Modeling Elastically Coupled Rigid Bodies with Geometric Algebra

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Abstract. This work is an introduction to a new method for modeling elastically coupled rigid bodies, with application to modeling and computer simulation of spatial, flexural mechanisms. Geometric algebra is used in an essential way to provide a coordinate-free approach to Euclidean geometry and rigid body mechanics that fully integrates rotational and translational dynamics. Finite, elastic displacements of rigid bodies are associated naturally with screw displacements that are simply represented as bivectors in geometric algebra.

The potential energy of an arbitrary elastic coupling in internal equilibrium has been investigated previously using dual-number, matrix methods. Geometric algebra provides an invariant formulation that is less ambiguous, easier to interpret geometrically, and manifestly more efficient in symbolic computation.

1. Introduction

Modeling elastically coupled rigid bodies is an important problem in multibody dynamics. This work concerns the problem of modeling what can be called flexural joints, where two essentially rigid bodies are coupled by a substantially more elastic body. Such an idealized system is shown in Fig. 1. The geometry of the elastic body is not important, even though the geometry depicted is that of an axisymmetric beam.

This work sets the stage for modeling elastically coupled rigid bodies with a new *homogeneous* formulation for mechanics introduced by Hestenes (2001). The term "homogeneous" refers to the fact that all points of physical space are treated equally, without designating any one of them as the origin of a coordinate system. The method can be regarded as refining the classical idea of homogeneous coordinates. The homogeneous formulation is achieved by using geometric algebra in an essential way. Geometric points are represented by null vectors in a 5-dimensional metric vector space with Minkowski signature (4,1). The geometric algebra generated by this vector space provides the essential mathematical apparatus for a completely coordinate-free treatment of finite rigid displacements and motions that fully integrates rotations and translations. The notion of a *twist* introduced in screw theory to describe a coupled rotation/translation is simply represented as a bivector variable, and the Lie algebra of twists emerges automatically as a bivector algebra. The elastic strain potential energy function can then be expressed as a function of the twist variable. However, we shall see that geometric algebra suggests a better choice for dynamical variable.

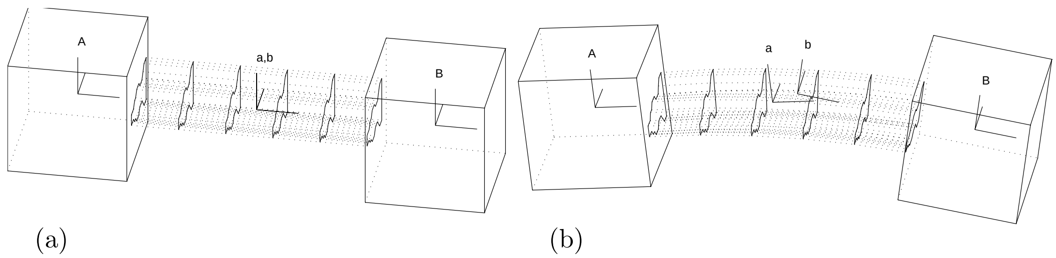

Figure 1. Elastically coupled rigid bodies shown in (a) Undeformed, relaxed configuration and (b) Deformed, strained configuration. In this case two bodies are coupled by a compliant strut.

For an in-depth review of prior work on elastically coupled rigid bodies, readers are referred to (Fasse 2000; Fasse 2001). Briefly, the geometry of elastically coupled rigid bodies has been extensively reported in the literature. Most of the literature is on (1) the analysis of linear elastic properties (stiffness and compliance) and (2) the synthesis of arbitrary linear elastic properties using combinations of simple compliant elements. Regarding geometrical methods, much of the relevant prior work has used screw theory and matricies. Readers are directed to such works as (Griffis and Duffy 1991; Patterson and Lipkin 1993; Lončarić 1987; Huang and Schimmels 1998; Ciblak and Lipkin 1998; Maschke 1996; Žefran and Kumar 1999; Fasse and Breedveld 1998). Happily, these methods and results are easily translated into geometric algebra where they can be evaluated and often improved. One purpose of this paper is to provide a guide for such translations.

Most of the paper is devoted to elaborating the homogeneous method as a foundation for practical applications. It introduces fundamentally new ways to formulate and analyze Euclidean geometry, rigid body displacements, kinematics and dynamics. A general approach to elastic interactions is developed with reference to practical applications to modeling and computer simulation of complex flexural mechanisms. The reader is presumed to be familiar with the basics of geometric algebra presented in the references and elsewhere in this book.

