3. Spherical Conformal Geometry with Geometric Algebra†

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3.1 Introduction

The recorded study of spheres dates back to the first century in the book *Sphaerica* of Menelaus. Spherical trigonometry was thoroughly developed in modern form by Euler in his 1782 paper [E1782]. Spherical geometry in n-dimensions was first studied by Schläfli in his 1852 treatise, which was published posthumously in [S1901]. The most important transformation in spherical geometry, the Möbius transformation, was considered by Möbius in his 1855 paper [M1855].

Hamilton was the first to apply vectors to spherical trigonometry. In 1987 Hestenes [H98] formulated a spherical trigonometry in terms of Geometric Algebra, and that remains a useful supplement to the present treatment.

This chapter is a continuation of the preceding chapter. Here we consider the homogeneous model of spherical space, which is similar to that of Euclidean space. We establish conformal geometry of spherical space in this model, and discuss several typical conformal transformations.

Although it is well known that the conformal groups of n-dimensional Euclidean and spherical spaces are isometric to each other, and are all isometric to the group of isometries of hyperbolic $(n+1)$-space [K1872], [K1873] spherical conformal geometry has its unique conformal transformations, and it can provide good understanding for hyperbolic conformal geometry. It is an indispensible part of the unification of all conformal geometries in the homogeneous model, which is addressed in the next chapter.

3.2 Homogeneous model of spherical space

In the previous chapter, we saw that, given a null vector $e \in \mathcal{R}^{n+1,1}$, the intersection of the null cone $\mathcal{N}^n$ of $\mathcal{R}^{n+1,1}$ with the hyperplane $\{x \in \mathcal{R}^{n+1,1} \mid x\cdot e = -1\}$

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represents points in $\mathcal{R}^n$. This representation is established through the projective split of the null cone with respect to null vector e.

What if we replace the null vector e with any nonzero vector in $\mathcal{R}^{n+1,1}$? This section shows that when e is replaced by a unit vector p_0 of negative signature, then the set

$$\mathcal{N}^n_{p_0} = \{x \in \mathcal{N}^n | x \cdot p_0 = -1\} \tag{3.1}$$

represents points in the n-dimensional spherical space

$$\mathcal{S}^n = \{x \in \mathcal{R}^{n+1} | x^2 = 1\}. \tag{3.2}$$

The space dual to p_0 corresponds to $\mathcal{R}^{n+1} = \tilde{p}_0$, an $(n+1)$-dimensional Euclidean space whose unit sphere is $\mathcal{S}^n$.

Applying the orthogonal decomposition

$$x = P_{p_0}(x) + P_{\tilde{p}_0}(x) \tag{3.3}$$

to vector $x \in \mathcal{N}^n_{p_0}$, we get

$$x = p_0 + \mathbf{x} \tag{3.4}$$

where $\mathbf{x} \in \mathcal{S}^n$. This defines a bijective map $i_{p_0} : \mathbf{x} \in \mathcal{S}^n \longrightarrow x \in \mathcal{N}^n_{p_0}$. Its inverse map is $P^{\perp}_{p_0} = P_{\tilde{p}_0}$.

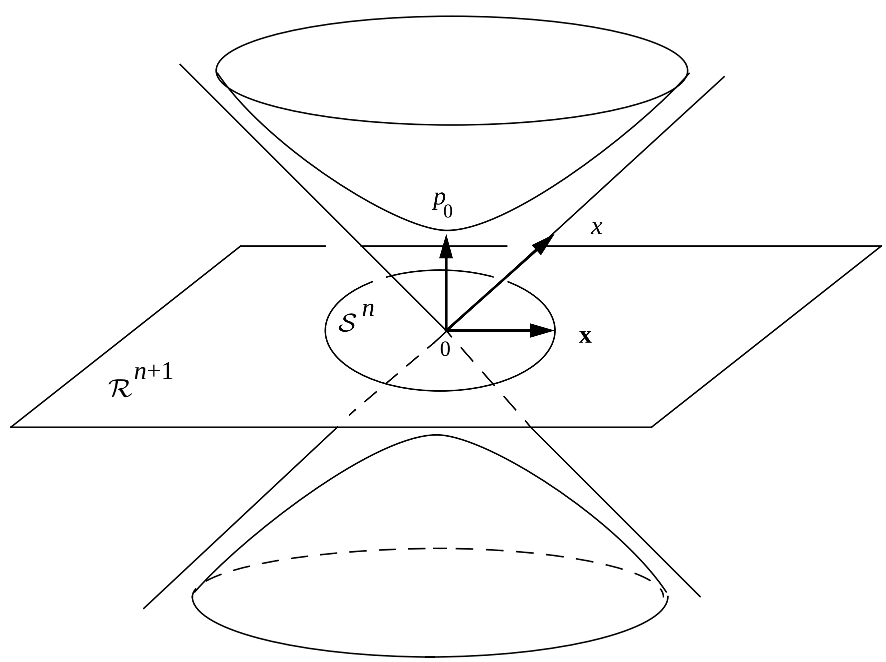

Figure 1: The homogeneous model of $\mathcal{S}^n$.

Theorem 1.

$$\mathcal{N}^n_{p_0} \simeq \mathcal{S}^n. \tag{3.5}$$

We call $\mathcal{N}^n_{p_0}$ the *homogeneous model* of $\mathcal{S}^n$. Its elements are called *homogeneous points*.

