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On Decoupling Probability from Kinematics in Quantum Mechanics

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Abstract.A means for separating subjective and objective aspects of the electron wave function is suggested, based on a reformulation of the Dirac Theory in terms of Spacetime Algebra. The reformulation admits a separation of the Dirac wave function into a two parameter probability factor and a six parameter kinematical factor. The complex valuedness of the wave function as well as its bilinearity in observables have perfect kinematical interpretations independent of any probabilistic considerations. Indeed, the explicit unit imaginary in the Dirac equation is automatically identified with the electron spin in the reformulation. Moreover, the canonical momentum is seen to be derived entirely from the rotational velocity of the kinematical factor, and this provides a geometrical interpretation of energy quantization. Exact solutions of the Dirac equation exhibit circular zitterbewegung in exact agreement with the classical Wessenhoff model of a particle with spin. Thus, the most peculiar features of quantum mechanical wave functions have kinematical explanations, so the use of probability theory in quantum mechanics should not differ in any essential way from its use in classical mechanics.

Introduction

I believe that quantum mechanics, as generally understood and practiced today, intermixes subjective and objective components of human knowledge, and furthermore, that we will not understand the subject fully until those components can be cleanly separated. The main purpose of this article is to propose a means by which that separation might be effected. As will be seen, my proposal has many specific and surprising consequences as well as possibilities for further development.

I regard the Dirac electron theory as the fundamental core of current quantum mechanics. It is from the Dirac theory that the most precise and surprising consequences of quantum mechanics have been derived. Some would claim that quantum field theory is more fundamental, but one can argue that field theory is merely a formal device for imposing boundary conditions of the single particle theory to accommodate particle creation and annihilation along with the Pauli principle [1]. For these reasons, it is to the Dirac theory that I look to understand the role of probability in quantum mechanics. We shall see that the Dirac theory supplies insights into the significance of quantum mechanical wave functions that could not possibly be derived from the Schrödinger theory.

To separate subjective and objective components of the Dirac Theory I suggest that we need two powerful conceptual tools. The first tool is the *Universal Probability Calculus* which has been synthesized and expounded so clearly by Ed Jaynes and amply justified by many applications discussed in these Workshops and elsewhere. Fortunately, this calculus is so familiar to workshop participants that I need not spell out any of the details, though the calculus is still not appreciated by most physicists. However, I should reiterate the major claims for the calculus which explains its relevance to the interpretation of quantum mechanics. First of all, the calculus is universal in the sense that it is applicable to *any* problem involving interpretions and explanations of experimental data with mathematical models. Thus, the calculus provides a universal interface between theory and experiment. Secondly, the calculus provides the basis for unambiguous distinctions between subjective and objective knowledge. Probabilities are always subjective; they describe limitations on objective knowledge about the real world rather than properties of the real world itself. Jaynes has applied the calculus brilliantly to cleanly separate subjective and objective components of statistical mechanics. It should be possible to do the same in quantum mechanics.

The very generality of the probability calculus is one of its inherent limitations. Objective knowledge enters the calculus only through the Boolean algebra of propositions, and this does not take into account the general implications of spacetime structure for probabilistic reasoning about the physical world. To remedy this deficiency, I propose to employ another powerful conceptual tool which I call the *Universal Geometric Calculus*. We shall see that it suggests some extensions of the probability calculus.

Geometric Calculus is a universal mathematical language for expressing geometrical relations and deducing their consequences. As such, it is a natural language for most of mathematics and perhaps all of physics ([2], [3], [4]). Of special interest here is a portion of the general geometric calculus called Spacetime Algebra (STA). It can be regarded as the minimally complete algebra of geometrical relations in spacetime.

When the Dirac theory is expressed in terms of STA a hidden geometric structure is revealed, and natural explanations appear for some of the most peculiar features of quantum mechanics [5]. For example, it can be seen that there are geometrical reasons for the appearance of complex probability amplitudes and the bilinear dependence of observables on them. The main ideas and insights of this approach are reviewed below. Then they are applied to achieve the proposed separation between subjective and objective features of the Dirac theory, resulting in a new concept of "pure state." A new class of solutions to the Dirac equations is identified, solutions which clearly exhibit circular zitterbewegung and suggest that it is an objective property of electron motion independent of probabilistic aspects of the theory, in general agreement with earlier speculations [6]. This is illustrated by a new kind of free particle solution to the Dirac equation.

All of this together indicates a need to integrate probability calculus with geometric calculus to form a single coherent conceptual system, and it suggests how that might be accomplished. Geometric calculus is needed to represent objective properties of real objects in mathematical models and theories. Probability calculus is needed to relate such models to real phenomena. Thus, the two must be integrated to achieve an integrated world view.

Spacetime Algebra

For the purposes of this paper, we adopt a flat space model of physical spacetime, so each point event can be uniquely represented by an element x in a 4-dimensional vector space.

We call x the *location* of the event, and we call the vector space of all locations *spacetime*. Thus, we follow the usual practice of conflating our mathematical model with the physical reality it supposedly represents.

To complete the mathematical characterization of spacetime, we define a geometric product among spacetime vectors u, v, w by the following rules:

$$u(vw) = (uv)w\,, \tag{1.1}$$

$$u(v+w) = uv + uw\,, \tag{1.2a}$$

$$(v+w)u = vu + wu\,. \tag{1.2b}$$

For every vector u and scalar λ

$$u\lambda = \lambda u\,, \tag{1.3}$$

$$u^2 = \text{a scalar (a real number)}\,. \tag{1.4}$$

The metric of spacetime is specified by the allowed values for u^2. As usual, a vector u is said to be *timelike*, *lightlike* or *spacelike* if $u^2 > 0$, $u^2 = 0$, $u^2 < 0$ respectively.

Under the geometric product defined by these rules, the vectors of spacetime generate a real associative algebra, which I call the *Spacetime Algebra* (STA), because all its elements and operations have definite geometric interpretations, and it suffices for the description of geometric structures on spacetime. An account of STA is given in [2], and extended in the other references. Only a few features of STA with special relevance to the problem at hand can be reviewed here.