2. Homogeneous Euclidean Geometry

Newtonian physics presumes that physical objects are composed of particles whose motions can be represented as curves in a 3-dimensional *Euclidean space* $\mathcal{E}^3$. This has been called the *Zeroth Law* of Newtonian Theory (Hestenes 1986, 1992), because Newton's other Laws are ill-defined without it. The "standard model" for $\mathcal{E}^3$ in classical physics and engineering is the real 3-dimensional vector space $\mathcal{R}^3 = \mathcal{R}^{3,0}$ with *Euclidean signature* (3,0), as expressed by the isomorphism

$$\mathcal{E}^3 \cong \mathcal{R}^3, \tag{1}$$

where each *point* x in $\mathcal{E}^3$ corresponds to a unique *vector* $\mathbf{x}$ in $\mathcal{R}^3$. An advantage of this correspondence is that the geometric product defined on $\mathcal{R}^3$ generates a geometric algebra $\mathcal{R}_{3,0} = \mathcal{G}(\mathcal{R}^{3,0})$. Geometric Algebra greatly facilitates the description of geometric objects and inferences therefrom. An extensive account of Newtonian physics in the language of $\mathcal{R}_3$ is given in (Hestenes 1986).

A drawback of the "vector space model" $\mathcal{R}^3$ for $\mathcal{E}^3$ is that it singles out a particular point, say e_0, to be designated as *the origin* and represented by the zero vector $\mathbf{0}$. Consequently, in the analysis of mathematical models it is often necessary to shift the origin to simplify calculations, to avoid dividing by zero, or to prove that results are independent of the origin choice. We avoid this drawback and find many surprising benefits by employing the homogeneous model for $\mathcal{E}^3$ introduced in (Hestenes 2001).

As a foundation for modeling mechanical systems, we need to formulate the rudiments of Euclidean geometry in terms of the homogeneous model. We begin with the *Minkowski vector space* $\mathcal{R}^{4,1}$ and its geometric algebra $\mathcal{R}_{4,1} = \mathcal{G}(\mathcal{R}^{4,1})$. The reference to Minkowski recalls the similarity of the signature (4,1) to the signature (3,1) introduced by Minkowski in his original vector space model of spacetime. This brings to light subtle similarities of structure between spacetime geometry and Euclidean geometry, especially as regards the *null cone* (often called the *light cone* in spacetime). However, the physical interpretation is vastly different in the two cases. In our modeling with $\mathcal{R}^{4,1}$ none of the dimensions is associated with time.

A vector x is said to be a *null vector* if $x^2 = x \cdot x = 0$. The set of all null vectors in $\mathcal{R}^{4,1}$ is said to be a *null cone.* Now, it is a remarkable fact that $\mathcal{E}^3$ can be identified with the set of all null vectors in $\mathcal{R}^{4,1}$ satisfying the constraint

$$x \cdot e = 1, \tag{2}$$

where e is a distinguished null vector called the *point at infinity.* This constraint is the equation for a *hyperplane* with normal e. Thus, we identify $\mathcal{E}^3$ with the intersection of a hyperplane and the null cone in $\mathcal{R}^{4,1}$, as expressed by

$$\mathcal{E}^3 = \{x \,|\, x^2 = 0,\ x \cdot e = 1\}, \tag{3}$$

where each x designates a *point* in $\mathcal{E}^3$ and

$$(x - y)^2 = -2x \cdot y \tag{4}$$

is the squared *Euclidean* distance between points x and y. It is not difficult to prove that the triangular inequality and the Pythagorean theorem for Euclidean distances follow as theorems.

The first surprise is that eqn. (4) tells us that Euclidean distances can be computed directly from inner products between points. Of course, this product vanishes if the points are one and the same. The second surprise is that the (oriented) line A through two distinct points a and b is completely characterized by the *line vector*

$$A = a \wedge b \wedge e. \tag{5}$$

This conforms to the classical notion of a line vector or *sliding vector* as an oriented line (or axis) with a magnitude. The simple trivector $a \wedge b \wedge e$ can be regarded as the *moment* of line segment $a \wedge b$ with respect to the point at infinity as well as a continuation of the line through infinity. The *tangent vector* n for the line is

$$n \equiv (a \wedge b) \cdot e = a - b\,. \tag{6}$$

Using

$$(a \wedge b \wedge e)^2 = (a \wedge b) \cdot [e \cdot (a \wedge b \wedge e)] = (a \wedge b) \cdot [\, e \wedge (e \cdot (a \wedge b))\,] = [\,(a \wedge b) \cdot e\,]^2,$$

we see that the *length* of the line segment is given by

$$A^2 = n^2 = (a - b)^2 = -2a \cdot b. \tag{7}$$

It can be shown that the line vector A is equivalent to the classical representation of a line by Plücker coordinates with respect to an origin (Hestenes 2001), but we have no need for that fact.