Distances

For two points $\mathbf{a}, \mathbf{b} \in \mathcal{S}^n$, their *spherical distance* $d(\mathbf{a}, \mathbf{b})$ is defined as

$$d(\mathbf{a}, \mathbf{b}) = \cos^{-1}\big(\mathbf{a} \cdot \mathbf{b}\big). \tag{3.6}$$

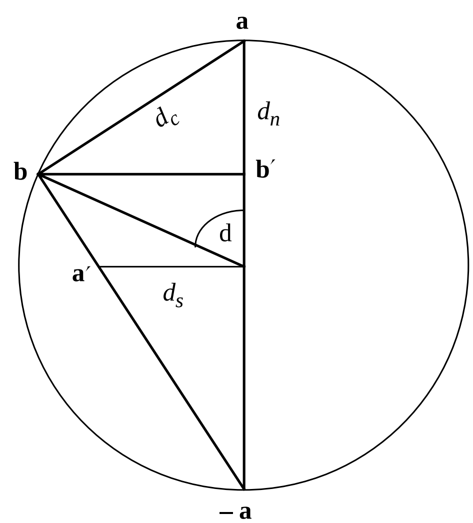

Figure 2: Distances in $\mathcal{S}^n$.

We can define other equivalent distances. Distances d_1, d_2 are said to be equivalent if for any two pairs of points $\mathbf{a}_1, \mathbf{b}_1$ and $\mathbf{a}_2$, $\mathbf{b}_2$, then $d_1(\mathbf{a}_1, \mathbf{b}_1) = d_1(\mathbf{a}_2, \mathbf{b}_2)$ if and only if $d_2(\mathbf{a}_1, \mathbf{b}_1) = d_2(\mathbf{a}_2, \mathbf{b}_2)$. The *chord distance* measures the length of the chord between $\mathbf{a}, \mathbf{b}$:

$$d_c(\mathbf{a}, \mathbf{b}) = |\mathbf{a} - \mathbf{b}|. \tag{3.7}$$

The *normal distance* is

$$d_n(\mathbf{a}, \mathbf{b}) = 1 - \mathbf{a} \cdot \mathbf{b}. \tag{3.8}$$

It equals the distance between points $\mathbf{a}, \mathbf{b}'$, where $\mathbf{b}'$ is the projection of $\mathbf{b}$ onto $\mathbf{a}$. The *stereographic distance* measures the distance between the origin 0 and $\mathbf{a}'$, the intersection of the line connecting $-\mathbf{a}$ and $\mathbf{b}$ with the hyperspace of $\mathcal{R}^{n+1}$ parallel with the tangent hyperplane of $\mathcal{S}^n$ at $\mathbf{a}$:

$$d_s(\mathbf{a}, \mathbf{b}) = \frac{|\mathbf{a} \wedge \mathbf{b}|}{1 + \mathbf{a} \cdot \mathbf{b}}. \tag{3.9}$$

Some relations among these distances are:

$$\begin{aligned} d_c(\mathbf{a},\mathbf{b}) &= 2\sin\frac{d(\mathbf{a},\mathbf{b})}{2}, \\ d_n(\mathbf{a},\mathbf{b}) &= 1-\cos d(\mathbf{a},\mathbf{b}), \\ d_s(\mathbf{a},\mathbf{b}) &= \tan\frac{d(\mathbf{a},\mathbf{b})}{2}, \\ d_s^2(\mathbf{a},\mathbf{b}) &= \frac{d_n(\mathbf{a},\mathbf{b})}{2-d_n(\mathbf{a},\mathbf{b})}. \end{aligned} \tag{3.10}$$

For two points $\mathbf{a}, \mathbf{b}$ in $\mathcal{S}^n$, we have

$$a \cdot b = \mathbf{a}\cdot\mathbf{b} - 1 = -d_n(\mathbf{a},\mathbf{b}). \tag{3.11}$$

Therefore the inner product of two homogeneous points a, b characterizes the normal distance between the two points.

Spheres and hyperplanes

A sphere is a set of points having equal distances with a fixed point in $\mathcal{S}^n$. A sphere is said to be *great*, or *unit*, if it has normal radius 1. In this chapter, we call a great sphere a *hyperplane*, or an $(n-1)$-*plane*, of $\mathcal{S}^n$, and a non-great one a *sphere*, or an $(n-1)$-*sphere*.

The intersection of a sphere with a hyperplane is an $(n-2)$-dimensional sphere, called $(n-2)$-*sphere*; the intersection of two hyperplanes is an $(n-2)$-dimensional plane, called $(n-2)$-*plane*. In general, for $1 \le r \le n-1$, the intersection of a hyperplane with an r-sphere is called an $(r-1)$-*sphere*; the intersection of a hyperplane with an r-plane is called an $(r-1)$-*plane*. A 0-plane is a pair of antipodal points, and a 0-sphere is a pair of non-antipodal ones.

We require that the normal radius of a sphere be less than 1. In this way a sphere has only one center. For a sphere with center $\mathbf{c}$ and normal radius ρ, its *interior* is

$$\{\mathbf{x} \in \mathcal{S}^n | d_n(\mathbf{x},\mathbf{c}) < \rho\}; \tag{3.12}$$

its *exterior* is

$$\{\mathbf{x} \in \mathcal{S}^n | \rho < d_n(\mathbf{x},\mathbf{c}) \le 2\}. \tag{3.13}$$

A sphere with center $\mathbf{c}$ and normal radius ρ is characterized by the vector

$$s = c - \rho p_0 \tag{3.14}$$

of positive signature. A point $\mathbf{x}$ is on the sphere if and only if $x \cdot c = -\rho$, or equivalently,

$$x \wedge \tilde{s} = 0. \tag{3.15}$$

Form (3.14) is called the *standard form* of a sphere.

A hyperplane is characterized by its *normal vector* $\mathbf{n}$; a point $\mathbf{x}$ is on the hyperplane if and only if $\mathbf{x} \cdot \mathbf{n} = 0$, or equivalently,

$$x \wedge \tilde{\mathbf{n}} = 0. \tag{3.16}$$

Theorem 2. *The intersection of any Minkowski hyperspace $\tilde{s}$ with $\mathcal{N}^n_{p_0}$ is a sphere or hyperplane in $\mathcal{S}^n$, and every sphere or hyperplane of $\mathcal{S}^n$ can be obtained in this way. Vector s has the standard form*

$$s = \mathbf{c} + \lambda p_0, \tag{3.17}$$

where $0 \leq \lambda < 1$. It represents a hyperplane if and only if $\lambda = 0$.