The geometric product of vectors u and v can be decomposed into a symmetric part $u \cdot v$ and an antisymmetric part $u \wedge v$ as defined by

$$u \cdot v = \tfrac{1}{2}(uv + vu) \tag{1.5}$$

$$u \wedge v = \tfrac{1}{2}(uv - vu) \tag{1.6}$$

$$uv = u \cdot v + u \wedge v \tag{1.7}$$

One can easily prove that the symmetric product $u \cdot v$ defined by (1.5) is scalar-valued. Thus, $u \cdot v$ is the usual *inner product* (or metric tensor) on spacetime. The quantity $u \wedge v$ is neither scalar nor vector, but a new entity called a *bivector* (or 2-vector). It represents an oriented segment of the plane containing u and v in much the same way that a vector represents a directed line segment.

Let $\{\gamma_\mu,\ \mu = 0,1,2,3\}$ be a *righthanded orthonormal frame* of vectors; so

$$\gamma_0^2 = 1 \quad \text{and} \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1\,, \tag{1.8}$$

and it is understood that γ_0 points into the forward light cone. In accordance with (1.5), we can write

$$g_{\mu\nu} \equiv \gamma_\mu \cdot \gamma_\nu = \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)\,, \tag{1.9}$$

defining the components of the metric tensor $g_{\mu\nu}$ for the frame $\{\gamma_\mu\}$.

Representations of the vectors γ_μ by 4×4 matrices are called *Dirac matrices*. The *Dirac algebra* is the matrix algebra over the field of the complex numbers generated by the

Dirac matrices. The conventional formulation of the Dirac equation in terms of the Dirac algebra can be replaced by an equivalent formulation in terms of STA. This has important implications. First, a representation of the γ_μ by matrices is completely irrelevant to the Dirac theory; the physical significance of the γ_μ is derived entirely from their representation of geometrical properties of spacetime. Second, imaginaries in the complex number field of the Dirac algebra are superfluous, and we can achieve a geometrical interpretation of the Dirac wave function by eliminating them. For these reasons we eschew the Dirac algebra and stick to STA.

A generic element of the STA is called a *multivector*. Any multivector M can be written in the *expanded form*

$$M = \alpha + \mathbf{a} + F + i\mathbf{b} + i\beta\,, \tag{1.10}$$

where α and β are scalars, $\mathbf{a}$ and $\mathbf{b}$ are vectors, and F is a bivector. The special symbol i will be reserved for the *unit pseudoscalar*, which has the following three basic algebraic properties:

(a) it has negative square,

$$i^2 = -1\,, \tag{1.11a}$$

(b) it anticommutes with every vector $\mathbf{a}$,

$$i\mathbf{a} = -\mathbf{a}i\,, \tag{1.11b}$$

(c) it factors into the ordered product

$$i = \gamma_0\gamma_1\gamma_2\gamma_3\,. \tag{1.11c}$$

Geometrically, the pseudoscalar i represents a unit oriented 4-volume for spacetime.

By multiplication the γ_μ generate a complete basis for the STA consisting of

$$1, \quad \gamma_\mu, \quad \gamma_\mu\wedge\gamma_\nu, \quad i\gamma_\mu, \quad i\,. \tag{1.12}$$

These elements comprise a basis for the 5 invariant components of M in (1.10), the scalar, vector, bivector, pseudovector and pseudoscalar parts respectively. Thus, they form a basis for the space of completely antisymmetric tensors on spacetime. It will not be necessary for us to employ a basis, however, because the geometric product enables us to carry out computations without it.

Computations are facilitated by the operation of *reversion*. For M in the expanded form (1.10), the *reverse* M can be defined by

$$\widetilde{M} = \alpha + \mathbf{a} - F - i\mathbf{b} + i\beta\,. \tag{1.13}$$

Note, in particular, the effect of reversion on scalars, vectors, bivectors and pseudoscalars:

$$\widetilde{\alpha} = \alpha, \quad \widetilde{\mathbf{a}} = \mathbf{a}, \quad \widetilde{F} = -F, \quad \widetilde{i} = i\,.$$

Reversion has the general property

$$(MN)^{\sim} = \widetilde{N}\widetilde{M}\,, \tag{1.14}$$

which holds for arbitrary multivectors M and N.

Having completed the preliminaries, we are now equipped to state a powerful theorem of great utility: Every Lorentz transformation of an orthonormal frame $\{\gamma_\mu\}$ into a frame $\{e_\mu\}$ can be expressed in the *canonical form*

$$e_\mu = R\gamma_\mu \widetilde{R}\,, \tag{1.15}$$

where R is a *unimodular spinor*, which means that R is an even multivector satisfying the unimodularity condition

$$R\widetilde{R} = 1\,. \tag{1.16}$$

A multivector is said to be *even* if its vector and trivector parts are zero. The spinor R is commonly said to be a spin *representation* of the Lorentz transformation (1.15).

The set $\{R\}$ of all unimodular spinors is a group under multiplication. In the theory of group representations it is called SL(2, C) or "the spin-1/2 representation of the Lorentz group." However, group theory alone does not specify its invariant imbedding in the STA. It is precisely this imbedding that makes it so useful in the applications to follow.

Classical Electrodynamics and Particle Mechanics

The electromagnetic field $F = F(x)$ is a bivector-valued function on spacetime. The expansion of F in a bivector basis,

$$F = \tfrac{1}{2}F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu \tag{2.1}$$

shows its relation to the usual field by tensor components $F^{\mu\nu}$. However, because STA enables to coordinate-free manner.

The derivative with respect vector differential operator $\Box$ coordinate derivatives by

$$\Box = \gamma^\mu \partial_\mu \tag{2.3}$$

where

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \gamma_\mu \cdot \Box\,. \tag{2.4}$$

It will recognized that the matrix representation of (2.3) where the γ^μ are replaced by Dirac matricies is the famous "Dirac operator." However, STA reveals that the significance of this operator lies in its role as the fundamental differential operator on spacetime, rather than in any special role in quantum mechanics or even spinor mechanics.

Since $\Box$ is a vector operator, we can use (1.7) to decompose the derivative of a vector field $A = A(x)$ into *divergence* $\Box \cdot A$ and *curl* $\Box \wedge A$; thus

$$\Box A = \Box \cdot A + \Box \wedge A\,. \tag{2.5}$$

Taking A to be the electromagnetic *vector potential* and imposing the "Lorentz condition"

$$\Box \cdot A = 0\,, \tag{2.6}$$

we have

$$F = \Box \wedge A = \Box A\,. \tag{2.7}$$

Maxwell's equation for the E.M. field can then be written

$$\Box F = \Box^2 A = J_e, \tag{2.8}$$

where $J_e = J_e(x)$ is the (electric) *charge current* (density).