A point x lies on the line A if and only if

$$x \wedge A = x \wedge a \wedge b \wedge e = 0. \tag{8}$$

This is a non-parametric equation for the line. It can be solved for a parametric representation of the line. Two different ways to do that are worth noting here. First, we can use (8) to write

$$xA = x \cdot A = [x \cdot (a \wedge b)] \wedge e + a \wedge b.$$

Whence

$$x = (x \cdot A)A^{-1} = \frac{[\,(x \cdot a)b \wedge e - (x \cdot b)a \wedge e + a \wedge b\,] \cdot A}{(a-b)^2}, \tag{9}$$

where further reduction of the right side is possible, but this suffices to express x as a function of its distance from points a and b.

Alternatively, we recognize (8) as saying that x is linearly dependent on the three vectors a, b, e, so we can write $x = \alpha a + \beta b + \gamma e$, where α, β, γ are scalar coefficients subject to the following two constraints: $x \cdot e = 1 = \alpha + \beta$, and $x^2 = 0 = 2\gamma e \cdot (\alpha a + \beta b) + 2\alpha\beta a \cdot b$, so $\gamma = -\alpha\beta a \cdot b$. Eliminating redundant parameters, we obtain a one-parameter equation for the line:

$$x = b + \alpha(a-b) + \tfrac{1}{2}\alpha(1-\alpha)(a-b)^2 e. \tag{10}$$

For α in the interval [0, 1], this parametrizes the line segment from b to a.

To relate our homogenous methods to the vast literature on geometry and mechanics, we need to relate our homogeneous model for $\mathcal{E}^3$ to the standard vector space model. Happily, this can be done in a straightforward way with an elegant device called the *conformal split*. The essential idea is to parametrize all the points in Euclidean space by the family (or pencil) of lines through a single point.

The *pencil of lines* through a *fixed point* e_0 can be characterized by the variable line vector

$$\mathbf{x} = x \wedge e_0 \wedge e = x \wedge E. \tag{11}$$

where

$$E = e_0 \wedge e \quad \Longrightarrow \quad E^2 = 1 \tag{12}$$

This determines a unique correspondence between each Euclidean point x and a line segment $\mathbf{x}$. That can be proved by inverting (11) to express x as a function of $\mathbf{x}$ in much the same way that (9) was obtained from (8). Thus,

$$\mathbf{x}E = (x \wedge e_0 \wedge e) \cdot (e_0 \wedge e) = (x \wedge e_0) \cdot e + (x \wedge e_0) \cdot (e_0 \wedge e)e = x - e_0 - (x \cdot e_0)e\,.$$

Since $\mathbf{x}^2 = (x \wedge e_0 \wedge e)^2 = (x - e_0)^2 = -2x \cdot e_0$, we have the desired result

$$x = \mathbf{x}E - \tfrac{1}{2}\mathbf{x}^2 e + e_0. \tag{13}$$

The line vectors specified by (11) form a 3-dimensional vector space

$$\mathcal{R}^3 = \{\mathbf{x}\}, \tag{14}$$

which can be identified with the standard vector space model of $\mathcal{E}^3$, wherein the distinguished point e_0 is represented by the zero vector.

The *conformal split* of Euclidean points specified by (13) generates a split of the entire geometric algebra into a commutative product of subalgebras:

$$\mathcal{R}_{4,1} = \mathcal{R}_3 \otimes \mathcal{R}_{1,1}, \tag{15}$$

where $\mathcal{R}_3 = \mathcal{G}(\mathcal{R}^3)$ as before, and $\mathcal{R}_{1,1}$ is the Minkowski geometric algebra generated by the vectors e_0 and e. The identification in (14) of certain 3-vectors in $\mathcal{R}_{4,1}$ with vectors in $\mathcal{R}^3$ can be described as *regrading* a subalgebra, that is, redesignating generating elements of the subalgebra as vectors.