The dual theorem is:

Theorem 3. *Given homogeneous points $a_0, \ldots, a_n$ such that*

$$\tilde{s} = a_0 \wedge \cdots \wedge a_n, \tag{3.18}$$

then the $(n+1)$-blade $\tilde{s}$ represents a sphere in $\mathcal{S}^n$ if

$$p_0 \wedge \tilde{s} \neq 0, \tag{3.19}$$

or a hyperplane if

$$p_0 \wedge \tilde{s} = 0. \tag{3.20}$$

The above two theorems also provide an approach to compute the center and radius of a sphere in $\mathcal{S}^n$. Let $\tilde{s} = a_0 \wedge \cdots \wedge a_n \neq 0$, then it represents the sphere or hyperplane passing through points $\mathbf{a}_0, \ldots, \mathbf{a}_n$. When it represents a sphere, let $(-1)^\epsilon$ be the sign of $s \cdot p_0$. Then the center of the sphere is

$$(-1)^{\epsilon+1} \frac{P^\perp_{p_0}(s)}{|P^\perp_{p_0}(s)|}, \tag{3.21}$$

and the normal radius is

$$1 - \frac{|s \cdot p_0|}{|s \wedge p_0|}. \tag{3.22}$$

3.3 Relation between two spheres or hyperplanes

Let $\tilde{s}_1, \tilde{s}_2$ be two distinct spheres or hyperplanes in $\mathcal{S}^n$. The signature of the blade

$$s_1 \wedge s_2 = (\tilde{s}_1 \vee \tilde{s}_2)^\sim \tag{3.23}$$

characterizes the relation between the two spheres or hyperplanes:

Theorem 4. *Two spheres or hyperplanes $\tilde{s}_1, \tilde{s}_2$ intersect, are tangent, or do not intersect if and only if $(s_1 \wedge s_2)^2$ is less than, equal to or greater than 0, respectively.*

There are three cases:

Case 1. For two hyperplanes represented by $\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2$, since $\mathbf{n}_1 \wedge \mathbf{n}_2$ has Euclidean signature, the two hyperplanes always intersect. The intersection is an $(n-2)$-plane, and is normal to both $P^{\perp}_{\mathbf{n}_1}(\mathbf{n}_2)$ and $P^{\perp}_{\mathbf{n}_2}(\mathbf{n}_1)$.

Case 2. For a hyperplane $\tilde{\mathbf{n}}$ and a sphere $(\mathbf{c} + \lambda p_0)^{\sim}$, since

$$(\mathbf{n} \wedge (\mathbf{c} + \lambda p_0))^2 = (\lambda + |\mathbf{c} \wedge \mathbf{n}|)(\lambda - |\mathbf{c} \wedge \mathbf{n}|), \tag{3.24}$$

then:

- If $\lambda < |\mathbf{c} \wedge \mathbf{n}|$, they intersect. The intersection is an $(n-2)$-sphere with center $\dfrac{P^{\perp}_{\mathbf{n}}(\mathbf{c})}{|P^{\perp}_{\mathbf{n}}(\mathbf{c})|}$ and normal radius $1 - \dfrac{\lambda}{|\mathbf{c} \wedge \mathbf{n}|}$.
- If $\lambda = |\mathbf{c} \wedge \mathbf{n}|$, they are tangent at the point $\dfrac{P^{\perp}_{\mathbf{n}}(\mathbf{c})}{|P^{\perp}_{\mathbf{n}}(\mathbf{c})|}$.
- If $\lambda > |\mathbf{c} \wedge \mathbf{n}|$, they do not intersect. There is a pair of points in $\mathcal{S}^n$ which are inversive with respect to the sphere, while at the same time symmetric with respect to the hyperplane. They are $\dfrac{P^{\perp}_{\mathbf{n}}(\mathbf{c}) \pm \mu\mathbf{n}}{\lambda}$, where $\mu = \sqrt{\lambda^2 + (\mathbf{c} \wedge \mathbf{n})^2}$.

Case 3. For two spheres $(\mathbf{c}_i + \lambda_i p_0)^{\sim}$, $i = 1, 2$, since

$$((\mathbf{c}_1 + \lambda_1 p_0) \wedge (\mathbf{c}_2 + \lambda_2 p_0))^2 = (\mathbf{c}_1 \wedge \mathbf{c}_2)^2 + (\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2)^2, \tag{3.25}$$

then:

- If $|\mathbf{c}_1 \wedge \mathbf{c}_2| > |\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2|$, they intersect. The intersection is an $(n-2)$-sphere on the hyperplane of $\mathcal{S}^n$ represented by

  $$(\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2)^{\sim}. \tag{3.26}$$

  The intersection has center

  $$\frac{\lambda_1 P^{\perp}_{\mathbf{c}_2}(\mathbf{c}_1) + \lambda_2 P^{\perp}_{\mathbf{c}_1}(\mathbf{c}_2)}{|\lambda_1 \mathbf{c}_2 - \lambda_2 \mathbf{c}_1||\mathbf{c}_1 \wedge \mathbf{c}_2|} \tag{3.27}$$

  and normal radius

  $$1 - \frac{|\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2|}{|\mathbf{c}_1 \wedge \mathbf{c}_2|}. \tag{3.28}$$

- If $|\mathbf{c}_1 \wedge \mathbf{c}_2| = |\lambda_2 \mathbf{c}_1 - \lambda_1 \mathbf{c}_2|$, they are tangent at the point

  $$\frac{\lambda_1 P^{\perp}_{\mathbf{c}_2}(\mathbf{c}_1) + \lambda_2 P^{\perp}_{\mathbf{c}_1}(\mathbf{c}_2)}{|\mathbf{c}_1 \wedge \mathbf{c}_2|^2}. \tag{3.29}$$