Equation (2.8) describes the production and propagation of E.M. fields equally well in classical and quantum theories, but it must be complemented by an equation of motion for charged particles which describes the effects of the E.M. field. The ordinary classical equation of motion for a charged particle seems so different from the quantum mechanical equation that they are difficult even to compare. However, STA admits a new formulation of the classical equation which greatly clarifies its relation to quantum theory.

The equation

$$e_\mu = R\gamma_\mu \widetilde{R} \tag{2.9}$$

can be used to describe the relativistic kinematics of a rigid body (with negligible dimensions) traversing a world line $x = x(\tau)$ with proper time τ, if we identify e_0 with the proper velocity v of the body (or particle), so that

$$\frac{dx}{d\tau} = v = e_0 = R\gamma_0 \widetilde{R}\,. \tag{2.10}$$

Then $e_\mu = e_\mu(\tau)$ is a comoving frame traversing the world line along with the particle, and the spinor R must also be a function of proper time, so that, at each time τ, equation (2.9) describes a Lorentz transformation of some fixed frame $\{\gamma_\mu\}$ into the comoving frame $\{e_\mu(\tau)\}$. Thus, we have a spinor-valued function of proper time $R = R(\tau)$ determining a 1-parameter family of Lorentz transformations.

The spacelike vectors $e_k = R\gamma_k \widetilde{R}$ (for k = 1,2,3) can be identified with the principal axes of the body in some applications, but for a particle with an intrinsic angular momentum or *spin*, it is most convenient to identify e_3 with the spin direction $\hat{s}$; so we write

$$\hat{s} = e_3 = R\gamma_3 \widetilde{R}\,. \tag{2.11}$$

From the fact that R is an even multivector satisfying $R\widetilde{R} = 1$, it can be proved that $R = R(\tau)$ must satisfy a *spinor equation of motion* of the form

$$\dot{R} = \tfrac{1}{2}\Omega R\,, \tag{2.12}$$

where the dot represents the proper time derivative, and $\Omega = \Omega(\tau) = -\widetilde{\Omega}$ is a bivector-valued function. Differentiating (2.9) and using (2.12), we see that the equations of motion for the comoving frame must be of the form

$$\dot{e}_\mu = \tfrac{1}{2}(\Omega\, e_\mu - e_\mu \Omega) \equiv \Omega \cdot e_\mu\,. \tag{2.13}$$

Clearly Ω can be interpreted as a generalized *rotational velocity* of the comoving frame.

For a classical particle of mass m and charge e moving in an external E.M. field F, we take $\Omega = em^{-1}F$ and from (2.13) with $\mu = 0$ we get (in units with the speed of light $c = 0$)

$$m\dot{v} = eF \cdot v\,. \tag{2.14}$$

This is the classical equation of motion where the right side is the *Lorentz force*. On the other hand, (2.12) gives us

$$\dot{R} = \frac{e}{2m}\,FR\,. \tag{2.15}$$

This spinor equation of motion implies the conventional equation (2.14), so it determines the same world line, but it gives us much more. First, it is easier to solve; solutions for various external fields are given in [7]. Second, it gives immediately a classical model for a particle with spin; by (2.11) and (2.13) with $\mu = 3$, the equation of motion for the spin is

$$\dot{s} = \frac{e}{m}\,F \cdot s\,. \tag{2.16}$$

This is, in fact, the spin precession equation for a particle with gyromagnetic ratio $g = 2$, exactly the basic value of g implied by the Dirac theory. This is no accident, for the greatest advantage of (2.15) is that the spinor R can be related directly to the Dirac wave function, thus enabling a close comparison of classical and quantum equations of motion, as shown below.

The Real Dirac Theory

This section summarizes (without proof) the formulation of the Dirac electron theory in terms of STA. Proofs that this is equivalent to the conventional matrix formulation are given in [8] and [5].

The Dirac wave function $\psi = \psi(x)$ is an even multivector-valued function on spacetime. It has the Lorentz invariant decomposition

$$\psi = (\rho e^{i\beta})^{\frac{1}{2}}R\,, \tag{3.1}$$

where $R = R(x)$ is a unimodular spinor, i is the unit pseudoscalar (1.11c), and $\rho = \rho(x)$, $\beta = \beta(x)$ are scalar-valued functions.

The Dirac equation has the *real form*

$$\Box\psi\mathbf{i}\hbar + eA\psi = m\psi\gamma_0\,, \tag{3.2}$$

where $A = A(x)$ is the vector potential, $\hbar$ is Planck's constant over 2π, and the boldface

$$\mathbf{i} = \gamma_2\gamma_1 = i\gamma_3\gamma_0 \tag{3.3}$$

is a constant bivector which corresponds to the unit imaginary in the matrix formulation of the Dirac equation. Equation (3.2) is called the *real form* of the Dirac equation, because it does not involve complex numbers; it involves only elements of STA, which all have definite geometrical meaning.

The physical meaning of ψ is determined by assumptions which relate it to observables. From (3.1) we can construct the invariant

$$\psi\widetilde{\psi} = \rho e^{i\beta}\,, \tag{3.4}$$

but this does not elucidate physical meaning. The *Dirac probability current* J is defined by

$$J = \psi\gamma_0\widetilde{\psi} = \rho v\,, \tag{3.5}$$

where

$$v = R\gamma_0\widetilde{R} \tag{3.6}$$

is the local velocity, so $\rho = \rho(x)$ can be interpreted as the proper probability density. The Dirac equation implies the *probability conservation* law

$$\Box \cdot J = \Box \cdot (\rho v) = 0\,. \tag{3.7}$$

The wave function determines everywhere a "comoving frame" $\{e_\mu = e_\mu(x)\}$ by

$$\psi\gamma_\mu\widetilde{\psi} = \rho R\gamma_\mu\widetilde{R} = \rho e_\mu\,, \tag{3.8}$$

which, of course, includes (3.5). The *spin vector* $s = s(x)$ is defined by

$$s = \frac{\hbar}{2}\,R\gamma_3\widetilde{R} = \frac{\hbar}{2}\,e_3\,. \tag{3.9}$$

Actually, angular momentum is a bivector quantity and (as demonstrated in [8]) the appropriate spin bivector $S = S(x)$ is given by

$$S = \frac{1}{2}\,R\mathbf{i}\hbar\widetilde{R} = \frac{\hbar}{2}\,R\gamma_2\gamma_1\widetilde{R} = \frac{\hbar}{2}\,e_2e_1 = isv\,. \tag{3.10}$$

This shows exactly the sense in which the imaginary factor $\mathbf{i}\hbar$ in the Dirac theory (and hence in the Schrödinger Theory) *can be interpreted as a representation of the electron spin*. The right side of (3.10) relates the spin vector to the spin bivector, so either of them can be used to describe spin.