The conformal split (14) has deep similarities with the *spacetime split* of the geometric algebra $\mathcal{R}_{3,1}$ for spacetime, originally introduced by Hestenes (1966) and given its name in (Hestenes 1974). Just as the conformal split refers homogeneous Euclidean geometry to a single point, the spacetime split refers homogeneous spacetime geometry to a single inertial reference frame. In both cases, the split introduces unnecessary complications, so it should be avoided whenever possible. Long experience with the spacetime algebra shows that it is invariably best practice to carry out all calculations with invariant equations and perform a spacetime split only at the end if it is needed to compare with other results in the literature or to relate to empirical data. Likewise, we shall see how homogeneous methods can simplify the formulation and solution of problems in geometry and mechanics without resorting to the conformal split.

3. Rigid Displacements

By definition, a *rigid displacement* $\underline{D}$ of points in a material body leaves the Euclidean distance between body points invariant, as expressed by $(x - y)^2 = -2x \cdot y$ in our homogeneous model. Invariance of the inner product $x \cdot y$ is the defining property of orthogonal transformations on the vector space $\mathcal{R}^{4,1}$. It is a general theorem of geometric algebra (Hestenes 1991, Hestenes and Sobczyk 1984) that every such transformation $\underline{D}$ taking a generic point x_0 to the point x can be expressed in the canonical form

$$x = \underline{D}x_0 = Dx_0D^{-1} \tag{16}$$

where D is an invertible multivector in $\mathcal{R}_{4,1}$ called a *versor*. The great advantage of this result is that rotations and translations have simple representations as versors, and the composition of rigid displacements is reduced to versor multiplication. Our main problem will be to determine the form of D for various rigid body motions.

To represent rigid displacements uniquely, we must impose some general restrictions on the form of D. To exclude reflections, which are also orthogonal transformations, D must be an even multivector. Since (16) is bilinear in D, we can normalize D to unity and identify D^{-1} with its *reverse* $D^\dagger$, so that

$$DD^\dagger = DD^{-1} = 1. \tag{17}$$

One consequence of this is that D can be written in the exponential form

$$D = e^{\frac{1}{2}S} \qquad \text{with} \qquad D^\dagger = e^{-\frac{1}{2}S}, \tag{18}$$

where S is a bivector called a *twist* or *screw*. Screws are often normalized to unity, so twists are scalar multiples of screws.

To preserve our definition of homogeneous Euclidean space, the point at infinity must be an invariant of any rigid displacement, as expressed by

$$DeD^{\dagger} = e \qquad \text{or} \qquad De = eD, \tag{19}$$

or, equivalently, by

$$S \cdot e = 0 \qquad \text{or} \qquad Se = eS. \tag{20}$$

In other words, e commutes with every twistor and its twist. This completes our list of general restrictions on rigid displacement versors. Henceforth, it will be convenient to designate such versors by the suggestive name *twistor*, as there is little danger of confusion with Roger Penrose's use of the same term in relativity theory.

To enable comparison with other works on screw theory and robotics, we need the conformal split of a twist (or screw) S. As before, we pick a convenient point e_0 to serve as origin and employ the bivector $E = e_0 \wedge e$ to make the split. The product SE is decomposed into scalar (0-vector), bivector (2-vector) and quadravector (4-vector) parts by the identity

$$SE = S \cdot E + S \times E + S \wedge E, \tag{21}$$

where $S \times E \equiv \frac{1}{2}(SE - ES)$ is the commutator product. We consider each term in turn. It follows from (20) that the scalar part vanishes, as shown by

$$S \cdot E = S \cdot (e_0 \wedge e) = e_0 \cdot (e \cdot S) = 0. \tag{22}$$

When working with bivectors, it is often helpful to express inner and outer products with vectors as commutator products so we can use the *Jacobi identity* like this:

$$S \times E = S \times (e_0 \times e) = (S \times e_0) \times e + e_0 \times (S \times e) = (S \cdot e_0)e, \tag{23}$$

where the term with $S \times e = S \cdot e$ again vanishes. The right side of (23) determines a vector $n = S \cdot e_0 + \lambda e$ up to a scalar component λ along e. Note that

$$ne = n \wedge e = -en \qquad \text{or} \qquad n \cdot e = 0, \tag{24}$$

so the vector n cannot represent a point. Nevertheless, we can apply the conformal split (13) to get

$$n = \mathbf{n}E + (n \cdot e_0)e \qquad \text{where} \qquad \mathbf{n} = n \wedge E. \tag{25}$$

Using

$$eE = e = -Ee, \tag{26}$$

we see that

$$en = e\mathbf{n} = \mathbf{n}e = -ne. \tag{27}$$

In general, we can write $S \wedge E = a \wedge b \wedge E = (a \wedge b)E$, where a and b are vectors satisfying $a \cdot E = b \cdot E = 0$ (so, like n, they cannot be regarded as points). Treating a and b like n in (25) to (27), we have

$$(S \wedge E)E = a \wedge b = \mathbf{a} \wedge \mathbf{b} = -i\mathbf{a} \times \mathbf{b}, \tag{29}$$

where the last term involves the conventional cross product (Hestenes 1986), and i is the the unit pseudoscalar for $\mathcal{R}_3$, which a conformal split identifies with the unit pseudoscalar for $\mathcal{R}_{4,1}$ as well.