- If $|\mathbf{c}_1 \wedge \mathbf{c}_2| < |\lambda_2\mathbf{c}_1 - \lambda_1\mathbf{c}_2|$, they do not intersect. There is a pair of points in $\mathcal{S}^n$ which are inversive with respect to both spheres. They are

$$\frac{\lambda_1 P^{\perp}_{\mathbf{c}_2}(\mathbf{c}_1) + \lambda_2 P^{\perp}_{\mathbf{c}_1}(\mathbf{c}_2) \pm \mu(\lambda_2\mathbf{c}_1 - \lambda_1\mathbf{c}_2)}{(\lambda_2\mathbf{c}_1 - \lambda_1\mathbf{c}_2)^2}, \tag{3.30}$$

  where $\mu = \sqrt{(\mathbf{c}_1 \wedge \mathbf{c}_2)^2 + (\lambda_2\mathbf{c}_1 - \lambda_1\mathbf{c}_2)^2}$. The two points are called the *Poncelet points* of the spheres.

The scalar

$$s_1 * s_2 = \frac{s_1 \cdot s_2}{|s_1||s_2|} \tag{3.31}$$

is called the *inversive product* of vectors s_1 and s_2. Obviously, it is invariant under orthogonal transformations in $\mathcal{R}^{n+1,1}$. We have the following conclusion for the geometric interpretation of the inversive product:

Theorem 5. *Let* $\mathbf{a}$ *be a point of intersection of two spheres or hyperplanes* $\tilde{s}_1$ *and* $\tilde{s}_2$*, let* m_i*,* $i = 1, 2$*, be the respective outward unit normal vector at* $\mathbf{a}$ *of* $\tilde{s}_i$ *if it is a sphere, or* $s_i/|s_i|$ *if it is a hyperplane, then*

$$s_1 * s_2 = m_1 \cdot m_2. \tag{3.32}$$

Proof. Given that s_i has the standard form $\mathbf{c}_i + \lambda_i p_0$. When $\tilde{s}_i$ is a sphere, its outward unit normal vector at point $\mathbf{a}$ is

$$\mathbf{m}_i = \frac{\mathbf{a}(\mathbf{a} \wedge \mathbf{c}_i)}{|\mathbf{a} \wedge \mathbf{c}_i|}, \tag{3.33}$$

which equals $\mathbf{c}_i$ when $\tilde{s}_i$ is a hyperplane. Point $\mathbf{a}$ is on both $\tilde{s}_1$ and $\tilde{s}_2$ and yields

$$\mathbf{a} \cdot \mathbf{c}_i = \lambda_i, \text{ for } i = 1, 2, \tag{3.34}$$

so

$$\mathbf{m}_1 \cdot \mathbf{m}_2 = \frac{\mathbf{c}_1 - \mathbf{a} \cdot \mathbf{c}_1\mathbf{a}}{\sqrt{1 - (\mathbf{a} \cdot \mathbf{c}_1)^2}} \cdot \frac{\mathbf{c}_2 - \mathbf{a} \cdot \mathbf{c}_2\mathbf{a}}{\sqrt{1 - (\mathbf{a} \cdot \mathbf{c}_2)^2}} = \frac{\mathbf{c}_1 \cdot \mathbf{c}_2 - \lambda_1\lambda_2}{\sqrt{(1 - \lambda_1^2)(1 - \lambda_2^2)}}. \tag{3.35}$$

On the other hand,

$$s_1 * s_2 = \frac{(\mathbf{c}_1 + \lambda_1 p_0) \cdot (\mathbf{c}_2 + \lambda_2 p_0)}{|\mathbf{c}_1 + \lambda_1 p_0||\mathbf{c}_2 + \lambda_2 p_0|} = \frac{\mathbf{c}_1 \cdot \mathbf{c}_2 - \lambda_1\lambda_2}{\sqrt{(1 - \lambda_1^2)(1 - \lambda_2^2)}}. \tag{3.36}$$

□

An immediate corollary is that any orthogonal transformation in $\mathcal{R}^{n+1,1}$ induces an angle-preserving transformation in $\mathcal{S}^n$. This conformal transformation will be discussed in the last section.

3.4 Spheres and planes of dimension r

We have the following conclusion similar to that in Euclidean geometry:

Theorem 6. *For $2 \leq r \leq n+1$, every r-blade A_r of Minkowski signature in $\mathcal{R}^{n+1,1}$ represents an $(r-2)$-dimensional sphere or plane in $\mathcal{S}^n$.*

Corollary *The $(r-2)$-dimensional sphere passing through r points $\mathbf{a}_1, \ldots, \mathbf{a}_r$ in $\mathcal{S}^n$ is represented by $a_1 \wedge \cdots \wedge a_r$; the $(r-2)$- plane passing through $r-1$ points $\mathbf{a}_1, \ldots, \mathbf{a}_{r-1}$ in $\mathcal{S}^n$ is represented by $p_0 \wedge a_1 \wedge \cdots \wedge a_{r-1}$.*

There are two possibilities:

Case 1. When $p_0 \wedge A_r = 0$, A_r represents an $(r-2)$-plane in $\mathcal{S}^n$. After normalization, the *standard form* of the $(r-2)$-plane is

$$p_0 \wedge \mathbf{I}_{r-1}, \tag{3.37}$$

where $\mathbf{I}_{r-1}$ is a unit $(r-1)$-blade of $\mathcal{G}(\mathcal{R}^{n+1})$ representing the minimal space in $\mathcal{R}^{n+1}$ supporting the $(r-2)$-plane of $\mathcal{S}^n$.

Case 2. A_r represents an $(r-2)$-dimensional sphere if

$$A_{r+1} = p_0 \wedge A_r \neq 0. \tag{3.38}$$

The vector

$$s = A_r A_{r+1}^{-1} \tag{3.39}$$

has positive square and $p_0 \cdot s \neq 0$, so its dual $\tilde{s}$ represents an $(n-1)$-dimensional sphere. According to Case 1, A_{r+1} represents an $(r-1)$-dimensional plane in $\mathcal{S}^n$, therefore

$$A_r = sA_{r+1} = (-1)^{\epsilon} \tilde{s} \vee A_{r+1}, \tag{3.40}$$

where $\epsilon = \frac{(n+2)(n+1)}{2} + 1$ represents the intersection of $(n-1)$-sphere $\tilde{s}$ with $(r-1)$-plane A_{r+1}.