The conservation law (3.7) implies that through each point x where $\psi(x) \neq 0$ there is a unique world line (or *bicharacteristic*) of ψ with tangent $v(x)$. Along each world line, by (3.9) and (3.6) the unimodular spinor $R(x)$ uniquely determines the direction of electron spin as well as the velocity, exactly as in the classical case discussed previously. Thus, the spinor R can be given the same well-defined kinematic interpretation in both classical and quantum theory.

The assumptions made so far provide a kinematic interpretation for the factor R in the wave function. As detailed in [8], one more assumption is needed to complete the Dirac theory. That is the crucial assumption introducing the *energy-momentum* operator $\underline{p}_\mu$ defined in STA by

$$\underline{p}_\mu\psi = \partial_\mu\psi\mathbf{i}\hbar - eA_\mu\psi\,. \tag{3.11}$$

(The underbar in $\underline{p}_\mu$ is meant to indicate a linear operator.) This operator can be introduced by the following definition of the electron *energy-momentum tensor* components

$$T_{\nu\mu} = \langle\, \gamma_0 \widetilde{\psi} \gamma_\nu \underline{p}_\mu \psi \,\rangle = \langle\, \gamma_\nu (\underline{p}_\mu \psi) \gamma_0 \widetilde{\psi} \,\rangle\,, \tag{3.12}$$

where $\langle\, M \,\rangle$ means "scalar part of M." Accordingly, the energy momentum flux through a hyper-surface with normal γ_ν is $\underline{T}\gamma_\nu = T_{\nu\mu}\gamma^\mu$. So the proper *energy momentum density* ρp, defined as the flux in the direction v, is given by

$$\rho p = \underline{T} v = v^\nu T_{\nu\mu} \gamma^\mu\,, \tag{3.13}$$

where $v^\nu = \gamma^\nu \cdot v$. The components of the energy-momentum density are therefore given by

$$\rho p_\mu = \rho p \cdot \gamma_\mu = v^\nu T_{\nu\mu} = \langle\, v (\underline{p}_\mu \psi) \gamma_0 \widetilde{\psi} \,\rangle\,. \tag{3.14}$$

Insight into the kinematic significance of this relation is obtained by introducing the canonical decomposition for ψ.

From the fact that R is a unimodular spinor, it can be proved that its partial derivatives have the form

$$\partial_\mu R = \gamma_\mu \cdot \Box R = \tfrac{1}{2} \Omega_\mu R\,, \tag{3.15}$$

where $\Omega_\mu = -\widetilde{\Omega}_\mu$ is a bivector quantity representing the rate of rotation for a displacement in the direction γ_μ. The derivatives of the frame $\{e_\mu\}$ in (3.9) can therefore be put in the form

$$\partial_\mu e_\nu = \Omega_\mu \times e_\nu = \Omega_\mu \cdot e_\nu\,, \tag{3.16}$$

where $\times$ signifies the commutator product defined by $A \times B = \frac{1}{2}(AB - BA)$. Similarly, the derivatives of the spin (3.10) have the form

$$\partial_\mu S = \Omega_\mu \times S\,. \tag{3.17}$$

Now with the help of (3.10), (3.15) and (3.17) we can write

$$\partial_\mu R \mathbf{i}\hbar \widetilde{R} = \Omega_\mu S = P_\mu + i q_\mu + \partial_\mu S\,, \tag{3.18}$$

where P_μ and q_μ are defined by

$$P_\mu = \Omega_\mu \cdot S\,, \tag{3.19}$$

$$i q_\mu = \Omega_\mu \wedge S\,. \tag{3.20}$$

Finally, using (3.18) to evaluate $\partial_\mu \psi$ in (3.14) we get

$$p_\mu = P_\mu - eA_\mu = \Omega_\mu \cdot S - eA_\mu\,. \tag{3.21}$$

This result is striking, because it shows that p_μ has a purely kinematical dependence on the wave function independent of the parameters ρ and β. It shows that p_μ depends on the rotation rate Ω_μ only through its projection onto the spin plane. This, in turn, gives insight into the geometrical meaning of quantization in stationary states; for the requirement that the wave function be single-valued implies that the comoving frame (3.8) be single-valued, so along any closed curve it must rotate an integral number of times; according to (3.21), therefore, we should have

$$\oint (p + eA) \cdot dx = \oint P \cdot dx = \oint (dx^\mu \Omega_\mu) \cdot S = \tfrac{1}{2} n\hbar\,, \tag{3.22}$$

where n is an integer. Thus, we have a geometric interpretation of the electron energy-momentum vector and its relation to quantization. That is what we need to separate probabilistic and kinematical components of the Dirac theory in the next section.

Separating Kinematics from Probabilities

The STA formulation of the Dirac theory in the preceding section suggests that the wave function decomposition $\psi = R(\rho e^{i\beta})^{\frac{1}{2}}$ has a fundamental physical significance besides being "relativistically invariant." For it reveals that the kinematics of electron motion (specifically the values of velocity, spin and energy-momentum) are completely determined by the unimodular spinor R and its derivatives $\partial_\mu R = \frac{1}{2}\Omega R$. On the other hand, ρ has an obvious probabilistic interpretation and we shall see how β might be given one as well. Thus, it appears that ψ can be decomposed into a purely kinematic factor R and a probabilistic factor $(\rho e^{i\beta})^{\frac{1}{2}}$. These factors are coupled by the Dirac equation in a way which is difficult to interpret physically [8]. However, it appears possible to decouple them completely by using the superposition principle as explained below.