Writing $\mathbf{m} = \mathbf{a} \times \mathbf{b}$ and combining splits for each or the terms in (21), we obtain

$$S = (S \wedge E + S \times E)E = -i\mathbf{m} - e\mathbf{n}. \tag{30}$$

This is the general form for the *conformal split* of any twist, indeed, of any bivector satisfying $S \cdot e = 0$. Thus, we can write (18) in the form

$$D = e^{-\frac{1}{2}(i\mathbf{m}+e\mathbf{n})}, \tag{31}$$

but it must be remembered that the values for $\mathbf{m}$ and $\mathbf{n}$ depend on the arbitrary choice of base point e_0.

When $\mathbf{n} = 0$, the twistor (31) reduces to a *rotor*

$$R = e^{-\frac{1}{2}i\mathbf{m}} \tag{32}$$

representing a rotation through angle $|\,\mathbf{m}\,|$ about an axis directed along $\mathbf{m}$, as explained in (Hestenes 1986). The minus sign in (32) is to conform to the usual right hand rule for representing rotation angles by vectors.

When $\mathbf{m} = 0$, the twistor (32) reduces to the special form

$$T = e^{-\frac{1}{2}e\mathbf{n}} = e^{\frac{1}{2}ne} = 1 + \tfrac{1}{2}ne, \tag{33}$$

which represents a translation, as we shall see.

If $\mathbf{m}$ and $\mathbf{n}$ are collinear, that is, if $\mathbf{mn} = \mathbf{n} \cdot \mathbf{n} = \mathbf{nm}$, then we recover the displacement

$$TR = RT = e^{-\frac{1}{2}i\mathbf{m}}e^{-\frac{1}{2}e\mathbf{n}} = e^{-\frac{1}{2}(i\mathbf{m}+e\mathbf{n})} = D. \tag{34}$$

This represents a *screw displacement* along a line with direction $n = \mathbf{n}E$ through the point e_0. It consists of a translation through distance $|n| = |\mathbf{n}|$ composed with a rotation through angle $|\mathbf{m}|$ about the line. Every displacement can be expressed as a screw displacement along some line, but finding a point e_0 on that line to give it the screw form (34) is often impractical or unnecessary.

Any displacement $\underline{D}$ can be decomposed into a rotation $\underline{R}$ about a line through a given point x_0 followed by a translation $\underline{T}$, as expressed by the operator equation

$$\underline{D} = \underline{T}\,\underline{R}. \tag{35}$$

Given $\underline{D}$ defined by (16), we want to express (36) as a twistor equation

$$D = TR. \tag{36}$$

Displacement of a given body point from x_0 to x determines a translation defined by

$$x = \underline{T}x_0 = Tx_0T^{\dagger}. \tag{37}$$

Assuming that T has the canonical form (33), from (37) we obtain

$$xT = x(1 + \tfrac{1}{2}ne) = (1 + \tfrac{1}{2}ne)x_0. \tag{38}$$

This can be solved for n, with the help of $ex_0 + x_0e = 2e \cdot x_0 = 2$, to get the expected result:

$$n = x - x_0. \tag{39}$$

Finally, with D and T known, we can find R easily from (37). Since $\underline{D}x_0 = \underline{T}x_0$ by definition, we have

$$\underline{R}x_0 = Rx_0R^\dagger = x_0 \qquad \text{or} \qquad Rx_0 = x_0R. \tag{40}$$

In other words, x_0 is a fixed point for the rotation.

For practical applications we need general methods for determining the twistor D from measurements on body points. As shown in the preceding paragraph, position measurements on a conveniently chosen body point enable one to compute a translation versor T. According to (37), therefore, this reduces the problem of finding D to finding R. Identifying x_0 with e_0 in our conformal split of a twist, it is easy to show that (40) implies that R can be parametrized by an angle vector $\mathbf{m}$ as in (32). Many other parametrizations of R, such as Euler angles, are are given in (Hestenes 1986). In general, the best choice of parameters is determined by the problem-at-hand, especially by the form in which data is given and by symmetries of the system being modeled. It is most important to recognize that R itself is a computationally-efficient, coordinate-free representation of a rotation, so we aim to relate it to data as directly as possible.