With suitable normalization, we can write $s = c - \rho p_0$. Since $s \wedge A_{r+1} = p_0 \wedge A_{r+1} = 0$, the sphere A_r is also centered at c and has normal radius ρ. Accordingly we represent an $(r-2)$-dimensional sphere in the *standard form*

$$(c - \rho p_0)\ (p_0 \wedge \mathbf{I}_r), \tag{3.41}$$

where $\mathbf{I}_r$ is a unit r-blade of $\mathcal{G}(\mathcal{R}^{n+1})$ representing the minimal space in $\mathcal{R}^{n+1}$ supporting the $(r-2)$-sphere of $\mathcal{S}^n$.

This completes our classification of standard representations for spheres and planes in $\mathcal{S}^n$.

Expanded form

For $r+1$ homogeneous points $a_0, \ldots, a_r$ "in" $\mathcal{S}^n$, where $0 \leq r \leq n+1$, we have

$$A_{r+1} = a_0 \wedge \cdots \wedge a_r = \mathbf{A}_{r+1} + p_0 \wedge \mathbf{A}_r, \tag{3.42}$$

where

$$\begin{array}{lcl} \mathbf{A}_{r+1} & = & \mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_r, \\ \mathbf{A}_r & = & \partial\!\!\!/ \mathbf{A}_{r+1}. \end{array} \tag{3.43}$$

When $\mathbf{A}_{r+1} = 0$, A_{r+1} represents an $(r-1)$-plane, otherwise A_{r+1} represents an $(r-1)$-sphere. In the latter case, $p_0 \wedge \mathbf{A}_{r+1} = p_0 \wedge A_{r+1}$ represents the support plane of the $(r-1)$-sphere in $\mathcal{S}^n$, and $p_0 \wedge \mathbf{A}_r$ represents the $(r-1)$-plane normal to the center of the $(r-1)$-sphere in the support plane. The center of the $(r-1)$-sphere is

$$\frac{\mathbf{A}_r \mathbf{A}_{r+1}^\dagger}{|\mathbf{A}_r||\mathbf{A}_{r+1}|}, \tag{3.44}$$

and the normal radius is

$$1 - \frac{|\mathbf{A}_{r+1}|}{|\mathbf{A}_r|}. \tag{3.45}$$

Since

$$\begin{array}{lcl} A_{r+1}^\dagger \cdot A_{r+1} & = & \det(a_i \cdot a_j)_{(r+1)\times(r+1)} \\ & = & (-\frac{1}{2})^{r+1} \det(|\mathbf{a}_i - \mathbf{a}_j|^2)_{(r+1)\times(r+1)}; \end{array} \tag{3.46}$$

thus, when $r = n+1$, we obtain Ptolemy's Theorem for spherical geometry:

Theorem 7 (Ptolemy's Theorem). *Let $\mathbf{a}_0, \cdots, \mathbf{a}_{n+1}$ be points in $\mathcal{S}^n$, then they are on a sphere or hyperplane of $\mathcal{S}^n$ if and only if* $\det(|\mathbf{a}_i - \mathbf{a}_j|^2)_{(n+2)\times(n+2)} = 0$.

3.5 Stereographic projection

In the homogeneous model of $\mathcal{S}^n$, let $\mathbf{a}_0$ be a fixed point on $\mathcal{S}^n$. The space $\mathcal{R}^n = (\mathbf{a}_0 \wedge p_0)^\sim$, which is parallel to the tangent spaces of $\mathcal{S}^n$ at points $\pm\mathbf{a}_0$, is Euclidean. By the stereographic projection of $\mathcal{S}^n$ from point $\mathbf{a}_0$ to the space $\mathcal{R}^n$, the ray from $\mathbf{a}_0$ through $\mathbf{a} \in \mathcal{S}^n$ intersects the space at the point

$$j_{\mathcal{SR}}(\mathbf{a}) = \frac{\mathbf{a}_0(\mathbf{a}_0 \wedge \mathbf{a})}{1 - \mathbf{a}_0 \cdot \mathbf{a}} = 2(\mathbf{a} - \mathbf{a}_0)^{-1} + \mathbf{a}_0. \tag{3.47}$$

Many advantages of Geometric Algebra in handling stereographic projections are demonstrated in [HS84].

We note the following facts about the stereographic projection:

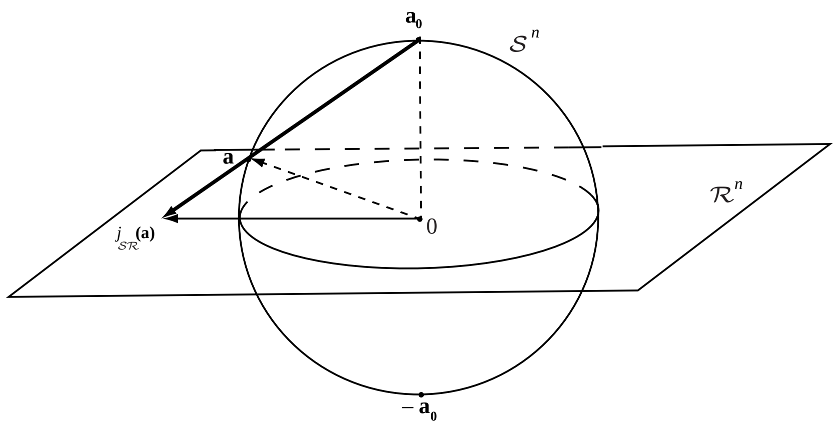

Figure 3: Stereographic projection of $\mathcal{S}^n$ from $\mathbf{a}_0$ to the space normal to $\mathbf{a}_0$.