Let us suppose that unimodular solutions of the Dirac equation have a fundamental physical significance and sanctify this by calling them *pure states*. A pure state $\psi = R$ satisfies $\psi\widetilde{\psi} = 1$. The most common example of a pure state state is a plane wave, but there are more interesting examples as we shall see. For a plane wave it is ordinarily supposed that the probability density $\rho(x)$ is uniform, hence unnormalizable, so there is some question as to its physical significance. Anyway, it is argued that in a plane wave state the particle position is indeterminate so that, in accordance with the uncertainty principle, the momentum and velocity have definite values. We shall introduce a different interpretation of pure states.

Each pure state $\psi(x)$ determines a unique velocity $v(x)$, spin $S(x)$ and momentum $p(x)$ at each spacetime location x. Let us assume that the electron is a point particle and this is the state of motion to be attributed to it if it is at a given location x. Thus, the pure state assigns a definite state of motion to every possible electron location. The dynamics of electron motion are incorporated into the pure state by requiring that it be a solution of the Dirac equation. This much is a purely deterministic model of electron motion.

Probabilities enter the theory by assuming that we do not know precisely the electron's pure state or location so the state of our knowledge is best described as a weighted average of pure states indexed by some parameter λ, e.g.

$$\psi = \int d\lambda\, w_\lambda \psi_\lambda = R(\rho e^{i\beta})^{\frac{1}{2}} \tag{4.1}$$

It is easy to prove that, in general, a superposition of pure states (with $\psi\widetilde{\psi} = 1$) produces a state with $\psi\widetilde{\psi} = \rho e^{i\beta}$ so the factor $e^{i\beta}$ arises naturally along with the probability density. Another way to see that the parameter ρ alone is not enough to describe the result of averaging process is to suppose that an expected velocity $\bar{v}(x)$ at x is to be obtained by averaging velocities $\bar{v}_\lambda(x)$. To satisfy the relativistic constraint $\bar{v}^2 = v_\lambda^2 = 1$, the average must have the form

$$\int d\lambda\, \bar{v}_\lambda = \bar{v}\cos\alpha\,, \tag{4.2}$$

where α could possibly be identified with β. This argument is meant to be suggestive only. The main idea is that the $e^{i\beta}$ in (3.1) is the *statistical factor* that arises from the unimodularity constraint on the kinematical factor of the wave function.

A similarity to Feynman's path integral formulation of quantum mechanics appears by interpreting the integral in (4.1) as a sum over paths with unit weight factor $w_\lambda = 1$. Feynman assumed

$$\psi = e^{\mathbf{i}\phi_\lambda/\hbar}\,,$$

where ϕ_λ is the classical action along the λ-path. Our theory suggests that this phase factor should be generalized to a unimodular spinor to account for spin. In that case also the "statistical factor" on the right side of (4.1) would arise from superposition. This promising generalization of the Feynman approach will not be pursued here, although the following sections contain hints on how to carry it out. The main point is that the unimodular spinors play a fundamental role in the Feynman approach just as they do in the present approach. The basic statistical problem is how to assign appropriate statistical weights to alternative world lines.

It remains to be seen whether the statistical notions suggested here can be accommodated by the Universal Probability Calculus in a natural way. This issue should be set within the general question of how best to reconcile probability calculus with relativity. The probability densities ordinarily used in probability calculus are not relativistically invariant, so they must be replaced by probability currents which are. This generalization from scalar densities to vectorial currents raises questions about how such quantities as entropy should be defined in terms of currents.

On the other hand, since a probability current $J = J(x)$ is a timelike vector field, it can always be written in the form

$$J = \psi\gamma_0\widetilde{\psi}\,, \tag{4.3}$$

so the spinor field $\psi = \psi(x)$ can be taken as the fundamental descriptor of probabilistic state. This applies to classical as well as quantum statistical mechanics. The bilinearity of (4.3) is a consequence of spacetime geometry alone. However, it raises a question about how to compute statistical averages. In accordance with the superposition principle, or at least with the linearity of the Dirac equation, quantum mechanics constructs composite states by averaging over spinor wave functions, whereas classical statistical mechanics takes averages over vector and tensor quantities. What general principle of probability theory will tell us which kind of average to consider?

Let me suggest that the answer is to be found in the way that physical information about pure states is specified. In The Dirac Theory the relation among pure state observables is specified by the Dirac equation. Therefore, *in constructing a composite state to express our uncertainty* as to which pure state describes a given electron, we should *require that the form of the Dirac equation be preserved*. Considering the linearity of the Dirac equation, there is evidently no alternative to the linear superposition (4.1). This makes the superposition principle of quantum mechanics seem much less fundamental, and it directs us to a study of the pure state Dirac equation to understand the underlying dynamics of quantum mechanics.

Unimodular Solutions of the Dirac Equation

In this section we study general properties of unimodular (pure state) solutions of the Dirac equation. In particular, we seek to determine the equations of motion for a particle with spin moving along a bicharacteristic of the wave function.

As noted before, the unimodular condition $\psi\widetilde{\psi} = 1$ implies that

$$\partial_\mu \psi = \tfrac{1}{2}\Omega_\mu \psi \,, \tag{5.1}$$

where the Ω_μ are bivectors. Consequently the rate of change of along a bicharacteristic is given by

$$\dot{\psi} = v \cdot \Box\psi = \tfrac{1}{2}\Omega\psi \,, \tag{5.2}$$

where $\Omega = v^\mu \Omega_\mu$. The equations of motion (2.13) for a comoving frame $\{e_\mu = \psi\gamma_\mu\widetilde{\psi}\}$ on a bicharacteristic $x = x(\tau)$ are determined by expressing Ω as a function of $x(\tau)$. To that end, we note that Ω can be obtained from the identity (derived in [8])

$$\Omega = 2\dot{\psi}\widetilde{\psi} = (\Box\psi)\widetilde{\psi}v + v(\Box\psi)\widetilde{\psi} - \Box v \,. \tag{5.3}$$

Now, using (3.6) and (3.10), the Dirac equation (3.2) can be put in the form

$$\Box\psi\mathbf{i}\hbar\psi = 2(\Box\psi)\widetilde{\psi}S = mv - eA \,. \tag{5.4}$$

Using this in (5.3) we obtain

$$\Omega = -\Box v + (m - ev \cdot A)S^{-1} \,. \tag{5.5}$$

Since $\Box v = \Box \wedge v + \Box \cdot v$, the scalar part of (5.5) gives the Dirac current conservation equation

$$\Box \cdot v = 0 \,. \tag{5.6}$$

Thus (5.5) reduces the problem of evaluating Ω to evaluating $\Box \wedge v$.