The most commonly used representations for rotations are matrices of direction cosines. However, the matrix method is computationally inefficient, largely because matrices are redundant representations of the information. Its main advantage is the availability of highly developed software for matrix computations. For these reasons alone, we need efficient algorithms for interconverting rotation matrices and rotors. That has been fully worked out by (Hestenes 1986), but a brief review is in order here.

For a given rigid body, a *body frame* $\{e_k, k = 1, 2, 3\}$ can be defined by

$$e_k = T^{-1}(x_k - x)T = T^{-1}x_kT - x_0, \tag{41}$$

where the x_k and x are four distinct points affixed to the body. The e_k are related to a *reference frame* σ_k based at the point x_0 by the rotation

$$e_k = \underline{R}\sigma_k = R\sigma_kR^\dagger = \sum_j \alpha_{jk}\sigma_j. \tag{42}$$

Though by no means essential, it is often convenient to select the body points so that $\{e_k\}$ is an orthonormal set. Then the

$$\alpha_{jk} = \sigma_j \cdot e_k = \langle\, \sigma_j R\sigma_k R^\dagger \,\rangle \tag{43}$$

are *direction cosines* relating the two frames. Equation (42) gives each element of the rotation matrix as a function of the rotor R. Equation (43) can be inverted to express R as a function of the e_k or the α_{jk}, with the elegant result

$$\pm 4\alpha R = 1 + \Sigma, \tag{44}$$

where

$$\Sigma = \sum_k \sigma_k e_k = \sum_{jk} \alpha_{jk}\sigma_j\sigma_k, \tag{45}$$

and

$$\alpha = \langle\, R \,\rangle = \tfrac{1}{2}(1 + \sum_k \alpha_{kk})^{1/2}. \tag{46}$$

It bears repeating, though, that there are more direct ways to relate rotors to data.

4. Kinematics

The orbit $x = x(t)$ of any particle in a rigid body is determined by a time dependent twistor $D = D(t)$ in equation (16). Therefore, the kinematics of a rigid body is completely characterized by a twistor differential equation of the form

$$\dot{D} = \tfrac{1}{2} V D, \tag{47}$$

where the overdot indicates a time derivative. From

$$V = 2\dot{D}D^{\dagger} = -2D\dot{D}^{\dagger} = -V^{\dagger}, \tag{48}$$

it follows that V is necessarily a bivector, and (19) implies that

$$V \cdot e = 0. \tag{49}$$

By differentiating (16) and using (48), we obtain a kinematical equation for the motion of a body point:

$$\dot{x} = V \cdot x. \tag{50}$$

Its equivalence to the usual vectorial equation is easily established by a conformal split.

Since their algebraic properties are analogous, the conformal split of V must have the same form as the conformal split (31) for a twist, so we can write

$$V = -i\boldsymbol{\omega} - e\mathbf{v}. \tag{51}$$

Inserting this along with (13) into (50), we obtain

$$\begin{aligned} \dot{x} = \dot{\mathbf{x}}E - (\dot{\mathbf{x}} \cdot \mathbf{x})e &= -[i\boldsymbol{\omega} + e\mathbf{v}] \times [\mathbf{x}E + e_0 - \tfrac{1}{2}\mathbf{x}^2 e] \\ &= (-i\boldsymbol{\omega} \wedge \mathbf{x})E - \mathbf{v}(e \wedge e_0) - (\mathbf{v} \cdot \mathbf{x})eE = [\boldsymbol{\omega} \times \mathbf{x} + \mathbf{v}]E - (\mathbf{v} \cdot \mathbf{x})e. \end{aligned}$$

Equating linearly independent parts, we obtain

$$\dot{\mathbf{x}} = \boldsymbol{\omega} \times \mathbf{x} + \mathbf{v} \tag{52}$$

as well as $\dot{\mathbf{x}} \cdot \mathbf{x} = \mathbf{v} \cdot \mathbf{x}$ which, of course, is not an independent equation. Equation (52) confirms an interpretation of $\boldsymbol{\omega}$ as *angular velocity* and $\mathbf{v}$ as *translational velocity*, expressed in (51) as components of a generalized velocity V. Henceforth, it will be convenient to refer to V as the *screw velocity* of the rigid displacement.