1. A hyperplane passing through $\mathbf{a}_0$ and normal to $\mathbf{n}$ is mapped to the hyperspace in $\mathcal{R}^n$ normal to $\mathbf{n}$.

2. A hyperplane normal to $\mathbf{n}$ but not passing through $\mathbf{a}_0$ is mapped to the sphere in $\mathcal{R}^n$ with center $c = \mathbf{n} - \dfrac{\mathbf{a}_0}{\mathbf{n}\cdot\mathbf{a}_0}$ and radius $\rho = \dfrac{1}{\sqrt{|\mathbf{n}\cdot\mathbf{a}_0|}}$. Such a sphere has the feature that

$$\rho^2 = 1 + c^2. \tag{3.48}$$

   Its intersection with the unit sphere of $\mathcal{R}^n$ is a unit $(n-2)$-dimensional sphere. Conversely, given a point c in $\mathcal{R}^n$, we can find a unique hyperplane in $\mathcal{S}^n$ whose stereographic projection is the sphere in $\mathcal{R}^n$ with center c and radius $\sqrt{1+c^2}$. It is the hyperplane normal to $a_0 - c$.

3. A sphere passing through $\mathbf{a}_0$, with center $\mathbf{c}$ and normal radius ρ, is mapped to the hyperplane in $\mathcal{R}^n$ normal to $P^{\perp}_{\mathbf{a}_0}(\mathbf{c})$ and with $\dfrac{1-\rho}{\sqrt{1-(1-\rho)^2}}$ as the signed distance from the origin.

4. A sphere not passing through $\mathbf{a}_0$, with center $\mathbf{c}$ and normal radius ρ, is mapped to the sphere in $\mathcal{R}^n$ with center $\dfrac{(1-\rho)\mathbf{p}_0 + P^{\perp}_{\mathbf{a}_0}(\mathbf{c})}{d_n(\mathbf{c},\mathbf{a}_0)-\rho}$ and radius $\dfrac{\sqrt{1-(1-\rho)^2}}{|d_n(\mathbf{c},\mathbf{a}_0)-\rho|}$.

It is a classical result that the map $j_{\mathcal{SR}}$ is a conformal map from $\mathcal{S}^n$ to $\mathcal{R}^n$. The conformal coefficient λ is defined by

$$|j_{\mathcal{SR}}(\mathbf{a}) - j_{\mathcal{SR}}(\mathbf{b})| = \lambda(\mathbf{a},\mathbf{b})|\mathbf{a}-\mathbf{b}|, \text{ for } \mathbf{a},\mathbf{b} \in \mathcal{S}^n. \tag{3.49}$$

We have

$$\lambda(\mathbf{a},\mathbf{b}) = \frac{1}{\sqrt{(1-\mathbf{a}_0\cdot\mathbf{a})(1-\mathbf{a}_0\cdot\mathbf{b})}}. \tag{3.50}$$

Using the null cone of $\mathcal{R}^{n+1,1}$ we can construct the conformal map $j_{\mathcal{SR}}$ trivially: it is nothing but a rescaling of null vectors.

Let

$$e = \mathbf{a}_0 + p_0, \quad e_0 = \frac{-\mathbf{a}_0 + p_0}{2}, \quad E = e \wedge e_0. \tag{3.51}$$

For $\mathcal{R}^n = (e \wedge e_0)^{\sim} = (\mathbf{a}_0 \wedge p_0)^{\sim}$, the map $i_E : x \in \mathcal{R}^n \mapsto e_0 + x + \frac{x^2}{2}e \in \mathcal{N}_e^n$ defines a homogeneous model for the Euclidean space.

Any null vector h in $\mathcal{S}^n$ represents a point in the homogeneous model of $\mathcal{S}^n$, while in the homogeneous model of $\mathcal{R}^n$ it represents a point or point at infinity of $\mathcal{R}^n$. The rescaling transformation $k_{\mathcal{R}} : \mathcal{N}^n \longrightarrow \mathcal{N}_e^n$ defined by

$$k_{\mathcal{R}}(h) = -\frac{h}{h\cdot e}, \text{ for } h \in \mathcal{N}^n, \tag{3.52}$$

induces the conformal map $j_{\mathcal{SR}}$ through the following commutative diagram:

$$\begin{array}{ccc} \mathbf{a}+p_0 \in \mathcal{N}_{p_0}^n & \overset{k_{\mathcal{R}}}{- - - - \longrightarrow} & \dfrac{\mathbf{a}+p_0}{1-\mathbf{a}\cdot\mathbf{a}_0} \in \mathcal{N}_e^n \\ i_{p_0} \Big\uparrow & & \Big\downarrow P_E^{\perp} \\ \mathbf{a} \in \mathcal{S}^n & \overset{j_{\mathcal{SR}}}{- - - - \longrightarrow} & \dfrac{\mathbf{a}_0(\mathbf{a}_0\wedge\mathbf{a})}{1-\mathbf{a}\cdot\mathbf{a}_0} \in \mathcal{R}^n \end{array} \tag{3.53}$$

i.e., $j_{\mathcal{SR}} = P_E^{\perp} \circ k_{\mathcal{R}} \circ i_{p_0}$. The conformal coefficient λ is derived from the following identity: for any vector x and null vectors h_1, h_2,

$$\left| -\frac{h_1}{h_1\cdot x} + \frac{h_2}{h_2\cdot x} \right| = \frac{|h_1 - h_2|}{\sqrt{|(h_1\cdot x)(h_2\cdot x)|}}. \tag{3.54}$$

The inverse of the map $j_{\mathcal{SR}}$, denoted by $j_{\mathcal{RS}}$, is

$$j_{\mathcal{RS}}(u) = \frac{(u^2-1)\mathbf{a}_0 + 2u}{u^2+1} = 2(u-\mathbf{a}_0)^{-1} + \mathbf{a}_0, \text{ for } u \in \mathcal{R}^n. \tag{3.55}$$

According to [HS84], (3.55) can also be written as

$$j_{\mathcal{RS}}(u) = -(u-\mathbf{a}_0)^{-1}\mathbf{a}_0(u-\mathbf{a}_0). \tag{3.56}$$

From the above algebraic construction of the stereographic projection, we see that the null vectors in $\mathcal{R}^{n+1,1}$ have geometrical interpretations in both $\mathcal{S}^n$ and $\mathcal{R}^n$, as do the Minkowski blades of $\mathcal{R}^{n+1,1}$. Every vector in $\mathcal{R}^{n+1,1}$ of positive signature can be interpreted as a sphere or hyperplane in both spaces. We will discuss this further in the next chapter.