We can express the Dirac equation as a constitutive relation among observables by using (3.18) and

$$p = P + eA \tag{5.7}$$

to put the Dirac equation (5.4) in the form

$$p - iq + \Box S = mv \,. \tag{5.8}$$

Separately equating vector and trivector parts, we obtain

$$p = mv - \Box \cdot S \,, \tag{5.9}$$

and

$$iq = \Box \wedge S. \tag{5.10}$$

We can regard (5.9) as a constitutive equation for the momentum p in terms of the velocity and spin.

Further properties of observables can be derived by diffentiating the Dirac relation (5.8) to get

$$\Box p + i\Box q + \Box^2 S = m\Box v \,. \tag{5.11}$$

Separating this into scalar, pseudoscalar and bivector parts and using (5.6), (5.7) we obtain

$$\Box \cdot p = \Box \cdot P + e\Box \cdot A = m\Box \cdot v = 0 \,, \tag{5.12}$$

$$\Box \cdot q = 0\,, \tag{5.13}$$

$$m\Box \wedge v = \Box^2 S - eF + (\Box \wedge P + i\Box \wedge q)\,. \tag{5.14}$$

This last equation shows explicity how $\Box \wedge v$ depends on the E.M. field $F = \Box \wedge A$, but we still need to evaluate the terms in parenthesis. That can be done as follows.

By (5.1), the "integrability condition"

$$\partial_\nu \partial_\mu \psi = \partial_\mu \partial_\nu \psi$$

can be cast in the form

$$\partial_\nu \Omega_\mu - \partial_\mu \Omega_\nu = \Omega_\nu \times \Omega_\mu\,. \tag{5.15}$$

And by (3.18) it can be cast in the alternative form

$$\partial_\nu P_\mu - \partial_\mu P_\nu + i(\partial_\nu q_\mu - \partial_\mu q_\nu) = (\partial_\mu S \times \partial_\nu S) S^{-1}\,. \tag{5.16}$$

Whence

$$\Box \wedge P = (\partial_\mu S \times \partial_\nu S) \cdot S^{-1}(\gamma^\mu \wedge \gamma^\nu)\,, \tag{5.17}$$

$$i\Box \wedge q = (\partial_\mu S \times \partial_\nu S) \wedge S^{-1}(\gamma^\mu \wedge \gamma^\nu)\,. \tag{5.18}$$

This enables the far right part of (5.14) to be expressed solely in terms of the spin S and its derivatives. To make further simplifications we need to know something specific about the spin derivatives $\partial_\mu S$. That's next.

Classical Solutions of the Dirac Equation

Now we look for a special class of pure state solutions to the Dirac equation by requiring that variations in the spin $S = S(x)$ be due solely to motion along the bicharacteristics. This can be expressed mathematically by the requirement

$$\partial_\mu S = v_\mu v \cdot \Box S = v_\mu \dot{S}\,. \tag{6.1}$$

Let us call this the *decoupling condition*, because it implies that neighboring bicharacteristics are not coupled by an exchange of spin angular momentum. It follows that

$$\Box S = v\dot{S}\,, \tag{6.2}$$

$$\Box^2 S = v\ddot{S}\,. \tag{6.3}$$

Furthermore, from (5.17) and (5.18)

$$\Box \wedge P = 0\,, \tag{6.4}$$

$$\Box \wedge q = 0\,. \tag{6.5}$$

It must be understood that these are local conditions which may be violated at singular points; otherwise (6.4) is inconsistent with the quantization condition (3.23) because of Stokes' Law

$$\int d^2x \cdot (\Box \wedge P) = \oint dx \cdot P\,, \tag{6.6}$$

where d^2x is a bivector-valued directed area element.

Substituting (6.3) to (6.5) into (5.14) we get

$$m\Box \wedge v = \ddot{S} - eF\,, \tag{6.7}$$

so (5.5) yields the desired result

$$m\Omega = eF - \ddot{S} + m(m - eA\cdot v)S^{-1}\,. \tag{6.8}$$

This gives us immediately the bicharacteristic equations of motion for velocity and spin:

$$m\dot{v} = (eF - \ddot{S})\cdot v\,, \tag{6.9}$$

$$m\dot{S} = (eF - \ddot{S})\times S\,. \tag{6.10}$$

Since (6.9) and (6.10) are well-defined, deterministic equations of motion along any specific bicharacteristic, they may be regarded as classical equations of motion for a point particle. Accordingly, let us refer to the corresponding pure state spinor ψ as a *classical solution* of the Dirac equation.

To find out more about these classical solutions, we look to the Dirac equation itself. Applying (6.2) to (5.9) we get the "Wessenhoff relation"

$$p = mv + \dot{S}\cdot v\,. \tag{6.11}$$

This shows that the momentum of the particle is not generally collinear with the velocity, because it includes a contribution from the spin. Nevertheless, (6.11) implies

$$p\cdot v = m\,. \tag{6.12}$$

Using this along with (5.7) we can now put (6.8) in the form

$$m\Omega = eF - \ddot{S} + m(v\cdot P)S^{-1}\,. \tag{6.13}$$

Now $\Box \wedge p = 0$ implies $p = \Box\phi$, where $\phi = \phi(x)$ may be recognized as the *phase* of the wave function ψ. Consequently $v\cdot P = v\cdot\Box\phi = \dot{\phi}$ is the rate of phase change along a bicharacteristic. Since

$$\dot{\phi} = v\cdot P = m - eA\cdot v\,, \tag{6.14}$$

we can find the phase change by direct integration after the bicharacteristics have been found.

Equations (5.13) and (6.5) combine to give

$$\Box q = 0\,, \tag{6.15}$$

while (6.2) and (5.10) give

$$iq = v\wedge\dot{S}\,. \tag{6.16}$$

It can be shown that equation (6.15) implies that q is constant if and only if (6.15) holds everywhere without any singularity. It seems likely, therefore, that we can impose the simplifying condition

$$q = 0 \tag{6.17a}$$

without unduly restricting the class of physically significant solutions. So let us consider its implications. According to (6.16), then, we have

$$v \wedge \dot{S} = 0\,. \tag{6.17b}$$

Adding this to (6.11), we obtain

$$p = mv + \dot{S}v\,. \tag{6.18}$$

Multiplying by v and taking the bivector part, we obtain

$$\dot{S} = p \wedge v\,. \tag{6.19}$$

Equations of motion for the spin and velocity of the form (6.19) and (6.9) have been derived by Wessenhoff [9] and Corben [10] as a classical model of a particle with spin analogous to the Dirac electron. Here we have ascertained for the first time the conditions under which these equations may be *exact* consequences of the Dirac theory. However, the present theory differs from their model in some important respects. First, the mass m, which is rigorously constant in (6.9) and (6.18), is allowed to be variable in the Wessenhoff-Corben equations. Second, we have here the additional feature of a wave function with variable phase determined by the last term on the right side of (6.8).