With the relation to conventional kinematics established, we can deal directly with the fundamental twistor equation (48) without resorting to conformal splits. The first problem is to decide what form for the twistor D is most suitable for computational purposes. The exponential form $D = e^{\frac{1}{2}S}$ has the drawback that it is defined by a power series, so it is computationally expensive to apply and invert. Secondly, the simple relation $V = \dot{S}$ holds only when $\dot{S}$ commutes with S.

Heinz Krüger (private communication to DH) has demonstrated how these problems can be addressed by putting D in the form

$$D = \frac{1+B}{1-B}\,, \tag{53}$$

where B is a bivector. First, note that this form automatically ensures the normalization $DD^{\dagger} = 1$ whatever the value of B. Second by differentiating (51) with due attention to noncommutivity of B and $\dot{B}$, we find a closed algebraic relation between V and $\dot{B}$:

$$V = 2\dot{D}D^{\dagger} = \frac{4}{1-B}\dot{B}\frac{1}{1+B}. \tag{54}$$

For dynamics we need one more derivative, which can be put in the form

$$\tfrac{1}{4}(1-B)\dot{V}(1+B) = \ddot{B} - \dot{B}(\frac{2B}{1-B^2})\dot{B}. \tag{55}$$

This makes it possible to express the dynamical equation for a rigid body in terms of B and its derivatives.

Equation (53) can be solved for

$$B = \frac{D - D^{\dagger}}{2 + D + D^{\dagger}} = \frac{\langle\, D\,\rangle_2}{1 + \langle\, D\,\rangle_0 + \langle\, D\,\rangle_4} \tag{56}$$

This will be recognized as generalizing the trigonometric formula for the tangent of a half angle. It can be shown that $\langle\, D\,\rangle_4$ is proportional to $ie = eI$, so it commutes with $\langle\, D\,\rangle_2$ and plays the role of a null pseudoscalar. Accordingly, we say that (56) expresses B as *dually proportional* to the bivector part of D.

5. Dynamics

For the purposes of dynamics a *comomentum* P for the rigid body is defined by

$$P = \underline{M}V = i\underline{I}\,\boldsymbol{\omega} - m\mathbf{v}e_0 = i\boldsymbol{\ell} - \mathbf{p}e_0. \tag{57}$$

This also defines a generalized "mass tensor" $\underline{M}$ in terms of the inertia tensor $\underline{I}$ and the body mass m. The conformal split for P is given to define its relation to standard quantities. Note that this split differs in form from the split (51) for V in having the null point e replaced by e_0. This difference is necessary to express the invariant kinetic energy by

$$K = \tfrac{1}{2}V \cdot P = \tfrac{1}{2}(\boldsymbol{\omega}\cdot\boldsymbol{\ell} + \mathbf{v}\cdot\mathbf{p}). \tag{58}$$

The *coforce* or *wrench* F acting on a rigid body is defined in terms of the torque $\boldsymbol{\Gamma}$ and net force $\mathbf{f}$ by

$$F = i\boldsymbol{\Gamma} + \mathbf{f}e_0. \tag{59}$$

The dynamical equation for combined rotational and translational motion then takes the compact form:

$$\dot{P} = F. \tag{60}$$

An immediate consequence is the energy conservation law

$$\dot{K} = V \cdot F = \boldsymbol{\omega} \cdot \boldsymbol{\Gamma} + \mathbf{v} \cdot \mathbf{f} \tag{61}$$

The problem remains to specify a net wrench F on the body so that (60) becomes a well-defined equation of motion.

6. Elastic Coupling

Previous sections characterize the motion of a single rigid body with respect to a fixed reference body. This generalizes immediately to a theory of two interacting rigid bodies by identifying the second body with the reference body. Then the twistor equation $D = D(t)$ describes the *relative motion* of the first body with respect to the second. The reverse twistor $D^{\dagger}$ describes the motion of the second body with respect to the first. Since we are concerned here only with relative motion, we may continue to regard the second body as fixed and concentrate on the motion of the first.