3.6 Conformal transformations

In this section we present some results on the conformal transformations in $\mathcal{S}^n$. We know that the conformal group of $\mathcal{S}^n$ is isomorphic with the Lorentz group of $\mathcal{R}^{n+1,1}$. Moreover, a Lorentz transformation in $\mathcal{R}^{n+1,1}$ is the product of at most $n+2$ reflections with respect to vectors of positive signature. We first analyze the induced conformal transformation in $\mathcal{S}^n$ of such a reflection in $\mathcal{R}^{n+1,1}$.

3.6.1 Inversions and reflections

After normalization, any vector in $\mathcal{R}^{n+1,1}$ of positive signature can be written as $s = \mathbf{c} + \lambda p_0$, where $0 \leq \lambda < 1$. For any point $\mathbf{a}$ in $\mathcal{S}^n$, the reflection of a with respect to s is

$$\frac{1+\lambda^2 - 2\lambda \mathbf{c}\cdot\mathbf{a}}{1-\lambda^2}\mathbf{b}, \tag{3.57}$$

where

$$\mathbf{b} = \frac{(1-\lambda^2)\mathbf{a} + 2(\lambda - \mathbf{c}\cdot\mathbf{a})\mathbf{c}}{1+\lambda^2 - 2\lambda\mathbf{c}\cdot\mathbf{a}}. \tag{3.58}$$

If $\lambda = 0$, i.e., if $\tilde{s}$ represents a hyperplane of $\mathcal{S}^n$, then (3.58) gives

$$\mathbf{b} = \mathbf{a} - 2\mathbf{c}\cdot\mathbf{a}\,\mathbf{c}, \tag{3.59}$$

i.e., $\mathbf{b}$ is the reflection of $\mathbf{a}$ with respect to the hyperplane $\tilde{\mathbf{c}}$ of $\mathcal{S}^n$.

If $\lambda \neq 0$, let $\lambda = 1 - \rho$, then from (3.58) we get

$$\left(\frac{\mathbf{c}\wedge\mathbf{a}}{1+\mathbf{c}\cdot\mathbf{a}}\right)^{\dagger}\left(\frac{\mathbf{c}\wedge\mathbf{b}}{1+\mathbf{c}\cdot\mathbf{b}}\right) = \frac{\rho}{2-\rho}. \tag{3.60}$$

Since the right-hand side of (3.60) is positive, $\mathbf{c}, \mathbf{a}, \mathbf{b}$ and $-\mathbf{c}$ are on a half great circle of $\mathcal{S}^n$. Using (3.9) and (3.10) we can write (3.60) as

$$d_s(\mathbf{a},\mathbf{c})d_s(\mathbf{b},\mathbf{c}) = \rho_s^2, \tag{3.61}$$

where ρ_s is the stereographic distance corresponding to the normal distance ρ. We say that $\mathbf{a}, \mathbf{b}$ are *inversive* with respect to the sphere with center $\mathbf{c}$ and stereographic radius ρ_s. This conformal transformation is called an *inversion* in $\mathcal{S}^n$.

An inversion can be easily described in the language of Geometric Algebra. The two inversive homogeneous points a and b correspond to the null directions in the 2-dimensional space $a \wedge (c - \rho p_0)$, which is degenerate when a is on the sphere represented by $(c - \rho p_0)^{\sim}$, and Minkowski otherwise.

Any conformal transformation in $\mathcal{S}^n$ is generated by inversions with respect to spheres, or reflections with respect to hyperplanes.

3.6.2 Other typical conformal transformations

Antipodal transformation

By an antipodal transformation a point $\mathbf{a}$ of $\mathcal{S}^n$ is mapped to point $-\mathbf{a}$. This transformation is induced by the versor p_0.

Rotations

A rotation in $\mathcal{S}^n$ is just a rotation in $\mathcal{R}^{n+1}$. Any rotation in $\mathcal{S}^n$ can be induced by a spinor in $\mathcal{G}(\mathcal{R}^{n+1})$.

Given a unit 2-blade $\mathbf{I}_2$ in $\mathcal{G}(\mathcal{R}^{n+1})$ and $0 < \theta < 2\pi$, the spinor $e^{\mathbf{I}_2\theta/2}$ induces a rotation in $\mathcal{S}^n$. Using the orthogonal decomposition

$$\mathbf{x} = P_{\mathbf{I}_2}(\mathbf{x}) + P_{\mathbf{I}_2}^{\perp}(\mathbf{x}), \text{ for } \mathbf{x} \in \mathcal{S}^n, \tag{3.62}$$

we get

$$e^{-\mathbf{I}_2\theta/2}\mathbf{x}e^{\mathbf{I}_2\theta/2} = P_{\mathbf{I}_2}(\mathbf{x})e^{\mathbf{I}_2\theta} + P_{\tilde{\mathbf{I}}_2}(\mathbf{x}). \tag{3.63}$$

Therefore when $n > 1$, the set of fixed points under this rotation is the $(n-2)$-plane in $\mathcal{S}^n$ represented by $\tilde{\mathbf{I}}_2$. It is called the *axis* of the rotation, where θ is the angle of rotation for the points on the line of $\mathcal{S}^n$ represented by $p_0 \wedge \mathbf{I}_2$. This line is called the *line of rotation*.