The momentum $p = P - eA$ is most easily related to the rotational velocity by (3.18), which gives us

$$\partial_\mu \psi \mathbf{i}\hbar\widetilde{\psi} = \Omega_\mu S = P_\mu + v_\mu \dot{S}\,, \tag{6.20}$$

whence

$$\Omega_\mu S = P \cdot v + \dot{S}\,. \tag{6.21}$$

For later use, it should be noted that the geometric product of two bivectors Ω and S can be decomposed into scalar, bivector and pseudoscalar parts by means of the identity $\Omega S = \Omega \cdot S + \Omega \times S + \Omega \wedge S$. On using (6.18) to eliminate $\dot{S}$ (6.21) yields

$$\Omega S = pv + eA \cdot v\,, \tag{6.22}$$

an algebraic relation among all the basic observables which is readily solved for any one of them in terms of the others. Alternatively, elimination of Ω from (6.21) by (6.13) yields

$$m\dot{S} = (eF - \ddot{S})S\,. \tag{6.23}$$

The bivector part of this has already been found in (6.10). However, the scalar and pseudoscalar parts yield the relations

$$\ddot{S} \cdot S = eF \cdot S = -\dot{S}^2\,, \tag{6.24}$$

$$\ddot{S} \wedge S = eF \wedge S\,. \tag{6.25}$$

The last equality in (6.24) follows from the fact that S^2 is constant so $S \cdot \dot{S} = 0$. The pseudoscalar part of (6.21) gives

$$\Omega \wedge S = 0\,. \tag{6.26}$$

This is a consequence of the condition $q = 0$, though it also follows from the weaker condition $q \cdot v = 0$, which is already entailed by (6.16). To understand its significance, consider the identity

$$v \cdot (\Omega \wedge S) = (v \cdot \Omega) \wedge S + \Omega \wedge (v \cdot S)\,.$$

Since $v \cdot S = 0$, this shows that (6.26) implies

$$\dot{v} \wedge S = 0\,. \tag{6.27}$$

This means that $\dot{v}$ lies in the S-plane. Combining (6.27) with (6.17b) we find that

$$v \cdot \Box\,(v \wedge S) = 0\,. \tag{6.28}$$

Thus, the condition $q = 0$ implies that the trivector $S \wedge v = Sv = is$ is a constant of motion along the bicharacteristics. Evidently it is too strong a condition for motion in arbitrary E.M. fields, but it may hold in special cases. It can be shown that (6.25) is a consequence of this condition.

To find an equation of motion for the momentum we differentiate (6.11) and use (6.9) to get

$$\dot{p} = eF \cdot v + \dot{S} \cdot \dot{v}\,. \tag{6.29}$$

The last term here is an unusual one in dynamics, but note that

$$v \cdot \dot{p} = v \cdot \dot{S} \cdot \dot{v} = \dot{S} \cdot (\dot{v} \wedge v)\,. \tag{6.30}$$

Moreover, from the condition (6.17b) we get

$$v \cdot (v \wedge \dot{S}) = \dot{S} - v \wedge (v \cdot \dot{S}) = 0\,, \tag{6.31}$$

whence

$$\dot{v} \cdot \dot{S} = (v \cdot S \cdot \dot{v})v\,. \tag{6.32}$$

So (6.29) becomes

$$\dot{p} = eF \cdot v + (\dot{p} \cdot v)v\,. \tag{6.33}$$

The absence of the last term in (6.33) is responsible for the variable mass in the Wessenhoff-Corben model.

Finally, to justify the interpretation of S as intrinsic angular momentum, we define the total angular momentum

$$M = x \wedge p + S\,, \tag{6.34}$$

where $x \wedge p$ is the orbital angular momentum. Differentiating with $v \cdot \Box$ and using (6.19) we obtain the equation for *angular momentum conservation*

$$\dot{M} = x \wedge \dot{p}\,, \tag{6.35}$$

the right side being the generalized torque.

The Zitterbewegung

Now we can throw new light on the most basic "classical solution" of the Dirac equation, the free particle. In the absence of an external E.M. field, we have immediately from (6.33) and (6.35) that the momentum p and the total angular momentum $M = x\wedge p + S$ are constants of motion. From (6.22) combined with (6.18) we have

$$\Omega S = pv = m + \dot{S}\,. \tag{7.1}$$

Its scalar part

$$\Omega\cdot S = p\cdot v = m \tag{7.2}$$

integrates immediately to

$$p\cdot x = m\tau\,. \tag{7.3}$$

This defines a proper time $\tau = \tau(x)$ on the bicharacteristic passing through any given location x.

Multiplying (7.1) by its reverse, we obtain

$$p^2 = m^2 - \dot{S}^2 = \Omega^2 S^2 = -\frac{\hbar^2}{4}\,\Omega^2\,. \tag{7.4}$$

This determines $|\,\Omega\,| = 2|\,p\,|\hbar^{-1}$ and incidentially shows that $\dot{S}^2$ is constant. Solving (7.1) for Ω we obtain

$$\Omega = pvS^{-1} = p(v\wedge S)S^{-2}\,. \tag{7.5}$$

According to (6.28), $v\wedge S$ is a constant of the motion, so Ω is constant as well. Consequently the spinor equation

$$\dot{\psi} = \tfrac{1}{2}\Omega\psi \tag{7.6}$$

integrates immediately to

$$\psi = e^{\frac{1}{2}\Omega\tau}R_0 = e^{\frac{1}{2}m^{-1}\Omega p\cdot x}\,R_0\,, \tag{7.7}$$

where R_0 is a constant unimodular spinor. Specifically, $\psi = R_0$ on the spacelike hyperplane $m\tau = p\cdot x = 0$. The velocity and spin are thus given by

$$v = e^{\frac{1}{2}\Omega\tau}v_0 e^{-\frac{1}{2}\Omega\tau}\,, \tag{7.8}$$

$$S = e^{\frac{1}{2}\Omega\tau}S_0 e^{-\frac{1}{2}\Omega\tau}\,, \tag{7.9}$$

where $v_0 = R_0\gamma_0\widetilde{R}_0$ and $S_0 = \frac{1}{2}R_0\mathbf{i}\hbar\widetilde{R}_0$. It should be mentioned that (7.7) is not necessarily a full solution of the Dirac equation, but only the result of integrating (7.6) along bicharacteristics.

