines of the fluid is the same as the equation for the trajectory of a point charge subject to the Lorentz force.

Though not needed in this paper, it is interesting to note in passing that the fluid viewpoint provides a direct physical interpretation of the vector potential $\mathbf{A}$ if equation (B.2) is interpreted as a gauge transformation. Thus, any particular solution of the wave equation

$$\Box^2\mathbf{A} = c^{-1}\mathbf{j} \tag{B.8}$$

determines a field $-(e/mc)\mathbf{A}$ which may be regarded, at each point, as the velocity of a test particle with charge e and mass m under the influence of the source $\mathbf{j}$. So the vector potential can be interpreted quite directly as the velocity field of a homogeneous fluid of test charges with unit charge and mass. The velocity field thus obtained is not unique, because the fluid of test charges can be introduced with a variety of initial conditions. But from any particular solution, any other solution can be obtained by the gauge transformation (B.2).

Now write the unitary spinor U defined in Sec. 1 in the form

$$U = U_0 \exp{(iW/\hbar)}\,. \tag{B.9}$$

If U_0 is slowly varying this gives a kind of "eikonal approximation" to the Pauli theory. If U_0 is supposed constant, then the local spin is

$$\hat{S} = U\mathbf{i}U^\dagger = U_0\mathbf{i}U_0^\dagger\,. \tag{B.10}$$

Furthermore, (2.8) gives

$$E = \hbar(\partial_t U\mathbf{i}U^\dagger)_S = -\partial_t W \tag{B.11}$$

and (2.11) gives

$$P_j = -\hbar(\partial_j U\mathbf{i}U^\dagger)_S = -\partial_j W\,. \tag{B.12}$$

This, of course, is just the "Schrödinger approximation" to the Pauli theory.

If now the magnitude of the spin is sufficiently small (i.e., in the limit $\hbar \to 0$), then $\mathbf{p}/m = \mathbf{v}$ and, by virtue of (B.11) and (B.12), Eq. (4.11) is equivalent to (B.3), (1.16) is equivalent to (B.2), (5.4) is equivalent to (B.7), and, of course, (4.9) agrees with (B.4) both in form and interpretation. In this way the classical theory of a “test charge” is obtained as a limiting case of the Pauli theory. There are important questions of interpretation in connection with this procedure, but they will not be broached here.

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