For example, the spinor $\mathbf{a}(\mathbf{a}+\mathbf{b})$ induces a rotation from point $\mathbf{a}$ to point $\mathbf{b}$, with $p_0 \wedge \mathbf{a} \wedge \mathbf{b}$ as the line of rotation. The spinor $(\mathbf{c} \wedge \mathbf{a})(\mathbf{c} \wedge (\mathbf{a}+\mathbf{b}))$, where $\mathbf{a}$ and $\mathbf{b}$ have equal distances from $\mathbf{c}$, induces a rotation from point $\mathbf{a}$ to point $\mathbf{b}$ with $p_0 \wedge P_{\mathbf{c}}^{\perp}(\mathbf{a}) \wedge P_{\mathbf{c}}^{\perp}(\mathbf{b})$ as the line of rotation.

Rotations belong to the orthogonal group $O(\mathcal{S}^n)$. A versor in $\mathcal{G}(\mathcal{R}^{n+1,1})$ induces an orthogonal transformation in $\mathcal{S}^n$ if and only if it leaves $\{\pm p_0\}$ invariant.

Tidal transformations

A tidal transformation of coefficient $\lambda \neq \pm 1$ with respect to a point $\mathbf{c}$ in $\mathcal{S}^n$ is a conformal transformation induced by the spinor $1 + \lambda p_0 \wedge \mathbf{c}$. It changes a point $\mathbf{a}$ to point

$$\mathbf{b} = \frac{(1-\lambda^2)\mathbf{a} + 2\lambda(\lambda\mathbf{a}\cdot\mathbf{c} - 1)\mathbf{c}}{1 + \lambda^2 - 2\lambda\mathbf{a}\cdot\mathbf{c}}. \tag{3.64}$$

Points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are always on the same line. Conversely, from $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ we obtain

$$\lambda = \frac{d_n^2(\mathbf{b}, \mathbf{a})}{d_n^2(\mathbf{b}-\mathbf{c}) - d_n^2(\mathbf{a}-\mathbf{c})}. \tag{3.65}$$

By this transformation, any line passing through point $\mathbf{c}$ is invariant, and any sphere with center $\mathbf{c}$ is transformed into a sphere with center $\mathbf{c}$ or $-\mathbf{c}$, or the hyperplane normal to $\mathbf{c}$. The name "tidal transformation" arises from interpreting points $\pm\mathbf{c}$ as the source and influx of the tide.

Given points $\mathbf{a}, \mathbf{c}$ in $\mathcal{S}^n$, which are neither identical nor antipodal, let point $\mathbf{b}$ move on line $\mathbf{ac}$ of $\mathcal{S}^n$, then $\lambda = \lambda(\mathbf{b})$ is determined by (3.65). This function has the following properties:

1. $\lambda \neq \pm 1$, i.e., $\mathbf{b} \neq \pm \mathbf{c}$. This is because if $\lambda = \pm 1$, then $1 + \lambda p_0 \wedge \mathbf{c}$ is no longer a spinor.
2. Let $\underline{\mathbf{c}}(\mathbf{a})$ be the reflection of $\mathbf{a}$ with respect to $\mathbf{c}$, then

$$\underline{\mathbf{c}}(\mathbf{a}) = \mathbf{a} - 2\mathbf{a} \cdot \mathbf{c}\mathbf{c}^{-1}, \tag{3.66}$$

and

$$\lambda(-\underline{\mathbf{c}}(\mathbf{a})) = \infty, \quad \lambda(\underline{\mathbf{c}}(\mathbf{a})) = \mathbf{a} \cdot \mathbf{c}. \tag{3.67}$$

3. When $\mathbf{b}$ moves from $-\underline{\mathbf{c}}(\mathbf{a})$ through $\mathbf{c}$, $\mathbf{a}$, $-\mathbf{c}$ back to $-\underline{\mathbf{c}}(\mathbf{a})$, λ increases strictly from $-\infty$ to ∞.
4. $$\lambda(-\underline{\mathbf{c}}(\mathbf{b})) = \frac{1}{\lambda(\mathbf{b})}. \tag{3.68}$$
5. When $\mathbf{c} \cdot \mathbf{a} = 0$ and $0 < \lambda < 1$, then $\mathbf{b}$ is between $\mathbf{a}$ and $-\mathbf{c}$, and

$$\lambda = d_s(\mathbf{a}, \mathbf{b}). \tag{3.69}$$

When $0 > \lambda > -1$, then $\mathbf{b}$ is between $\mathbf{a}$ and $\mathbf{c}$, and

$$\lambda = -d_s(\mathbf{a}, \mathbf{b}). \tag{3.70}$$

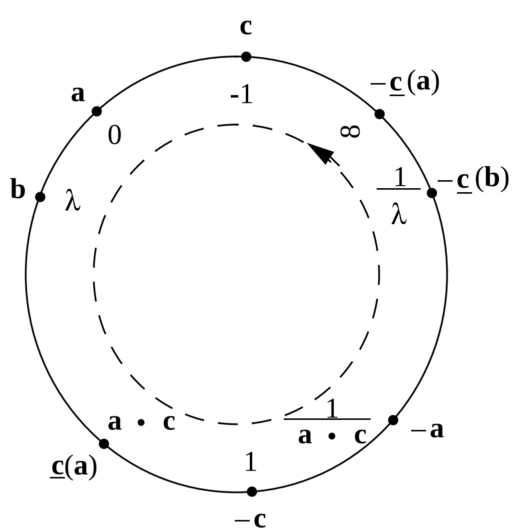

Figure 4: $\lambda = \lambda(\mathbf{b})$ of a tidal transformation.

When $|\lambda| > 1$, a tidal transformation is the composition of an inversion with the antipodal transformation, because

$$1 + \lambda p_0 \wedge \mathbf{c} = -p_0(p_0 - \lambda \mathbf{c}). \tag{3.71}$$

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