The bicharacteristics can be found by integrating (7.8), but a better way is to combine (7.3) with the total angular momentum as follows

$$x\wedge p + x\cdot p = xp = M - S + m\tau\,.$$

Multiplication by $p^{-1} = p/p^2$ gives

$$x = (M - S)p^{-1} + m\tau p^{-1}\,, \tag{7.10}$$

which becomes an explicit function $x = x(\tau)$ when (7.9) is inserted. To show the character of the solution more explicitly, note that (7.1) yields $p^{-1}\Omega = vS^{-1}$, so $v \cdot S = 0$ implies

$$p \cdot \Omega = 0, \quad \text{or} \quad p\Omega = \Omega p\,. \tag{7.11}$$

Consequently,

$$\begin{aligned} p \cdot S &= e^{\frac{1}{2}\Omega\tau} p \cdot S_0\, e^{-\frac{1}{2}\Omega\tau} \\ &= [(p \cdot S_0) \wedge \Omega + e^{\Omega\tau}(p \cdot S_0) \cdot \Omega]\,\Omega^{-1}\,. \end{aligned} \tag{7.12}$$

Noting that $M = x_0 p + S_0$ on the hyperplane $x_0 \cdot p = 0$ and defining

$$r_0 = [(p^{-1} \cdot S_0) \cdot \Omega]\,\Omega^{-1}\,, \tag{7.13}$$

we can put (7.10) in the form

$$x = x_0 - r_0 + e^{\Omega\tau} r_0 + m\tau p^{-1}\,. \tag{7.14}$$

This is the parametric equation $x = x(\tau)$ for a timelike helix with axis $x_0 - r_0 + m\tau p^{-1}$ and squared radius $|\, r_0 \,|^2 = -r_0^2$. This is the same type of helical orbit found by Wessenhoff and Corben.

This peculiar helical motion, unsupported by an external E.M. field, can be identified with the *zitterbewegung* originally attributed to the electron by Schrödinger [6]. Its physical significance is problematic. The radius of the zitterbewegung vanishes when $\Omega S = \Omega{\cdot}S = m$, so

$$\Omega = mS^{-1} = \frac{-2m}{\hbar}\, R_0 \mathbf{i} \widetilde{R}_0\,. \tag{7.15}$$

In that case (7.7) becomes

$$\psi = R_0 e^{-\mathbf{i}p\cdot x/\hbar}\,, \tag{7.16}$$

which will be recognized as the usual free particle solution to the Dirac equation. It has been suggested [6] that the phase factor in (7.16) also describes a helical zitterbewegung and that the Dirac theory should be modified to show it. If that is correct, then the zitterbewegung must be the source of the electron's magnetic moment and other feature of quantum theory [6]. The zitterbewegung must then be truly fundamental. That remains to be seen!

Conclusions

Reformulation of the Dirac Theory in terms of Spacetime Algebra reveals a hidden geometric structure and opens up a possibility for separating objective and subjective components of quantum mechanics. More specifically, we have noted the following:

(1) The Dirac wave function ψ has a Lorentz invariant decomposition

$$\psi = R(\rho e^{i\beta})^{\frac{1}{2}}$$

where R is a unimodular spinor which completely characterizes the kinematics of electron motion.

(2) The Dirac current J is given by

$$J = \psi\gamma_0\widetilde{\psi} = \rho R\gamma_0\widetilde{R} \, .$$

In fact, any timelike vector field $J = J(x)$ can be expressed in terms of a spinor field $\psi = \psi(x)$ in exactly this way. Therefore, the bilinear dependance of the probability density on the wave function is not a special feature of quantum mechanics. Rather, it is a consequence of spacetime geometry (as represented by the spacetime algebra). This decomposition into spinors could be applied as well to probability currents in classical relativistic statistical mechanics.

(3) The electron spin angular momentum S is given by

$$S = \tfrac{1}{2}R\mathbf{i}\hbar\widetilde{R} \, .$$

This reveals that the unit imaginary $\mathbf{i}$ in the quantum mechanics of electrons (at least!) is a bivector quantity. The ubiquitous factor $\frac{1}{2}\mathbf{i}\hbar$ represents the spin in a standard orientation, and the spinor field $R = R(x)$ rotates it into the local spin direction at each spacetime location. That is why $\mathbf{i}$ and $\hbar$ always appear together in the fundamental equations of the Dirac theory (and, perhaps, of quantum mechanics in general). This supports and expands Dirac's insight that the most fundamental aspect of quantum mechanics is the role of $i = \sqrt{-1}$ [11].

(4) The electron energy-momentum vector $p = p(x)$ is given by:

$$(p - eA)\cdot\gamma_\mu = \Omega_\mu \cdot S = 2[(\partial_\mu R)\widetilde{R}]\cdot S \, ,$$

where $A = A(x)$ is the external E.M. potential. Thus, it has a purely kinematic interpretation.

(5) The bicharacteristics of the Dirac wave function (tangent to the Dirac current) are interpreted as predicted electron world lines. This is not a new idea. Bohm and Hiley, among others, have argued forcefully that the identification of bicharacteristics of the Schrödinger wave function with possible electron paths leads to sensible particle interpretations of electron interference and tunneling as well as other aspects of Schrödinger electron theory [12].

(6) It is suggested that unimodular (pure state) solutions of the Dirac equation have a purely objective physical interpretation, while the factor $(\rho e^{i\beta})^{\frac{1}{2}}$ has a subjective probabilistic interpretation, and it arises from pure states by superposition.

(7) It is suggested that the superposition principle is simply a consequence of requiring that the form of the Dirac equation be preserved in the construction of statistical composites of pure states.

(8) Pure states exhibit circular zitterbewegung which may be the origin of the electron spin and magnetic moment, but the Dirac theory must be modified if that interpretation is to be upheld [6].

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