We are interested in modeling an elastic interaction between the two bodies. Assume that the state of elastic deformation is determined solely by the relative configuration of the rigid bodies. In this case the potential function of elastic deformation (strain energy) is a function of the relative displacement of the rigid bodies from the unstressed equilibrium configuration. Assume that such an unstressed equilibrium configuration exists and is locally unique. The displacement from equilibrium can be represented by the twist variable S, so the *elastic potential* U has the general form

$$U(S) = \tfrac{1}{2} S \cdot (\underline{K} S) + ..., \tag{62}$$

where $\underline{K}$ is a linear *stiffness* operator characterizing first order deviation from equilibrium (Hooke's Law). The potential determines an elastic wrench, which for small twists has the form

$$F(S) = -\partial_S U = -\tfrac{1}{2} \underline{K} S. \tag{63}$$

Fasse (2000) gives a general method for determining the wrench from a potential for arbitrary twists. Note that the potential must be an even function $U(-S) = U(S)$ so that the wrench is an odd function $F(-S) = -F(S)$ in accord with Newton's 3rd Law.

As argued in a preceding section, it may be better to use the variable B instead of S when solving the equations of motion. Just as B is dually proportional to the bivector part of the twistor D, it is dually proportional to the twist S, so the first order potential (62) can be re-expressed as a function of B with the same functional form:

$$U(B) = \tfrac{1}{2} B \cdot (\underline{K} B), \tag{64}$$

This being noted, we continue the discussion in terms of the twist variable S.

Inserting the conformal split (31) into (62) splits the first order potential into

$$U(S) = U(-i\mathbf{m} - e\mathbf{n}) = \mathbf{m} \cdot \underline{K}_O \mathbf{m} + \mathbf{n} \cdot \underline{K}_C \mathbf{m} + \mathbf{m} \cdot \bar{\underline{K}}_C \mathbf{n} + \mathbf{n} \cdot \underline{K}_t \mathbf{n}, \tag{65}$$

where $\underline{K}_O$ represents the rotational stiffness, $\underline{K}_t$ the translational stiffness and $\underline{K}_C$ with adjoint $\bar{\underline{K}}_C$ the coupling stiffness. This potential function is equivalent to the one introduced in (Fasse and Zhang 1999; Fasse, Zhang and Arabyan 2000).

If $tr(\underline{K}_t)$ is not an eigenvalue of $\underline{K}_t$, then there exist unique points x_a and x_b (coincident in equilibrium) on the two bodies at which the coupling stiffness $\underline{K}_C$ is symmetric (Lončarić 1987;

Brockett and Lončarić 1986). It is not assumed here that axes of the body frames 'a' and 'b' intersect at the *center of stiffness*. Nonetheless it is advisable to choose the center of stiffness as a reference. First, it is a unique, unambiguously defined point for most systems. Second, as shown by Ciblak (1998), any compliant axis decoupling translation and rotation must intersect the center of stiffness. It must also intersect the centers of compliance and elasticity. If two compliant axes exist, then the three centers coincide. Many manufactured compliant joints have compliant axes by design. Thus for these systems the center of stiffness has an intuitive physical significance.

Empirical estimation of the *stiffness tensor* $\underline{K}$ is nontrivial. Briefly, the following experimental procedure is envisioned: One of the two bodies is kept stationary while the other is displaced by some mechanism (Fig. 1), and measurements of the resulting displacements are correlated with the applied torque.

Viscous forces can be modeled to first order in analogy to elastic forces by introducing the generalized Rayleigh dissipation potential

$$\Pi(\dot{S}) = \tfrac{1}{2}\dot{S} \cdot \underline{\Pi}\,\dot{S}, \tag{66}$$

which generates the damping wrench

$$F_{\Pi}(\dot{S}) = -\partial_{\dot{S}}\Pi = -\underline{\Pi}\,\dot{S}. \tag{67}$$

This general form for damping forces has yet to be implemented in engineering design.

7. Conclusions

The homogeneous method introduced in this paper holds great promise for the design and analysis of mechanical devices. Geometric algebra provides an ideal language for the ideas of screw theory that evolved more than a century ago, but were imperfectly expressed in the coordinate-based mathematics of the day. The coordinate-free, homogeneous equations for twistor kinematics and dynamics determine the time evolution of a finite screw displacement in a way that leaves nothing to be desired. The formulation of twists as elements of a 6-dimensional bivector algebra automatically incorporates all the advantages of Lie algebra into rigid body theory, and it suggests that this is the best choice for the 6-dimensional configuration space of a rigid body.

Besides providing an optimally compact formulation for rigid body equations of motion, the homogeneous method provides complete flexibility in the choice of parameters for specific problems, and it opens up new computational possibilities. The task remains to develop software for modeling and simulation that takes full advantage of homogeneous methodology. The theory is sufficiently developed to make applications to problems throughout engineering fairly straightforward, but many details remain to be worked out.

Note: David Hestenes and Alyn Rockwood have applied for a patent on use of the homogeneous method in software for modeling and simulation.

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